AN

EASY INTRODUCTION

TO THE

MATHEMATICS;

IN WHICH

THE THEORY AND PRACTICE

ARE LAID DOWN AND FAMILIARLY EXPLAINED.

To each subject are prefixed,

A BRIEF POPULAR HISTORY OF ITS RISE AND PROGRESS, CONCISE MEMOIRS OF NOTED MATHEMATICAL AUTHORS ANCIENT AND MODERN, AND SOME ACCOUNT OF THEIR WORKS.

The whole forming

A COMPLETE AND EASY SYSTEM

OF

ELEMENTARY INSTRUCTION

IN THE

LEADING BRANCHES OF THE MATHEMATICS;

DESIGNED TO FURNISH STUDENTS WITH THE MEANS OF ACQUIRING CONSIDERABLE PROFICIENCY, WITHOUT THE NECESSITY OF VERBAL ASSISTANCE.

Adapted to the use of

SCHOOLS, JUNIOR STUDENTS AT THE UNIVERSITIES, AND PRIVATE LEARNERS,

ESPECIALLY THOSE WHO STUDY WITHOUT A TUTOR.

IN TWO VOLUMES.

BY CHARLES BUTLER.

Πάντ' ἐστὶν ἐξευρεῖν, ἐὰν μὴ τὸν πόνον
φεύγῃ τις, ὃς πρόσεστι τοῖς ζητουμένοις. ΦΙΛΗΜΩΝ.

Shake off your ease, and send your name
To immortality and fame,
By ev'ry hour that flies. WATTS.

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ERRATA.

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7 To the note at the bottom of the page add, "The sign $\because$ denotes *therefore*."

18 Last line, for =*the difference*, read d=*the difference.*

44 Line 8, for $n-1$ read $\overline{n-1}$.

Line 20, for $n.\overline{n-1.n-2.n-3.n-4}$, read $n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}$.

64 Line 3, for *Fo since*, read *For since ad.*

66 Art. 68, after the word CONVERTENDO, add, *Euclid pr. E. Book* 5.

71 The note at the bottom is useless here, as it occurs in the latter part of *The Properties of Numbers, pp.* 108, 109.

97 Last line, for $\overline{b=1}.q$, read $\overline{b+1}.q$.

123 Dele the third and five following lines.

252 Art. 15. line 2, dele "*or simple.*"

320 The three lines *AG, BD,* and *EC* in the figure, *should* intersect in the point *F* on the circumference. Two or three of the figures in Part X. are very indifferently cut, but it is hoped that there is nothing which can possibly mislead, or affect the demonstrations.

AN EASY INTRODUCTION TO THE MATHEMATICS, &c.

PART IV.

ALGEBRA.

GENERAL PROBLEMS.

ART. 1.

ALGEBRA is divided into two kinds, *numeral* and *literal*, both depending on the same principles and employing the same operations.

2. Numeral algebra [a] is that chiefly used in the solution of numeral problems, in which all the given quantities are expressed by numbers, the unknown quantities only being denoted by letters or other convenient symbols. This kind of algebra has been largely treated of in the preceding volume.

3. Literal or specious algebra [b] is that in which all the quan-

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[a] Numeral algebra is that part of the science, which the Europeans received from the Arabs, about the middle of the 15th century. It does not appear that the latter people, or even Diophantus, (who is the only Greek writer on the subject at present known,) understood any thing of the general methods now in use; accordingly we find but little attempted beyond the solution of numerical problems, in the writings of Lucas de Burgo, Cardan, Diophantus, Tartalea, Bombelli, Peletarius, Stevinus, Recorde, or any other of the early authors who treated on algebra.

[b] Vieta, the great improver of almost every branch of the Mathematics

tities, both known and unknown, are represented by letters and other general characters. This general mode of designation is of the greatest use; as every conclusion, and indeed every step by which it is obtained, becomes an universal rule for performing every possible operation of the kind.

4. In literal algebra, the initial letters *a*, *b*, *c*, *d*, &c. are usually employed to represent known or given quantities, and the final letters *x*, *y*, *z*, *w*, *v*, &c. to represent unknown quantities, whose values are required to be found.

5. A general algebraic problem is that in which all the quantities concerned, both known and unknown, are represented by letters or other general characters. Not only such problems as have their conditions proposed in general terms, are here implied, every particular numeral problem may be made general, by substituting letters for the known quantities concerned in it: when this is done, the problem which was originally proposed in a particular form, is now become a general problem.

6. Every problem consists of two parts, the *data*, and the *quæsita* [c]; the data include all the conditions and quantities given, and the quæsita the quantities sought.

7. The process by which the quæsita are obtained by means of the data, that is, by which the values of the unknown quantities are found, is called the ANALYSIS [d], or the ANALYTICAL

known in his time, is considered as the first who introduced the literal notation of given quantities into general practice, about the year 1600. Cardan had indeed given specimens of such an improvement, in his algebra, as early as 1545; but as the advantages of a general mode of notation were then in all probability not sufficiently understood, the method was not adopted until about the time we have mentioned. The improvement of Vieta was further advanced and applied by Thomas Harriot, the father of modern algebra, about 1620; likewise by Oughtred in 1631, Des Cartes in 1637, and afterwards by Wallis, Newton, Leibnitz, the Bernoullis, Baker, Raphson, Sterling, Euler, &c. and is justly perferred by all modern algebraists, on account of the universality of its application. The letters of the alphabet are called by Vieta, *species*; whence algebra has been named *arithmetica speciosa*: reasoning in species, as applied to the solution of mathematical problems, appears to have been borrowed from the Civilians, who determine cases at law between imaginary persons, representing them abstractedly by A and B; these they call cases in *species*: this is the more probable, as Vieta himself was a lawyer.

[c] The word *data* means things given, and *quæsita* things sought.

[d] The word analysis, (from the Greek *αναλυω*, *resolvo*,) in its general sense,

INVESTIGATION; it is also named the SOLUTION, or RESOLUTION of the problem.

8. When the values of the unknown quantities are found and expressed in known terms, the substituting these values, each for its respective unknown quantity in the given equations; that is, by reasoning in an order the converse of analysis, and thereby ultimately proving that the quantities thus assumed have the properties described in the problem, is called the SYNTHESIS [e], or SYNTHETICAL DEMONSTRATION of the problem, and frequently the COMPOSITION.

9. When the value of any quantity, which was at first unknown, is found and expressed in known terms, the translating of this value out of algebraic into common language, wherein the relation of the quantities concerned is simply declared, is called deducing a THEOREM [f]; but if the translation be exhibited in the form of a *precept*, it is called a CANON [g], or RULE.

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implies the resolving of any thing which is compounded, into its constituent simple elements: thus in algebra, several quantities, known and unknown, being compounded together, analysis is the disentangling of them; by its operation, each of the quantities included in the composition is disengaged from the rest, and its value found in terms of the known quantities concerned. This being the proper business of algebra, the science itself on that account is frequently termed analysis, which name however implies other branches besides algebra.

[e] Synthesis (from the Greek συνθεσις, *compositio*) is the converse of analysis. By analysis, as we have shewn, compound quantities are decompounded; by synthesis, the quantities disentangled and brought out by the analysis, are again compounded, by which operation the original compound quantity is reproduced; hence synthesis is called *the method of demonstration*, and analysis *the method of investigation*.

[f] A theorem (from the Greek θεωρημα, *a speculation*) is a proposition terminating in theory, in which something is simply affirmed or denied. Theorems, as we have observed before, are investigated or discovered by analysis, and their truth demonstrated by synthesis.

[g] A canon (from the Greek κανων) or rule (from the Latin *regula*) is a system of precepts directing what operations must be performed, in order to produce any proposed result; such are the rules of common arithmetic. It is noticed above, that a theorem, and a canon, are of nearly the same import, differing only in the form of words in which they are laid down: the distinction may appear trifling, but it is observed by writers, whose skill and judgment are unquestionable, and on that account we thought proper not entirely to omit it.

10. A COROLLARY[h] is a truth obtained intermediately, and by the bye; an additional truth, over and above what the problem proposed to search out, or prove.

11. A SCHOLIUM is a remark or explanatory observation, intended to illustrate something preceding.

12. To make what has been delivered perfectly plain, to the analytical investigation of several of the following problems, is added the synthetical demonstration; instances are given of deducing theorems and of deriving canons or rules from the analysis; examples are likewise proposed, where necessary, to shew the method of applying the general conclusions to particular cases; and finally, the manner of converting any particular numerical problem into a general form, and of substituting and deriving expressions for the unknown quantities, in a great variety of ways, are shewn and explained.

PROBLEM 1[i]. Given the sum and difference of two magnitudes, to find the magnitudes.

ANALYSIS. *Let* x = *the greater magnitude*, y = *the less*, s = *the given sum*, d = *the given difference*.

Then by the problem $x+y=s$.

And $x-y=d$.

Whence by addition $2x=s+d$, *or* $x=\frac{s+d}{2}$.

And by subtraction $2y=s-d$, *or* $y=\frac{s-d}{2}$. Q. E. I.[k]

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[h] The term corollary is derived from the Latin corollis, *something given over and above*; and scholium from σχόλιον, *a short comment*.

[i] Several of the problems here given, with others of the kind, may be found in Saunderson's Elements of Algebra, 2 vol. 4to. 1740. in the Abridgment of the same, and in Ludlam's Rudiments of Mathematics.

[k] In the technical language of the mathematicians, Q. E. I. denotes, quod erat investigandum, *which was to be investigated*; Q. E. D. quod erat demonstrandum, *which was to be demonstrated*; and Q. E. F. quod erat faciendum, *which was to be done*. The first is subjoined to analytical investigations, the second to synthetical demonstrations, and the third to the proof that a proposed practical operation is actually performed and done. We have adopted the distinctions of *analysis*, *synthesis*, *theorem*, *canon*, &c. and likewise the above abbreviations in a few instances, to assist the learner in a knowledge of their use, when any book containing them may happen to fall into his hands.

SYNTHESIS. *Because by the problem $x+y=s$, and $x-y=d$, if the values found by the analysis be really equivalent to x and y respectively, then those values being substituted for x and y in the given equations, and the latter value added to the former in the first equation, and subtracted from it in the second, the results will be s and d. Let us make the experiment.*

First $\frac{s+d}{2}+\frac{s-d}{2}=\frac{2s}{2}=s$, *which answers the first condition, namely that* $x+y=s$.

Secondly $\frac{s+d}{2}-\frac{s-d}{2}=\frac{2d}{2}=d$, *which answers the second condition, namely that* $x-y=d$; *wherefore the values of x and y found by the analysis, are those which the problem requires. Q. E. D.*

THEOREM 1. If the difference of any two magnitudes be added to their sum, half the result will be the greater magnitude; but if the difference be subtracted from the sum, half the result will be the less.

SCHOLIUM. The form of any general algebraic expression may be changed at pleasure, provided its value be not altered thereby: by this means a theorem may sometimes be laid down in a more convenient form than that derived immediately from the analysis. The value of x found above, viz. $\frac{s+d}{2}$ may be thus expressed, $\frac{s}{2}+\frac{d}{2}$; and the value of y, viz. $\frac{s-d}{2}$, thus, $\frac{s}{2}-\frac{d}{2}$: hence we obtain the above theorem in a more convenient form, viz.

THEOREM 2. Half the difference of two magnitudes being added to half their sum, the result will be the greater; and half the difference being subtracted from half the sum, the result will be the less.

COROLLARY. Hence it appears, that theorems and canons may be derived from any general algebraic investigation, which will solve every particular case subject to the same conditions with the general problem, to which that investigation belongs.

CANON 1. (From theorem 1.) Add the difference of any two magnitudes to their sum, and divide the result by 2, the quotient

will be the greater magnitude. Subtract the difference from the sum, and divide the result by 2, the quotient will be the less.

CANON 2. (from theorem 2.) Add half the difference of any two magnitudes to half their sum, and the result will be the greater magnitude. Subtract half the difference from half the sum, and the result will be the less.

EXAMPLES.—1. Given the sum of two numbers 20, and their difference 12, to find the numbers.

By canon 1. $\frac{20+12}{2}=\frac{32}{2}=16=$ *the greater number.*

$\frac{20-12}{2}=\frac{8}{2}=4=$ *the less number.*

By canon 2. $\frac{20}{2}+\frac{12}{2}=10+6=16=$ *the greater number.*

$\frac{20}{2}-\frac{12}{2}=10-6=4=$ *the less number, as before.*

2. If the sum of two numbers be 31, and their difference 14, what are the numbers?

By canon 1. $\frac{31+14}{2}=\frac{45}{2}=22\frac{1}{2}=$ *the greater.*

$\frac{31-14}{2}=\frac{17}{2}=8\frac{1}{2}=$ *the less.*

By canon 2. $\frac{31}{2}+\frac{14}{2}=15\frac{1}{2}+7=22\frac{1}{2}=$ *the greater.*

$\frac{31}{2}-\frac{14}{2}=15\frac{1}{2}-7=8\frac{1}{2}=$ *the less, as before.*

3. The sum of two numbers is 16, and their difference 6, to find the numbers? *Ans.* 11 *and* 5.

4. Given the sum 100, and the difference 51, to find the numbers? *Ans.* $75\frac{1}{2}$ *and* $24\frac{1}{2}$.

5. Given the sum of two numbers $4\frac{1}{2}$, and their difference $1\frac{1}{3}$, to find the numbers? *Ans.* $2\frac{11}{12}$ *and* $1\frac{7}{12}$.

6. Given the sum 123, and difference $10\frac{1}{2}$, to find the numbers?

PROBLEM 2. What magnitude is that, to which a given magnitude being added, and from it the same given magnitude being subtracted, the sum shall be to the remainder in a given ratio?

ANALYSIS. *Let x=the magnitude required, a=the given magnitude to be added and subtracted; r and s the terms of the given ratio; then by the problem,* $x+a : x-a :: r : s$, $\because rx-ar =sx+as$, $\because rx-sx=ar+as$, *and* $x=\frac{ar+as}{r-s}=\frac{r+s}{r-s}a$, *the magnitude required*[1]. *Q. E. I.*

SYNTHESIS. *First,* $\frac{ar+as}{r-s}+a=\frac{ar+as+ar-as}{r-s}=\frac{2ar}{r-s}$.

Secondly, $\frac{ar+as}{r-s}-a=\frac{ar+as-ar+as}{r-s}=\frac{2as}{r-s}$ $\because$

$\frac{2ar}{r-s} : \frac{2as}{r-s} :: \frac{2a}{r-s}\times r : \frac{2a}{r-s}\times s :: r : s$. *Q. E. D.*

EXAMPLES.—1. What number is that, which with 3 added to it, and also subtracted from it, the sum is to the remainder as 9 to 7?

Here $a=3$, $r=9$, $s=7$, *and* $x=\frac{9+7}{9-7}\times 3=\frac{16}{2}\times 3=8\times 3 =24$.

2. Required a number, which being increased and decreased by $\frac{1}{12}$, the sum is to the remainder as 3 to 1?

Here $a=\frac{1}{12}$, $r=3$, $s=1$, $\because x=\frac{3+1}{3-1}\times\frac{1}{12}=\frac{4}{2}\times\frac{1}{12}=\frac{4}{24} =\frac{1}{6}$.

3. If 10 be added to, and subtracted from, a certain number, the sum will be to the remainder as 11 to 9; what is the number? *Ans.* 100.

4. If $\frac{1}{4}$ be added to, and subtracted from, a required number, the results will be as 15 to 13; what is the number?

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[1] Here it is plain if $r=s$, then $x+a=x-a$, consequently $a=o$, whence any magnitude taken at pleasure for x will satisfy the conditions of the problem. If $r>s$ (the quantity $\frac{r+s}{r-s}a$, or) the value of x will be affirmative; but if $r<s$, the value of x will be negative: in the former case the ratio is that of *the greater inequality*, but in the latter, it is the ratio of *the lesser inequality*, and the given problem is changed into the following; "To find a magnitude, from and to which a given magnitude being subtracted and added, the remainder shall be to the sum as r to s."

PROBLEM 3. To divide a given magnitude into two parts in a given ratio.

ANALYSIS. *Let a=the given magnitude, x=one of the parts, then will $a-x$=the other part: also, let r and s represent the terms of the given ratio.*

Then by the problem $x : a-x :: r : s$, $\because sx=ar-rx$, *and* $rx+sx=ar$, $\therefore x=\frac{ar}{r+s}$, *and* $a-x=a-\frac{ar}{r+s}=\frac{ar+as-ar}{r+s}=\frac{as}{r+s}$. *Q. E. I.*

SYNTHESIS. *First,* $\frac{ar}{r+s}+\frac{as}{r+s}=\frac{ar+as}{r+s}=\frac{\overline{r+s}\,a}{r+s}=a$.

Secondly, $\frac{ar}{r+s}:\frac{as}{r+s}::ar:as::r:s$. *Q. E. D.*

EXAMPLES.—1. Divide the number 32 into two parts, in the ratio of 9 to 7.

Here $a=32$, $r=9$, $s=7$, *and* $x=\frac{32\times 9}{9+7}=2\times 9=18$, *and* $a-x=(\frac{as}{r+s}=)\ 32-18=14$.

2. Divide $\frac{3}{7}$ into two parts, in the ratio of $\frac{2}{5}$ to $\frac{4}{9}$.

Here $a=\frac{3}{7}$, $r=\frac{2}{5}$, $s=\frac{4}{9}$, *and* $x=\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}+\frac{4}{9}=\frac{6}{35}+\frac{38}{45}=\frac{6}{35}\times\frac{45}{38}=\frac{3}{7}\times\frac{9}{19}=\frac{27}{133}$, *and* $a-x=\frac{3}{7}-\frac{27}{133}=\frac{399-189}{931}=\frac{210}{931}=\frac{30}{133}$.

3. Divide 60 into two parts, in the ratio of 1 to 3. *Ans.* 15 *and* 45.

4. Divide 5 into two parts, in the ratio of 20 to 19.

PROBLEM 4. To divide a given number into two parts, such, that certain proposed multiples of the parts being taken, their sum shall equal another given number?

ANALYSIS. *Let a=the given number to be divided, x and y= the parts respectively, r=the multiplier of x, s=the multiplier of y, and b=the sum of the multiples of x and y; then by the problem, $x+y=a$, and $rx+sy=b$. From the first of these equations, we have $y=a-x$; and from the latter, $y=\frac{b-rx}{s}$; $\because a-x=$*

$\frac{b-rx}{s}$, $\because$ $as-sx=b-rx$, or $rx-sx=b-as$, $\because$ $x=\frac{b-as}{r-s}$, and y $=(a-x=)\ a-\frac{b-as}{r-s}=\frac{ar-as-b+as}{r-s}=\frac{ar-b}{r-s}$. Q. E. I.

SYNTHESIS. *First,* $\frac{b-as}{r-s}+\frac{ar-b}{r-s}=\frac{ar-as}{r-s}=\frac{\overline{r-s}.a}{r-s}=a.$

Secondly, $\frac{b-as}{r-s}\times r+\frac{ar-b}{r-s}\times s=(\frac{br-asr}{r-s}+\frac{asr-bs}{r-s}$ $=\frac{br-bs}{r-s}=)\frac{\overline{r-s}.b}{r-s}=b.$ *Q. E. D.*

EXAMPLES.—1. Let 100 be divided into two parts, so that four times one part being added to three times the other, the sum will be 355.

Here $a=100$, $r=4$, $s=3$, *and* $b=355$; $\because$ $x=\frac{b-as}{r-s}=$ $\frac{355-100\times 3}{4-3}=\frac{355-300}{1}=55$, *and* $y=\frac{ar-b}{r-s}=\frac{100\times 4-355}{4-3}=$ $\frac{400-355}{1}=45$.

2. To divide 13 into two parts, so that three times one part, added to five times the other, will make 47.

Here $a=13$, $r=3$, $s=5$, *and* $b=47$; $\because$ $\frac{47-13\times 5}{3-5}=$ $\frac{47-65}{-2}=\frac{-18}{-2}=9$, *and* $\frac{13\times 3-47}{3-5}=\frac{39-47}{-2}=\frac{-8}{-2}=4$.

3. To divide 23 into two parts, so that the sum of 9 times the first part, added to 7 times the second, may make 199.

PROBLEM 5. Given the sum and quotient of two numbers, to find them.

ANALYSIS. *Let* $s=$ *the given sum,* $q=$ *the given quotient,* x *and* $y=$ *the numbers required; then by the problem,* $x+y=s$, *and* $\frac{x}{y}=q$. *From the first* $x=s-y$, *and from the second* $x=qy$, $\because$ $qy=s-y$, *or* $qy+y=s$, $\because$ $y=\frac{s}{q+1}$, *and* $x=(qy=)\ \frac{qs}{q+1}$. *Q. E. I.*

SYNTHESIS. *First,* $\frac{qs}{q+1}+\frac{s}{q+1}=\frac{qs+s}{q+1}=\frac{\overline{q+1}.s}{q+1}=s.$

Secondly, $\frac{qs}{q+1}\div\frac{s}{q+1}=\frac{q}{1}=q.$ *Q. E. D.*

Examples.—1. The sum of two numbers is 54, and their quotient 8, to find the numbers?

Here $s=54$, $q=8$ $\because$ $x=\frac{qs}{q+1}=\frac{8\times 54}{8+1}=\frac{432}{9}=48$: *and* $y=\frac{s}{q+1}=\frac{54}{8+1}=\frac{54}{9}=6$.

2. Given the sum 3, and quotient 11, of two numbers, to find them?

Here $s=3$, $q=11$, $\because x=\frac{33}{12}=2\frac{3}{4}$, *and* $y=\frac{3}{12}=\frac{1}{4}$.

3. If the sum be 144, and quotient $2\frac{3}{11}$, what are the numbers? *Ans.* 100 *and* 44.

4. Let the sum be 91, and quotient 6; required the numbers?

Problem 6. The sum of two numbers and the difference of their squares being given, to find the numbers?

Analysis. *Let* $s=$ *the given sum*, $b=$ *the given difference of their squares*, x *and* $y=$ *the required numbers: then by the problem*, $x+y=s$, *and* $x^2-y^2=b$. *From the first*, $x=s-y$; *this value being substituted for* x *in the second, it becomes* $(\overline{s-y}|^2-y^2=s^2-2sy+y^2-y^2=)$ $s^2-2sy=b$, $\because$ $2sy=s^2-b$, *and* $y=\frac{s^2-b}{2s}$ [m];

whence $x=(s-y=)$ $s-\frac{s^2-b}{2s}=\frac{2s^2-s^2+b}{2s}=\frac{s^2+b}{2s}$. *Q. E. I.*

Synthesis. *First*, $\frac{s^2+b}{2s}+\frac{s^2-b}{2s}=\frac{2s^2}{2s}=s$.

Secondly, $\overline{\frac{s^2+b}{2s}}\Big|^2-\overline{\frac{s^2-b}{2s}}\Big|^2=\frac{s^4+2s^2b+b^2}{4s^2}-\frac{s^4-2s^2b+b^2}{4s^2}=\frac{4s^2b}{4s^2}=b$. *Q. E. D.*

Examples.—1. Given the sum 14, and the difference of the squares 28, of two numbers, to find them?

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[m] When $\overline{s}|^2 < b$, y will be negative, and the first given equation is changed into $x-y=s$, but the second remains the same; for the sign of y^2 is not altered by changing the sign of y. The problem by this change becomes the following; Given the difference, and the difference of the squares, to find the numbers. *See Ludlam*, p. 150.

Here $s=14$, $b=28$, $\because$ $x=\frac{14^2+28}{2\times14}=\frac{224}{28}=8$, *and* $y=\frac{14^2-28}{2\times14}=\frac{168}{28}=6$.

2. If the sum be 4, and the difference of the squares likewise 4, what are the numbers?

Here $s=4$, $b=4$, $\because$ $x=2\frac{1}{2}$, $y=1\frac{1}{2}$.

3. The sum is 101, and the difference of the squares 100, what are the numbers?

PROBLEM 7. Given the product and quotient of two numbers, to find the numbers?

ANALYSIS. *Let* $p=$ *the given product*, $q=$ *the given quotient*, x *and* $y=$ *the required numbers respectively; then by the problem,* $xy=p$, *and* $\frac{x}{y}=q$; *from the latter,* $x=qy$; *this substituted for* x *in the former, gives* $qy^2=p$ $\because$ $y^2=\frac{p}{q}$, *and* $y=\sqrt{\frac{p}{q}}$; $\because$ $x=qy=q\sqrt{\frac{p}{q}}=\sqrt{\frac{q^2p}{q}}=\sqrt{pq}$. *Q. E. I.*

SYNTHESIS. *First,* $\sqrt{pq}\times\sqrt{\frac{p}{q}}=\sqrt{\frac{p^2q}{q}}=\sqrt{p^2}=p$.

Secondly, $\sqrt{pq}\div\sqrt{\frac{p}{q}}=\sqrt{pq}\times\sqrt{\frac{q}{p}}=\sqrt{\frac{pq^2}{p}}=\sqrt{q^2}=q$. *Q. E. D.*

EXAMPLES.—1. Given the product 196, and quotient 4, to find the numbers?

Here $p=196$, $q=4$, $\because\sqrt{196\times4}=\sqrt{784}=28=x$; *and* $\sqrt{\frac{196}{4}}=\sqrt{49}=7=y$.

2. The product is $\frac{5}{9}$, and the quotient $1\frac{1}{4}$; required the numbers?

Here $p=\frac{5}{9}$, $q=\frac{5}{4}$, $\because$ $x=\sqrt{\frac{25}{36}}=\frac{5}{6}$, *and* $y=\sqrt{\frac{4}{9}}=\frac{2}{3}$.

3. If the product be 605, and the quotient 5, what are the numbers?

PROBLEM 8. Given the sum and product of two numbers, to find them?

ANALYSIS. *Let s=the given sum, p=the given product, x and y=the numbers required. Then by the problem, $x+y=s$, and $xy=p$; from the first $y=s-x$; this value substituted for y in the second, it becomes $sx-x^2=p$, $\because x^2-sx=-p$; complete the square, and* $x^2-sx+\frac{s^2}{4}=\frac{s^2}{4}-p=\frac{s^2-4p}{4}$, $\because$ $x-\frac{s}{2}=\pm\sqrt{\frac{s^2-4p}{4}}=\pm\frac{\sqrt{s^2-4p}}{2}$, $\because x=\frac{s\pm\sqrt{s^2-4p}}{2}$, *and* $y=(s-x=)\ s-\frac{s\pm\sqrt{s^2-4p}}{2}=\frac{2s-s\mp\sqrt{s^2-4p}}{2}=\frac{s\mp\sqrt{s^2-4p}}{2}$. *Q. E. I.*

SYNTHESIS. *First,* $\frac{s\pm\sqrt{s^2-4p}}{2}+\frac{s\mp\sqrt{s^2-4p}}{2}=\frac{2s}{2}=s$.

Secondly, $\frac{s\pm\sqrt{s^2-4p}}{2}\times\frac{s\mp\sqrt{s^2-4p}}{2}=\frac{4p}{4}=p$.

Q. E. D.

EXAMPLES.—1. Given the sum 17, and product 72, to find the numbers?

Here $s=17$, $p=72$, $\because x=\frac{17\pm\sqrt{289-288}}{2}=\frac{17\pm1}{2}=9$ *or* 8, *and* $y=\frac{17\mp\sqrt{289-288}}{2}=\frac{17\mp1}{2}=8$ *or* 9; *whence, if* $x=9$, *then* $y=8$; *but if* $x=8$, *then* $y=9$.

2. If the sum be $\frac{11}{12}$, and product $\frac{1}{6}$, what are the numbers?

Here $s=\frac{11}{12}$, $p=\frac{1}{6}$, $x=\frac{2}{3}$, $y=\frac{1}{4}$.

3. Let the sum be 21, and product 20, required the numbers?

PROBLEM 9. The sum of two numbers, and the sum of their squares being given, to find the numbers?

ANALYSIS. *Let s=the sum, a=the sum of the squares, x and y=the numbers sought. Then by the problem, $x+y=s$, and $x^2+y^2=a$; now from the first $y=s-x$, $\because y^2=s^2-2sx+x^2$; this value substituted for y^2 in the second equation, it becomes $x^2+s^2-2sx+x^2=a$; that is, $2x^2-2sx=a-s^2$, $\because x^2-sx=\frac{a-s^2}{2}$, $\because x^2-sx+\frac{s^2}{4}=(\frac{a-s^2}{2}+\frac{s^2}{4}=)\frac{2a-s^2}{4}$.*

When the square is completed, the process may be simplified by substituting a more convenient expression for the known side of the equation; thus, in the above equation, instead of $\frac{2a-s^2}{4}$, *let* $\frac{R^2}{4}$ [a] *be substituted, and it will become* $x^2-sx+\frac{s^2}{4}=\frac{R^2}{4}$; *whence by evolution,* $x-\frac{s}{2}=\pm\sqrt{\frac{R^2}{4}}=\pm\frac{R}{2}$, $\therefore x=(\frac{s}{2}\pm\frac{R}{2}=)\frac{s\pm R}{2}$, *and* $y=(s-x=)\ s-\frac{s\pm R}{2}=\frac{2s-s\mp R}{2}=\frac{s\mp R}{2}$. *Q. E. I.*

SYNTHESIS. *First,* $\frac{s\pm R}{2}+\frac{s\mp R}{2}=\frac{2s}{2}=s$.

Secondly, $\overline{\frac{s\pm R}{2}}\Big|^2+\overline{\frac{s\mp R}{2}}\Big|^2=\frac{s^2\pm 2sR+R^2}{4}+\frac{s^2\mp 2sR+R^2}{4}=\frac{2s^2+2R^2}{4}=\frac{s^2+R^2}{2}=$ *(by restoring the value of* R^2*)* $\frac{s^2+2a-s^2}{2}=\frac{2a}{2}=a$. *Q. E. D.*

EXAMPLES.—1. Let the sum 9, and the sum of the squares 45, be proposed, to find the numbers?

Here $s=9$, $a=45$, *then* $R=\sqrt{2a-s^2}=\sqrt{90-81}=\sqrt{9}=3$, *and* $x=\frac{9+3}{2}=\frac{12}{2}=6$, *and* $y=\frac{9-3}{2}=\frac{6}{2}=3$.

2. Let the sum 2.25, and the sum of the squares 2.5625, be given.

Here $s=2.25$, $a=2.5625$, $R=.25$, $x=1.25$, $y=1$.

3. Given the sum 15, and sum of the squares 137, to find the numbers?

PROBLEM 10. Given the product, and the sum of the squares of two numbers, to find them?

ANALYSIS. *Let* $p=$ *the product,* $a=$ *the sum of the squares,* x *and* $y=$ *the required numbers* [b]; *then by the problem,* $xy=p$, *and*

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[a] Since $2a-s^2=R^2$, it follows, that if $s^2>2a$, the problem will be impossible; because R^2 will be negative in that case, and consequently will have no square root.

[b] Let $x=$ the greater of two numbers, $y=$ the less, $s=$ their sum, $=$ the difference, $p=$ the product, $q=$ the quotient, $a=$ the sum of

$x^2+y^2=a$. *From the first,* $y=\frac{p}{x}$, $\because y^2=\frac{p^2}{x^2}$; *substitute this value for* y^2 *in the second, and* $x^2+\frac{p^2}{x^2}=a$, $\because x^4+p^2=ax^2$, *or* $x^4-ax^2=-p^2$, $\because x^4-ax^2+\frac{a^2}{4}=(\frac{a^2}{4}-p^2=\frac{a^2-4p^2}{4}$, *which by substitution*$=)\ \frac{R^2}{4}$, *whence* $x^2-\frac{a}{2}=(\pm\sqrt{\frac{R^2}{4}}=)\pm\frac{R}{2}$, $\because x^2=\frac{a\pm R}{2}$, *and* $x=\pm\sqrt{\frac{a\pm R}{2}}$; *also* $y=(\frac{p}{x}=)\ p\div\pm\sqrt{\frac{a\pm R}{2}}$; *but, in order to obtain the value of y in terms of a and R, we must substitute for p its equal* $\sqrt{\frac{a^2-R^2}{4}}$, *(which is derived from the above equation* $\frac{a^2-4p^2}{4}=\frac{R^2}{4}$,*) wherefore* $y=p\div\pm\sqrt{\frac{a\pm R}{2}}$, *becomes* $=\pm\sqrt{\frac{a^2-R^2}{4}}\div\pm\sqrt{\frac{a\pm R}{2}}=\pm\sqrt{\frac{a\mp R}{2}}$. *Q. E. I.*

SYNTHESIS. *First,* $\pm\sqrt{\frac{a\pm R}{2}}\times\pm\sqrt{\frac{a\mp R}{2}}=\sqrt{\frac{a^2-R^2}{4}}$, *(which by restoring the value of* R^2, *viz.* a^2-4p^2*)*$=\sqrt{\frac{a^2-a^2+4p^2}{4}}=\sqrt{\frac{4p^2}{4}}=p$.

Also, $\sqrt{\frac{a\pm R}{2}}\Big|^2+\sqrt{\frac{a\mp R}{2}}\Big|^2=\frac{a\pm R}{2}+\frac{a\mp R}{2}=\frac{2a}{2}=a$.

Q. E. D.

EXAMPLES.—1. If the product be 24, and the sum of the squares 52, what are the numbers?

Here $p=24$, $a=52$, $R=\overline{(a^2-4p^2)}|^{\frac{1}{2}}=)\ 20$, $x=\sqrt{\frac{52+20}{2}}=\sqrt{36}=6$; $y=\sqrt{\frac{52-20}{2}}=\sqrt{\frac{32}{2}}=4$.

2. Given the product 1.32, and the sum of the squares 2.65, to find the numbers?

Here $p=1.32$, $a=2.65$, $R=.23$, $x=1.2$, $y=1.1$.

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the squares, b=the difference of the squares; any two of these eight (x, y, s, d, p, q, a, and b) being given, the remaining six may thence be found, as was first shewn by Dr. Pell, in his Additions to Rhonius's Algebra, 1688. These problems may be found wrought out at length in *Ward's Young Mathematician's Guide*, 8th edition, London, 1724.

3. Given the product 117, and the sum of the squares 250, to find the numbers?

PROBLEM 11. A vintner makes a mixture of 100 gallons, with wine at 6 shillings a gallon, and wine at 10 shillings a gallon: what quantity of each sort must he put in, so as to afford to sell the compound at 7 shillings a gallon without loss?

ANALYSIS. *Let* $a=6$, $b=10$, $s=100$, $m=7$; $x=$ *the quantity at 6 shillings*, $y=$ *the quantity at 10 shillings. Then by the problem*, $x+y=s$, *and* $ax+by=ms$; *from the first*, $x=s-y$; *from the second*, $x=\frac{ms-by}{a}$, $\therefore s-y=\frac{ms-by}{a}$, *or* $as-ay=ms-by$, *or* $ay-by=as-ms$; *that is*, $\overline{a-b}.y=\overline{a-m}.s$, $\therefore y=\frac{a-m}{a-b}.s$, $\therefore x=(s-y=)\ s-\frac{a-m}{a-b}.s=\frac{as-bs}{a-b}-\frac{as-ms}{a-b}=\frac{ms-bs}{a-b}=\frac{m-b}{a-b}.s$. *Q. E. I.*

SYNTHESIS. *First*, $\frac{m-b}{a-b}.s+\frac{a-m}{a-b}.s=\frac{a-b}{a-b}.s=s$. *Likewise* $a\times\frac{m-b}{a-b}.s+b\times\frac{a-m}{a-b}.s=\frac{am-ab}{a-b}.s+\frac{ab-bm}{a-b}.s=\frac{am-bm}{a-b}.s=\frac{a-b}{a-b}.ms=ms$. *Q. E. D.*

The above problem resolved in numbers, gives $x=\frac{m-b}{a-b}.s=\frac{7-10}{6-10}\times100=\frac{-3}{-4}\times100=\frac{3}{4}\times100=75$ *gallons at* 6 *shillings*; *and* $y=\frac{a-m}{a-b}.s=\frac{6-7}{6-10}\times100=\frac{-1}{-4}\times100=\frac{1}{4}\times100=25$ *gallons at* 10 *shillings a gallon.*

PROBLEM 12. Towards the expense of building a bridge, A paid 1000*l.* more than B, and 2000*l.* more than C, and the square of A's payment equalled the sum of the squares of the other two; what sum did each contribute?

ANALYSIS. *Let* $a=1000$, *then* $2a=2000$, *also let* $x=$ *C's payment, then will* $x+a=$ *B's payment, and* $x+2a=$ *A's payment*; *whence by the problem* $\overline{x+2a}|^2=\overline{x+a}|^2+x^2$; *that is*, $x^2+4xa+4a^2=x^2+2xa+a^2+x^2$, *or* $3a^2=x^2-2xa$; *that is*, $x^2-2ax=3a^2$, $\therefore x^2-2ax+a^2=4a^2$, $\therefore x-a=\pm\sqrt{4a^2}=\pm2a$, *and* $x=3a=3000=$ *C's share*, $\therefore x+a=4a=4000=$ *B's share, and* $x+2a=5a=5000=$ *A's share. Q. E. I.*

SYNTHESIS. $\overline{5a}|^2=$ (*square of A's payment* $=$) $\overline{4a}|^2+\overline{3a}|^2=$ (*sum of the squares of B's and C's* $=$) $25a^2$. *Moreover A's payment* ($5a$) *exceeded B's* ($4a$) *by* a, *and C's* ($3a$) *by* $2a$. Q. E. D.

PROBLEM 13. It is required to divide 11 into two such parts, that the product of their squares may be 784.

ANALYSIS. *Let* $a=11$, $b=784$, x *and* $y=$ *the parts required; then by the problem,* $x+y=a$, *and* $x^2y^2=b$; *from the first,* $y=a-x$; *the square of this value substituted for* y^2 *in the second, gives* $\overline{a-x}|^2\times x^2=b$, *whence by evolution* $\overline{a-x}.x=\sqrt{b}$; *that is,* $ax-x^2=\sqrt{b}$, *or* $x^2-ax=-\sqrt{b}$, $\because x^2-ax+\frac{a^2}{4}=(\frac{a^2}{4}-\sqrt{b}=\frac{a^2-4\sqrt{b}}{4}=)$ $\frac{R^2}{4}$, $\because x-\frac{a}{2}=\pm\sqrt{\frac{R^2}{4}}=\pm\frac{R}{2}$, *and* $x=\frac{a\pm R}{2}$, *also* $y=(a-x=)$ $a-\frac{a\pm R}{2}=\frac{a\mp R}{2}$. Q. E. I.

SYNTHESIS. *First,* $\frac{a\pm R}{2}+\frac{a\mp R}{2}=\frac{2a}{2}=a$. *Then* $\overline{\frac{a\pm R}{2}}\Big|^2\times\overline{\frac{a\mp R}{2}}\Big|^2=\frac{a^2+2aR+R^2}{4}\times\frac{a^2-2aR+R^2}{4}=\frac{a^4-2a^2R^2+R^4}{16}=$ (*by restoring the value of* $R^2=a^2-4\sqrt{b}=$)
$\frac{a^4-2a^4+8a^2\sqrt{b}+a^4-8a^2\sqrt{b}+16b}{16}=\frac{16b}{16}=b$. Q. E. D.

The solution of the problem in numbers, is $x=\frac{a+R}{2}=\frac{a+\sqrt{a^2-4\sqrt{b}}}{2}=\frac{11+\sqrt{121-4\sqrt{784}}}{2}=7$, and $y=\frac{a-R}{2}=4$.

PROBLEM 14. Given the sum of two numbers 24, and the product equal thirty-five times their difference, to find the numbers?

ANALYSIS. *Let* x *and* y *be the numbers required,* $s=24$, $m=35$; *then by the problem,* $x+y=s$, *and* $xy=(m.\overline{x-y}=)$ $mx-my$. *From the first,* $y=s-x$; *this value substituted in the second, gives* $sx-x^2=(mx-ms+mx=)$ $2mx-ms$, *or* $x^2+\overline{2m-s}.x=ms$; *whence (putting* $a=2m-s$) $x^2+ax=ms$, $\because x^2+ax+\frac{a^2}{4}=(ms+\frac{a^2}{4}=)$ $\frac{4ms+a^2}{4}=\frac{R^2}{4}$, $\because x+\frac{a}{2}=\pm\frac{R}{2}$, *and* $x=\frac{\pm R-a}{2}$; *whence also* $y=(s-x=)$ $s-\frac{\pm R-a}{2}=\frac{2s\mp R+a}{2}$. Q. E. I.

SYNTHESIS. *First,* $\frac{R-a}{2}+\frac{2s-R+a}{2}=\frac{2s}{2}=s.$

Secondly, $\frac{R-a}{2}\times\frac{2s-R+a}{2}=\frac{2sR-R^2+2aR-2sa-a^2}{4}$

$=\frac{\overline{2a+2s}.R-R^2-2sa-a^2}{4}=$ (*since* $a+s=2m$)

$\frac{4mR-R^2-2sa-a^2}{4}$; (*which, because* $4ms+a^2=R^2$,) $=$

$\frac{4mR-4ma-4ms}{4}=m.\overline{R-a-s}=m.\frac{R-a}{2}-m.\frac{2s-R+a}{2}.$

Q. E. D.

The answer to this problem in numbers is, $x=\frac{R-a}{2}=\frac{74-46}{2}=\frac{28}{2}=14$, *and* $y=\frac{2s-R+a}{2}=\frac{48-74+46}{2}=\frac{20}{2}=10.$

TO REGISTER THE STEPS OF AN ALGEBRAIC OPERATION.

The register [p] is a method whereby the place from whence any step is derived, and the operation by which it is produced, are clearly pointed out, by means of symbols placed opposite the said step, in the margin.

The symbols employed are + for addition, — for subtraction, × for multiplication, ÷ for division, ⊛ for involution, ⁀ for evolution, ▭ for completing the square, = for equality, and *tr.* for transposition.

When the register is used in the solution of any problem, it requires three columns; the right hand column contains the alge-

[p] The register will be found to be a very convenient mode of reference, where an ample detail of the work is required; but as modern algebraists prefer noting down results, and omit as much as possible particularizing those intermediate steps which are in a great degree evident, the register is now less in use than formerly. We are indebted to Dr. John Pell, an eminent English mathematician, for the invention: it was first published in Rhonius's Algebra, translated out of the High Dutch into English by Thomas Brancker, altered and augmented by Dr. Pell, 4to. London, 1688. The learner will be enabled, by the specimen here given, to apply the method to other cases if he thinks proper; at least he should understand its use, as it is employed in the writings of Emerson, Ward, Carr, and some other books which are still read,

braic operation, in the next the steps are numbered, and in the left hand column opposite to each step are placed, first the number of the step from whence it is derived, and then the symbol denoting the operation by which it is obtained. And here it must be noted, that the numbers 1, 2, 3, &c. in the register column, always denote the numbers of the steps, as first, second, third, &c. but when a figure has a dash over it, as $\overline{3}$, it denotes a number concerned in the operation.

In the following example an additional column is placed on the left, for the purpose of explaining the process.

15. Given $\frac{x}{6}+\frac{y}{2}=7$, and $\sqrt{\frac{xy}{16}}=3$, to find x and y.

Let $b=6$, $d=2$, $m=7$, $c=16$, $n=3$.

Explanation.	*Register.*	*No.*	*Operation.*
	Given	1	$\frac{x}{b}+\frac{y}{d}=m.$
	Given	2	$\sqrt{\frac{xy}{c}}=n.$
In equation 1. *subtracting* $\frac{y}{d}$.	$1-\frac{y}{d}$	3	$\frac{x}{b}=m-\frac{y}{d}.$
Multiplying eq. 3. *into b.*	$3\times b$	4	$x=bm-\frac{by}{d}.$
Involving equation 2.	2 ⊙	5	$\frac{xy}{c}=n^2.$
Multiplying eq. 5. *into c.*	$5\times c$	6	$xy=cn^2.$
Dividing equation 6. *by y.*	$6\div y$	7	$x=\frac{cn^2}{y}.$
Equating the 4*th and* 7*th steps.*	$4=7$	8	$bm-\frac{by}{d}=\frac{cn^2}{y}.$
Multiplying eq. 8. *into d.*	$8\times d$	9	$bdm-by=\frac{cdn^2}{y}.$
Multiplying eq. 9. *into y.*	$9\times y$	10	$bdmy-by^2=cdn^2.$
Transposing in equation 10.	10 *tr.*	11	$by^2-bdmy=-cdn^2.$
Dividing equation 11. *by b.*	$11\div b$	12	$y^2-dmy=-\frac{cdn^2}{b}.$
Comp. the square, &c. in eq. 12.	12 □ &c.	13	$y^2-dmy+\frac{d^2m^2}{4}=\left(\frac{d^2m^2}{4}-\frac{cdn^2}{b}=\right)\frac{bd^2m^2-4cdn^2}{4b}=\frac{R^2}{4},$
Evolving the root of eq. 13.	13 ∽	14	$y-\frac{dm}{2}=\pm\frac{R}{2}.$
Adding $\frac{dm}{2}$ *to eq.* 14.	$14+\frac{dm}{2}$	15	$y=\frac{dm\pm R}{2}.$
From the 7*th and* 15*th eq.*	7 . . . 15	16	$x=cn^2\times\frac{2}{dm\pm R}=\frac{2cn^2}{dm+R}.$
By restitution in the 15*th eq.*	15 *restit.*	17	$y=\frac{2\times7\pm2}{2}=8$, *or* 6.
By restitution in the 16*th eq.*	16 *restit.*	18	$x=16\times9\times\frac{2}{2\times7\pm2}=144\times\frac{2}{16}$, *or* $144\times\frac{2}{12}=18$, *or* 24.

Wherefore if $y=8$, *then* $x=18$; *but if* $y=6$, *then* $x=24$.

16. Given the difference 9, and quotient 4, of two numbers, to find them?

Let x = the greater, y = the less, $d=9$, $q=4$.

Register.	*No.*	*Operation.*
Given	1	$x-y=d.$
	2	$\frac{x}{y}=q.$
$2\times y$	3	$x=qy.$
$1+y$	4	$x=d+y.$
$3=4$	5	$qy=d+y.$
$5-y$	6	$qy-y=d.$
$6\div\overline{q-1}$	7	$y=\frac{d}{q+1}=3.$
$1+7$	8	$x=(d+\frac{d}{q-1}=)\frac{qd}{q-1}=12.$

17. Given the sum of the squares of two numbers 61, and the difference of their squares 11, to find the numbers?

Let x = the greater, y = the less, $a=61$, $b=11$.

Register.	*No.*	*Operation.*
Given	1	$x^2+y^2=a.$
	2	$x^2-y^2=b.$
$1+2$	3	$2x^2=a+b.$
$3\div\bar{2}$	4	$x^2=\frac{a+b}{2}.$
4 w	5	$x=\sqrt{\frac{a+b}{2}}=6.$
$1-2$	6	$2y^2=a-b.$
$6\div\bar{2}$	7	$y^2=\frac{a-b}{2}.$
7 w	8	$y=\sqrt{\frac{a-b}{2}}=5.$

18. The difference of two numbers exceeds their quotient by 2, and their product exceeds their sum by 20: what are the numbers?

Let $x=$the greater, $y=$the less, $a=2$, $b=20$.

Register.	*No.*	*Operation.*
Given	1	$\frac{x}{y}=x-y-a.$
	2	$xy=x+y+b.$
$1\times y$	3	$x=xy-y^2-ay.$
3 *tr.*	4	$xy-x=y^2+ay.$
$2-x$	5	$xy-x=y+b.$
$4=5$	6	$y^2+ay=y+b.$
$6-y$	7	$y^2+\overline{a-1}\,y=b.$
7 $\square$ *and subst.*	8	$y^2+\overline{a-1}.y+\frac{\overline{a-1}\rvert^2}{4}=b+\frac{\overline{a-1}\rvert^2}{4}$ $=\frac{4b+a^2-2a+1}{4}=\frac{R^2}{4}.$
8 $\sqrt{}$	9	$y+\frac{a-1}{2}=\frac{R}{2}.$
9 *tr.*	10	$y=\frac{R-a+1}{2}=4.$
$5\div\overline{y-1}$	11	$x=\frac{y+b}{y-1}.$
11 *subst.* 10.	12	$x=\frac{\frac{R-a+1}{2}+b}{\frac{R-a+1}{2}-1}$ $=8.$

19. The square of the greater of two numbers multiplied into the less, produces 75; and the square of the less multiplied into the greater, 45: what are the numbers?

Let $x=$the greater, $y=$the less, $a=75$, $b=45$; then $x^2y=a$, and $xy^2=b$, by the problem; divide the first by the second, and $\frac{x}{y}=\frac{a}{b}$, $\therefore x=\frac{ay}{b}$; substitute this value for x in the second, and $\frac{ay^3}{b}=b$, or $ay^3=b^2$, $\therefore y^3=\frac{b^2}{a}$, and $y=\sqrt[3]{\frac{b^2}{a}}=3$, $\therefore x=(\frac{ay}{b}=)\ 5$.

20. To divide 100 into two parts, such, that their product may equal the difference of their squares.

Let $x=$the greater part, $y=$the less, $a=100$; then by the problem, $x+y=a$, and $xy=x^2-y^2$; from the first $x=a-y$; this substituted in the second, it becomes $ay-y^2=(x^2-y^2=)\ a^2-$

$2ay$ ∴ $y^2-3ay=-a^2$, *and* $y^2-3ay+\frac{9a^2}{4}=(\frac{9a^2}{4}-a^2=)$ $\frac{5a^2}{4}$, ∴ $y-\frac{3a}{2}=(\pm\sqrt{\frac{5a^2}{4}}=)\ \pm\frac{a\sqrt{5}}{2}$, *and* $y=\frac{3a\pm a\sqrt{5}}{2}$ [a] $=$ 38.19660112; ∴ $x=(a-y=a-\frac{3a\pm a\sqrt{5}}{2}=)\frac{-a\pm a\sqrt{5}}{2}$ [b] $=$ 61.80339888.

21. What two numbers are those, whose difference is 4, and the product of their cubes 9261?

Let $d=4$, $p=9261$, $x=$ *the less, then* $x+d=$ *the greater; whence by the problem,* $(x^3\times\overline{x+d}|^3=p$, ∴ $x\times\overline{x+d}=\sqrt[3]{p}$, *that is,* $x^2+dx=\sqrt[3]{p}$, ∴ $x^2+dx+\frac{d^2}{4}=\sqrt[3]{p}+\frac{d^2}{4}$; *this resolved, gives* $x=\frac{1}{2}\overline{\sqrt{d^2+4\sqrt[3]{p}}-d}=3$, *and* $x+d=7$.

22. The greater of two numbers is to the less as 3 to 2, and the sum of their squares is 208; required the numbers?

Let $a=3$, $b=2$, $c=208$, $x=$ *the greater number, then* $(a:b::x:)\frac{bx}{a}=$ *the less;* ∴ *by the prob.* $x^2+\frac{b^2x^2}{a^2}=c$, ∴ $x=\sqrt{\frac{a^2c}{a^2+b^2}}$ $=12$, *and* $\frac{bx}{a}=\frac{2x}{3}=8$.

23. Divide the number 25 into two such parts, that the sum of their square roots may be 7.

Let $a=25$, $b=7$, $x=$ *one part, then* $a-x=$ *the other,* ∴ *by the prob.* $\sqrt{x}+\sqrt{a-x}=b$; *square both sides, and* $x+2\sqrt{ax-x^2}+a-x=b^2$, ∴ $\sqrt{ax-x^2}=\frac{b^2-a}{2}$; *again square both sides, and* $ax-x^2=\left(\frac{b^2-a}{2}\right)^2$; *which resolved, gives* $x=\frac{a\pm\sqrt{a^2-\overline{b^2-a}|^2}}{2}=$ 16, *or* 9, *and* $a-x=9$, *or* 16.

24. What number is that, to which its biquadrate being added, and from the sum twice its cube subtracted, the remainder will be 1722?

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[a] Here we must evidently take the negative value of $\pm a\sqrt{5}$, otherwise y would come out greater than 100, and consequently x would be negative; which is contrary to what was proposed.

[b] Here the affirmative value of $\pm a\sqrt{5}$ must be taken.

Let $x+.5$ *= the number,* $a=1722$, *then* $\overline{x+.5}|^4-2.\overline{x+.5}|^3+x+.5=a$, *by the prob. whence* $x^4-1.5x^2+.3125=a$, ∴ *by completing the square, and reduction,* $x=\sqrt{.75+\sqrt{a+.25}}=6.5$, ∴ $x+.5=7$.

25. To find two numbers, such, that their sum, product, and the difference of their squares, may be equal to each other?

Let x *= the greater,* y *= the less, then by the prob.* $x+y=xy$, *and* $x+y=x^2-y^2$; *divide the latter by* $x+y$, *and* $1=x-y$, *or* $x=1+y$; *substitute this value for* x *in the first, and* $1+2y=y+y^2$, *whence* $y=\frac{1}{2}+\sqrt{\frac{5}{4}}=\frac{1+\sqrt{5}}{2}=1.6180339887$, &c. *and* $x=(1+y=)\frac{3}{2}+\sqrt{\frac{5}{4}}=\frac{3+\sqrt{5}}{2}=2.6180339887$, &c.

26. The product of two numbers is 1944; and the sursolid root of the greater is to the cube root of the less, as $1\frac{1}{2}$ to 1; what are the numbers?

Let x^5 *= the greater, then* $1\frac{1}{2}:1::x:(\frac{x}{1\frac{1}{2}}=)\frac{2x}{3}$, ∴ $\overline{\frac{2x}{3}}|^3=\frac{8x^3}{27}$ *= the less. Let* $c=\frac{8}{27}$, $p=1944$, *then by the problem* $(x^5\times cx^3=)\ cx^8=p$, ∴ $x=\sqrt[8]{\frac{p}{c}}=\sqrt[8]{6561}=3$, ∴ $x^5=243$ *= the greater, and* $cx^3=(\frac{8}{27}\times3^3=)\ 8$ *= the less.*

27. The sum and product of two numbers are equal, and if to either sum or product the sum of the squares be added, the result will be 12; what are the numbers?

Let x *and* y *represent the numbers,* $a=12$, *then* $x+y=xy$, *and* $x+y+x^2+y^2=a$, *by the prob. Take twice the first from the second, and* $(x^2+y^2-x-y=a-2xy$, *or)* $\overline{x+y}|^2-\overline{x+y}=a$; ∴ *by completing the square* $\overline{x+y}|^2-\overline{x+y}+\frac{1}{4}=(a+\frac{1}{4}=)\frac{4a+1}{4}$; ∴ *by evolution* $x+y-\frac{1}{2}=\pm\sqrt{\frac{4a+1}{4}}=\frac{\pm\sqrt{4a+1}}{2}$; ∴ $x+y=\frac{1\pm\sqrt{4a+1}}{2}=4$, *whence also* $xy=\frac{1\pm\sqrt{4a+1}}{2}=4$. *From the square of the last but one, take four times the last, and* $x^2-2xy+y^2=0$; ∴ *by evolution,* $x-y=0$, *and* $x=y$, ∴ $xy=x^2=y^2=4$, ∴ $x=2$, *and* $y=2$.

28. Given the product p (=12,) and the sum of the fourth powers s (=337,) of two numbers, to find them?

Let x=the greater, y=the less, then $xy=p$, and $x^4+y^4=s$; add twice the square of the first to, and subtract it from, the second, and extract the square root of the sum and difference, and there will arise $x^2+y^2=\sqrt{s+2p^2}$, and $x^2-y^2=\sqrt{s-2p^2}$: take the sum and difference of these two equations, and extract the square root from each, and $x=\sqrt{\frac{1}{2}\overline{\sqrt{s+2p^2}+\sqrt{s-2p^2}}}=4$, also $y=\sqrt{\frac{1}{2}\overline{\sqrt{s+2p^2}-\sqrt{s-2p^2}}}=3$.

29. The sum of two numbers is 25, and if they be divided alternately by each other, the sum of the quotients will be $4\frac{1}{4}$; required the numbers?

Let $a=25$, $b=4\frac{1}{4}$, x=the greater number, then $a-x$=the less, and by the problem $\frac{x}{a-x}+\frac{a-x}{x}=b$; whence $x^2+a^2-2ax+x^2=abx-bx^2$, or $2x^2+bx^2-2ax-abx=-a^2$; that is, $\overline{2+b}.x^2-\overline{2+b}.ax=-a^2$; divide this equation by $2+b$, and $x^2-ax=-\frac{a^2}{2+b}$; whence, by completing the square, and extracting the root, &c. $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a^2}{2+b}}=20$, or 5; and $a-x=\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{a^2}{2+b}}=5$, or 20.

30. Given the sum of two numbers 9, and the sum of their cubes 189, to find the numbers?

Let $2s=9$, $b=189$, $2x$=the difference of the required numbers; then (by prob. 1.) $s+x$=the greater, and $s-x$=the less, and by the prob. $(\overline{s+x}|^3+\overline{s-x}|^3=s^3+3s^2x+3sx^2+x^3+s^3-3s^2x+3sx^2-x^3=)$ $2s^3+6sx^2=b$, or $6sx^2=b-2s^3$; $\therefore x^2=\frac{b-2s^3}{6s}$, and $x=\sqrt{\frac{b-2s^3}{6s}}=\frac{1}{2}$, whence $s+x=\frac{9}{2}+\frac{1}{2}=5$, and $s-x=\frac{9}{2}-\frac{1}{2}=4$.

31. Given the sum 6, and the sum of the biquadrates 272, of two numbers, to find them?

Let $2s=6$, $2x$=the difference of the required numbers, $b=272$; then, as in the preceding problem, $s+x$=the greater, and $s-x$=the less, whence $\overline{s+x}|^4+\overline{s-x}|^4=b$; which involved and re-

duced, we have $2s^4+12s^2x^2+2x^4=b$, *or* $x^4+6s^2x^2=\frac{1}{2}b-s^4$; $\therefore$ *by completing the square,* $x^4+6s^2x^2+9s^4=\frac{1}{2}b+8s^4$; $\therefore$ *by evolution,* $x^2+3s^2=\pm\sqrt{\frac{1}{2}b+8s^4}$, $x^2=-3s^2\pm\sqrt{\frac{1}{2}b+8s^4}$, *and* $x=\pm\sqrt{-3s^2\pm\sqrt{\frac{1}{2}b+8s^4}}=1$; *whence* $s+x=4$, *and* $s-x=2$.

32. Given the sum 10, and the sum of the fifth powers 17050, of two numbers, to find them?

Let $2s=10$, $b=17050$, $2x=$*the difference of the numbers required; then* $s+x=$*the greater, and* $s-x=$*the less, and by proceeding as in problems 30 and 31, we have* $2s^5+20s^3x^2+10sx^4=b$, *whence* $x=\sqrt{\sqrt{\frac{b}{10s}+\frac{4s^4}{5}}-s^2}=2$, $\therefore s+x=7$, *and* $s-x=3$.

33. Given the product p, and the sum of the nth powers s, of two numbers, to find them?

Let x *and* y *represent the numbers, then by the problem* $x^n+y^n=s$, *and* $xy=p$; *from the second equation* $y=\frac{p}{x}$; *this value substituted for* y *in the first, gives* $x^n+\frac{p^n}{x^n}=s$, *or* $x^{2n}+p^n=sx^n$, *or* $x^{2n}-sx^n=-p^n$; *hence, completing the square,* $x^{2n}-sx^n+\frac{s^2}{4}=\frac{s^2}{4}-p^n$; *whence* $x^n-\frac{s}{2}=\pm\sqrt{\frac{s^2}{4}-p^n}$, $x^n=\frac{s}{2}\pm\sqrt{\frac{s^2}{4}-p^n}$, *and* $x=\sqrt[n]{\frac{s}{2}\pm\sqrt{\frac{s^2}{4}-p^n}}=\sqrt[n]{\frac{s\pm\sqrt{s^2-4p^n}}{2}}$; *and* $y=\frac{p}{x}=p\div\sqrt[n]{\frac{s\pm\sqrt{s^2-4p^n}}{2}}=\sqrt[n]{\frac{s\mp\sqrt{s^2-4p^n}}{2}}$.

34. Given the product p, and the difference of the nth powers d, of two numbers, to find them?

Let x *and* y *be the numbers, then* $xy=p$, *and* $x^n-y^n=d$; *whence by proceeding as in the foregoing problem,* $x=\sqrt[n]{\frac{d\pm\sqrt{d^2+4p^n}}{2}}$, *and* $y=p\div\sqrt[n]{\frac{d\pm\sqrt{d^2+4p^n}}{2}}=\sqrt[n]{\frac{d\mp\sqrt{d^2+4p^n}}{2}}$.

35. Required the values of x and y in the following equations, viz. $\sqrt[3]{x^2}\times\sqrt{y^3}=2y^2$, and $12\sqrt[3]{x}-\sqrt{y}=22$?

Let $u=\sqrt[3]{x}$, $z=\sqrt{y}$, *then* $u^3=x$, *and* $z^2=y$; $\because$ *the given equations become* $u^2z^3=2z^4$, *and* $12u-z=22$; *divide the last but one by* $2z^4$, *and* $z=\frac{u^2}{2}$; *this equation added to the preceding, gives* $12u=22+\frac{u^2}{2}$, *or* $u^2-24u=-44$; *this equation resolved, gives* $u=2$, $\because z=(\frac{u^2}{2}=)\,2$, $x=(u^3=)\,8$, *and* $y=(z^2=)\,4$.

36. If 18 oxen in 5 weeks can eat 6 acres of grass, and 45 oxen in 9 weeks eat 21 acres of the same, how many must there be to eat 38 acres in 19 weeks, the grass being allowed to grow uniformly?

Let $a=18$, $b=5$, $c=6$, $d=45$, $m=21$, $n=9$, $r=38$, $s=19$, $1=$ *the quantity eaten by an ox in a week,* $w=$ *the quantity on an acre at first,* $z=$ *the weekly increase on an acre after the first 5 weeks,* $x=$ *the number of oxen required,* $p=(n-b=)\,4$, $t=(s-b=)\,14$; *then will* $rw=$ *the grass on* r *acres at first, and* $rtz=$ *the increase on* r *acres in* t *weeks; the sum of these, by the problem, equals the quantity* x *oxen ate in* s *weeks, that is,* $sx=rw+rtz$; *again,* $mw=$ *the grass on* m *acres at first, and* $mpz=$ *the increase of the same in* p *weeks; the sum of these two equals what* d *oxen ate in* n *weeks, that is,* $mw+mpz=dn$; *also* $cw=$ (*the grass on* c *acres at first,* $=$) *the quantity* a *oxen can eat in* b *weeks, that is,* $cw=ab$, *whence* $w=\frac{ab}{c}$; *to* mp *times the first equation, add* rt *times the second, and* $mpsx+mrtw+mprtz=dnrt+mprw+mprtz$, *or* $mpsx=dnrt+mprw-mrtw$; *for* w *in this equation, substitute its equal* $\frac{ab}{c}$, *and the equation becomes* $mpsx=dnrt+\frac{abmpr}{c}-\frac{abmrt}{c}$, *or* $cmpsx=cdnrt+abmpr-abmrt$; *whence* $x=\frac{cdnt+abmp-abmt}{cmps}\times r=\frac{cdnt+abm\times\overline{p-t}}{cmps}\times r=\frac{34020+1890\times-10}{9576}\times 38=60$, *the answer.*

37. A waterman, who can row 11 miles an hour with the tide, and 2 miles an hour against it, rows 5 miles up a river and back

again in 3 hours; now supposing the tide to run uniformly the same way, required its velocity [1]?

Let $m=11$, $n=2$, $p=5$, $r=3$, $v=$ *the velocity required, and* $x=$ *the time he rowed with the tide, then will* $r-x=$ *the time he rowed against it; whence* ($x:p::1$ *hour* :) $\frac{p}{x}=$ *his velocity with the tide, and* ($r-x:p::1$ *hour* :) $\frac{p}{r-x}=$ *his velocity against tide; now since the tide assists him* $=v$ *when he goes with it, it must evidently retard him* $=v$ *when he goes against it; whence* $2v=$ *the difference of his velocity with, and against tide,* $\therefore \frac{p}{x}-\frac{p}{r-x}=2v$, *or* $v=\frac{p}{2x}-\frac{p}{2r-2x}$: *now because his velocity with, is to his velocity against, tide, as m to n; so his time of rowing with, is to his time of rowing against, tide, as n to m, since the time is inversely as the velocity; wherefore* $x:r-x::n:m$, $\therefore x=\frac{nr}{m+n}=\frac{6}{13}$ *of an hour* $=$ *the time he rowed with tide, and* $r-x=2\frac{7}{13}$ *hours* $=$ *the time he rowed against it; for x substitute its value* $\frac{6}{13}$ *in the equation above derived, and it becomes* $v=(\frac{p}{2x}-\frac{p}{2r-2x}=)$ $p\div\frac{12}{13}-p\div\overline{2r-\frac{12}{13}}=\frac{65}{12}-\frac{65}{66}=\frac{3510}{792}=4\frac{19}{44}$ *miles per hour* $=$ *the velocity of the tide.*

38. The ages of five persons, A, B, C, D, and E, are such, that the sum of the first four is 95, that of the three first and last 97, that of the two first and two last 103, that of the first and three last 106, and that of the four last 107; required the age of each?

Let $a=95$, $b=97$, $c=103$, $d=106$, $e=107$, $s=$ *the sum of all their ages, and let* x, y, z, v, w, *be put for their ages respectively; then will* $s-w=a$, $s-v=b$, $s-z=c$, $s-y=d$, *and* $s-x$

[1] Velocity (from the Latin *velox*, swift,) is that affection of motion, whereby a moving body passes over a certain space in a certain time; or in common language, it is the degree of swiftness with which a body moves: it is likewise named celerity, from the Latin *celer*, swift or nimble.

$=e$; *add these five equations together, and the sum is* $(5s-x-y-z-v-w=5s-s=)\ 4s=a+b+c+d+e$; *whence* $s=\frac{a+b+c+d+e}{4}$; *now if this value be substituted for* s *in the five preceding equations, we shall thence obtain the required numbers, viz.* $w=32$, $v=30$, $z=24$, $y=21$, *and* $x=20$, *being the ages of E, D, C, B, and A, respectively.*

39. To find a point in the straight line which joins two luminaries, or in the line produced, which is equally enlightened by both [*].

Let $a=$ *their distance apart,* $x=$ *the distance of the least of them from the required point, then* $a\pm x$ [†] $=$ *the distance of the other: let the quantity of light emitted by the first in a given time be to that emitted by the second in the same time, as* m *to* n; *then will* $\frac{1}{x^2}:\frac{1}{\overline{a\pm x}|^2}$ *be the ratio of the effects they produce, supposing* $m=n$, *and* $\frac{m}{x^2}:\frac{n}{\overline{a\pm x}|^2}$ *will be the ratio, supposing* m *and* n *unequal: but these effects are by hypothesis equal; whence* $\frac{m}{x^2}=\frac{n}{a^2\pm 2ax+x^2}$, $\therefore ma^2\pm 2amx+mx^2=nx^2$, *or* $\overline{m-n}.x^2\pm 2amx=-ma^2$, $\therefore x^2\pm\frac{2am}{m-n}x=-\frac{ma^2}{m-n}$, $\therefore x^2\pm\frac{2am}{m-n}x+\overline{\frac{am}{m-n}}|^2=\overline{\frac{am}{m-n}}|^2-\frac{ma^2}{m-n}$, $\therefore x\pm\frac{am}{m-n}=\pm\sqrt{\overline{\frac{am}{m-n}}|^2-\frac{ma^2}{m-n}}$, *and* $x=\left(\mp\frac{am}{m-n}\pm\sqrt{\overline{\frac{am}{m-n}}|^2-\frac{ma^2}{m-n}}=\frac{\mp am\pm\sqrt{mna^2}}{m-n}=\right)\frac{\mp a.m\pm\sqrt{mna^2}}{m-n}$ $=$ *the distance required.*

40. The weight w, and the specific gravity of a mixture, and the specific gravities a and b, of the two simples which compose it, being given, to find the quantity of each [u]?

[*] A luminary, (from the Latin *lumen*, light,) is a body that gives light, as the sun, moon, a planet, star, &c.

[†] The double sign serves both cases, viz. $a+x$ when the point required is *beyond* the smaller luminary, and $a-x$ when it is between them; also in the answer, the upper sign $-$ applies to the first case, and the lower sign $+$ to the second.

[u] The gravity of a body, (from the Latin *gravis*, heavy,) is its weight.

Let x = the weight of the simple, whose specific gravity is the greatest, then $w-x$ = the weight of the other.

$$\left.\begin{array}{r} \textit{But } \dfrac{x}{a}= \\ \dfrac{w-x}{b}= \\ \dfrac{w}{s}= \end{array}\right\} \begin{array}{c}\textit{the magnitude of the} \\ \textit{body, whose weight is}\end{array} \left\{\begin{array}{l} x \\ w-x \\ w \end{array}\right.$$

Whence $\dfrac{x}{a}+\dfrac{w-x}{b}=\dfrac{w}{s}$, *or* $bsx+asw-asx=abw$, $\because$ $bsx-asx=abw-asw$, *or* $x=\dfrac{abw-asw}{bs-as}=\dfrac{\overline{b-s}.aw}{\overline{b-a}.s}$.

41. Suppose two bodies, A and B, to move in opposite directions towards the same point with given velocities, the distance of the places from whence they set out, and the difference of the times in which they begin to move, being likewise given, thence to determine the point where they meet?

Let d = the distance from A to B at the time of setting out, x = A's distance from the point of meeting, then $d-x$ = B's distance from the point of meeting; let t = the difference between the times of their beginning to move, and suppose A moves through the space a in the time n, and B through the space b in the time m, then $(a:n::x:)\ \dfrac{nx}{a}$ = *the time of A's motion, and* $(b:m::d-x:)$ $\dfrac{\overline{d-x}.m}{b}$ = *the time of B's motion; whence by the problem,* $\dfrac{nx}{a}-\dfrac{\overline{d-x}.m}{b}=t$, $\because$ $x=\dfrac{bt+dm}{bn+am}.a$.

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and the specific gravity is its weight compared with that of a body of equal bulk, but of a different kind: thus, a cubic foot of common water weighs 1000 ounces avoirdupois, and a cubic inch of each of the following substances weighs as follows; viz. fine gold, 19640 oz. fine silver, 11091 oz. cork, 240 oz. new fallen snow, 86 oz. common air, 1.232 oz. &c. &c. these numbers, then, represent the specific gravities of the above-mentioned substances respectively, compared with common water.—Tables of the specific gravity of a great variety of bodies, both solid and fluid, may be found in the writings of Mersenne, Muschenbroeck, Ward, Cotes, Emerson, Hutton, Vyse, Martin, &c. and are useful for computing the weight of such bodies as are too large and unwieldy to be moved; by means of their kind and dimensions, which must be previously known.

EXAMPLES.—1. A sets out from London towards Durham distant 257 miles, and travels 11 miles in 4 hours; B sets out from Durham 8 hours later, and travels towards London at the rate of 10 miles in 3 hours: whereabouts on the road will they meet?

Here $d=257$, $t=8$, $a=11$, $n=4$, $b=10$, $m=3$.

$$Then\ x=\frac{10\times8+257\times3}{10\times4+11\times3}\times11=128\frac{17}{73}\ miles\ from\ London.$$

2. Supposing Africa to be 20,000 miles round, and a ship to sail from the Isthmus of Suez down the Red Sea, with intent to coast it round that vast continent, sailing on an average $2\frac{1}{4}$ miles an hour;—a week after another ship sails from the opposite side of the same Isthmus with the same intent, and passing the Straits of Gibraltar, sails at the rate of $3\frac{1}{4}$ miles an hour;—near what place on the coast will they meet?

42. If two bodies, A and B, move in the same direction and in the same straight line, their velocities, distance at setting out, and the interval between the times of their beginning to move, being given, thence to determine the point where they will come together.

Let A be the farthest from the required point, d= the distance from A to B, x= A's distance from the point, then will $x-d$= B's distance; also let t= the interval of time between their setting out, and let A move through the space a in the time r, and B through the space b in the time s; then will $(a:r::x:)\frac{rx}{a}$= *the time of A's motion, and* $(b:s::x-d:)\ \frac{\overline{x-d}.s}{b}$= *the time of B's motion; whence by the problem,* $\frac{rx}{a}-\frac{\overline{x-d}.s}{b}=t$, $\therefore x=\frac{bt-sd}{br-as}.a$, *when A sets out first; and* $\frac{\overline{x-d}.s}{b}-\frac{rx}{a}=t$, $\therefore x=\frac{bt+sd}{as-br}.a$, *when B sets out first.*

EXAMPLES.—1. A ship sails from the Downs, east, towards Petersburg, at the rate of 54 miles in 23 hours; 24 hours after another ship sails from Lisbon, distant from the Downs 550 miles west, in pursuit of her, and goes at the rate of 8 miles an hour: whereabouts will the latter ship overtake the former?

Here $d=550$, $t=24$, $a=8$, $r=1$, $b=54$, $s=23$; *and because B sets out first, therefore* $x=\frac{54\times24+23\times550}{8\times23-54\times1}\times8=$ 858.21538, *&c. miles from Lisbon, or* (858.21538, &c.$-550=$) 308.21538, *&c. miles from the Downs.*

2. Suppose the ship from Lisbon sets sail 24 hours before the other?

Then $x=\frac{54\times24-23\times550}{54\times1-8\times23}\times8=698.7138$, *&c. miles from Lisbon, or* (698.7138, &c.$-550=$) 148.7138, *&c. miles from the Downs.*

3. A is 100 miles south of London, and sets out on a journey northward, travelling 37 miles every 24 hours; B from London pursues the same rout, setting out 49 hours after A, and travelling at the rate of 11 miles every 8 hours: where will they be together?

43. Given the forces of several agents[*] separately, to determine their joint force?

Let A, B, C, D, &c. be the agents, and suppose

$$\left.\begin{matrix}A\\B\\C\\D\\\&c.\end{matrix}\right\}\text{ can produce an effect, }\left\{\begin{matrix}a\\b\\c\\d\\\&c.\end{matrix}\right\}\text{ times, in the time }\left\{\begin{matrix}m\\n\\r\\s\\\&c.\end{matrix}\right.$$

Call the given effect 1, *and let* $x=$ *the time in which they can produce it, all operating together:*

Then will

$$\left.\begin{array}{ccccccccc}m.(\textit{time}) & : & a.(\textit{effect}) & :: & x.(\textit{time} & : & \frac{ax}{m}\\ n & : & b & :: & x & : & \frac{bx}{n}\\ r & : & c & :: & x & : & \frac{cx}{r}\\ s & : & d & :: & x & : & \frac{dx}{s}\\ \&c. & & \&c. & & \&c. & & \&c.\end{array}\right\}\begin{array}{l}\textit{The effect produced}\\ \textit{in the time } x\textit{, by}\end{array}\left\{\begin{matrix}A\\B\\C\\D\\\&c.\end{matrix}\right.$$

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[*] An agent, (in Latin *agens*, from ἄγω to drive,) is that by which any thing is done or effected. Philosophers call that the *agent*, which is the immediate cause of any effect, and that on which the effect is produced they

But the sum of these effects is equal to the given effect 1, *produced by the joint operation of all the agents, in the time x; whence* $\frac{ax}{m}+\frac{bx}{n}+\frac{cx}{r}+\frac{dx}{s}$ &c.$=1$, *or* $x.\frac{a}{m}+\frac{b}{n}+\frac{c}{r}+\frac{d}{s}$ &c.$=1$, $\therefore x=\frac{1}{\frac{a}{m}+\frac{b}{n}+\frac{c}{r}+\frac{d}{s}}$, &c.

EXAMPLES.—1. A can reap 5 acres of wheat in 8 days, B can reap 4 acres in 7 days, and C 6 acres in 9 days; how long will they require to reap a field of 30 acres, all working together?

Here $m=8$, $a=5$, $n=7$, $b=4$, $r=9$, $c=6$.

Then $x=\frac{1}{\frac{a}{m}+\frac{b}{n}+\frac{c}{r}}\times 30=\frac{1}{\frac{5}{8}+\frac{4}{7}+\frac{6}{9}}\times 30=\frac{168}{313}\times 30=16\frac{32}{313}$ *days.*

2. A vessel has three cocks, A, B. and C; A can fill it twice in 3 hours, B 3 times in 4 hours, and C 4 times in 5 hours; in what time will it be filled with the three cocks all open together?

44. If two agents, A and B, can jointly produce an effect in the time m, A and C in the time n, and B and C in the time r; in what time will each alone produce the same effect?

Let $\left\{\begin{matrix} x \\ y \\ z \end{matrix}\right\}$ *=the time* $\left\{\begin{matrix} A \\ B \\ C \end{matrix}\right\}$ *would require to produce the given effect; and let the effect be called* 1.

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call the *patient;* the effect, as communicated by the agent, they call an *action;* but as received by the patient, a *passion:* a smith striking on an anvil has been frequently proposed as a proper example; thus the smith is the *superior agent*, the hammer with which he strikes is the *inferior agent*, the blow he strikes is the *action*, the anvil is the *patient*, and the blow it receives, the *passion*.

Then is

$$\left.\begin{array}{llllllll} x\ (time) & : & 1\ (effect) & :: & m\ (time) & : & \frac{m}{x} \\ y & : & 1 & :: & m & : & \frac{m}{y} \\ x & : & 1 & :: & n & : & \frac{n}{x} \\ z & : & 1 & :: & n & : & \frac{n}{z} \\ y & : & 1 & :: & r & : & \frac{r}{y} \\ z & : & 1 & :: & r & : & \frac{r}{z} \end{array}\right\} = \text{the effect produced by} \left\{\begin{array}{l} A \text{ in the time } m \\ B \ldots\ldots\ldots m \\ A \ldots\ldots\ldots n \\ C \ldots\ldots\ldots n \\ B \ldots\ldots\ldots r \\ C \ldots\ldots\ldots r \end{array}\right.$$

Whence $\frac{m}{x}+\frac{m}{y}=1$, *or* (1) $\frac{1}{x}+\frac{1}{y}=\frac{1}{m}$.

$\frac{n}{x}+\frac{n}{z}=1$, *or* (2) $\frac{1}{x}+\frac{1}{z}=\frac{1}{n}$.

$\frac{r}{y}+\frac{r}{z}=1$, *or* (3) $\frac{1}{y}+\frac{1}{z}=\frac{1}{r}$.

Add equations 1, 2, *and* 3 *together, and the sum will be* $\overline{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\times 2=\frac{1}{m}+\frac{1}{n}+\frac{1}{r}$, *or* (4) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2r}$; *from eq.* 4 *subtract eq.* 1, 2, *and* 3 *severally, and the remainders are*

$$\left.\begin{array}{l} \frac{1}{z}=\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2r}-\frac{1}{m} \\ \frac{1}{y}=\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2r}-\frac{1}{n} \\ \frac{1}{x}=\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2r}-\frac{1}{r} \end{array}\right\} \text{whence} \left\{\begin{array}{l} z=\frac{2mnr}{mr+mn-nr}. \\ y=\frac{2mnr}{nr+mn-mr} \\ x=\frac{2mnr}{nr+mr-mn}. \end{array}\right.$$

EXAMPLES.—1. A and B can unload a waggon in 3 hours, B and C in $2\frac{1}{4}$ hours, and A and C in $2\frac{3}{4}$ hours; how long will each be in doing the same by himself?

Here $m=3$, $n=2\frac{3}{4}$, $r=2\frac{1}{4}$, $x=\frac{2\times 3\times 2\frac{3}{4}\times 2\frac{1}{4}}{2\frac{3}{4}\times 2\frac{1}{4}+3\times 2\frac{1}{4}-3\times 2\frac{3}{4}}=$ $\frac{37.125}{4.6875}=7.92$ *hours.*

$$y=\frac{2\times3\times2\frac{3}{4}\times2\frac{1}{4}}{2\frac{3}{4}\times2\frac{1}{4}+3\times2\frac{3}{4}-3\times2\frac{1}{4}}=\frac{37.125}{7.6875}=4.82926829 \textit{ hours.}$$

$$z=\frac{2\times3\times2\frac{3}{4}\times2\frac{1}{4}}{3\times2\frac{1}{4}+3\times2\frac{3}{4}-2\frac{3}{4}\times2\frac{1}{4}}=\frac{37.125}{8.8125}=4.21276595 \textit{ hours.}$$

2. A quantity of provisions will serve A and B 8 months, A and C 9 months, and B and C 10 months; how long would the same quantity serve each person singly?

Ans. A 14 *m.* $20\frac{40}{49}$ *days, B* 17 *m.* $16\frac{34}{41}$ *days, C* 23 *m.* $6\frac{24}{31}$ *days, reckoning* 30 *days to a month.*

45. It is required to divide the number 22 into three such parts, that *once* the first, *twice* the second, and *thrice* the third being added together, the sum will be 47, and the sum of the squares of the parts 166?

Let x, y, *and* z, *denote the three parts respectively,* $a=22$, $b=47$, $c=166$; *then by the problem* $x+y+z=a$, $x+2y+3z=b$, *and* $x^2+y^2+z^2=c$; *subtract the first from the second, and* $y+2z=b-a$, *whence* $y=b-a-2z$; *subtract double the first from the second, and* $z-x=b-2a$, *whence* $x=z+2a-b$; *let* $f=b-a$, $g=b-2a$; *these values being substituted in the two latter equations, they become* $y=f-2z$, *and* $x=z-g$; *substitute these values for* y *and* x *in the third given equation, and it will become* $z^2-2gz+g^2+f^2-4fz+4z^2+z^2=c$, *or* $z^2-\frac{2f+g}{3}z=\frac{c-f^2-g^2}{6}$; *put* $h=\frac{2f+g}{3}$, *and the latter equation becomes* $z^2-hz=\frac{c-f^2-g^2}{6}$, *in which by completing the square, &c. it becomes* $z=\frac{h}{2}\pm\sqrt{\frac{c-f^2-g^2}{6}+\frac{h^2}{4}}$ (*which, by restoring the values of* $c, f, g,$ *and* h, *viz.* $c=166$, $f=b-a=47-22=25$, $g=b-2a=47-44=3$, *and* $h=\frac{2f+g}{3}=\frac{50+3}{3}=\frac{53}{3}$) $=\frac{53}{6}\pm\sqrt{\frac{166-625-9}{6}+\frac{2809}{36}}=9$, *whence* $x=(z-g=)\ 6$, *and* $y=(f-2z=)\ 7$.

46. Required the values of x and y in the following equations, viz. $x^4+x^3y+x^2y^2+xy^3+y^4=211=a$; and $x^8+x^6y^2+x^4y^4+x^2y^6+y^8=11605=b$?

Divide the second by the first, add the quotient to, and subtract it from the first, and the results will be $(2x^4+2x^2y^2+2y^4=$

$a+\frac{b}{a}$, *or*) $x^4+x^2y^2+y^4=\frac{1}{2}a+\frac{b}{2a}$, *and* ($2x^3y+2xy^3=a-\frac{b}{a}$, *or*) $\overline{x^2+y^2}.xy=\frac{1}{2}a-\frac{b}{2a}$; *let* $s=x^2+y^2$, $p=xy$, $m=\frac{1}{2}a+\frac{b}{2a}$, *and* $n=\frac{1}{2}a-\frac{b}{2a}$, *then will the two equations, above derived, become* ($x^4+x^2y^2+y^4=\overline{x^2+y^2}|^2-x^2y^2=$) $s^2-p^2=m$, *and* ($\overline{x^2+y^2}.xy=$) $sp=n$, $\therefore p=\frac{n}{s}$; *this being squared, and the square added to* $s^2-p^2=m$, *gives* $s^2=m+\frac{n^2}{s^2}$, *or* $s^4-ms^2=n^2$, $\therefore s=\sqrt{\frac{m}{2}+\sqrt{\frac{m^2}{4}+n^2}}=13$, *and* $p=(\frac{n}{s}=)\ 6$. *Now since* ($s=$) $x^2+y^2=13$, *and* ($p=$) $xy=6$, *if the square root of the sum and difference, of the former and double the latter be taken, we shall thence obtain* $x=3$, *and* $y=2$.

47. Given the sum$=s$, and the product$=p$, of any two numbers, to find the sum of their nth powers?

Let x *and* y *represent the two numbers, then will* $x+y=s$, *and* $xy=p$. *First,* ($\overline{x+y}|^2=$) $x^2+2xy+y^2=s^2$, *and* $2xy=2p$; *subtract the latter from the former, and* $x^2+y^2=s^2-2p=$ *the sum of the squares. Secondly,* $\overline{x^2+y^2}.\overline{x+y}=\overline{s^2-2p}.s$, *or* $x^3+xy.\overline{x+y}+y^3=s^3-2sp$, *which (by substituting* sp *for its equal* $xy.\overline{x+y}$) *becomes* $x^3+sp+y^3=s^3-2sp$, $\therefore x^3+y^3=s^3-3sp=$ *the sum of the cubes. Thirdly,* $\overline{x^3+y^3}.\overline{x+y}=\overline{s^3-3sp}.s$, *or* $x^4+xy.\overline{x^2+y^2}+y^4=s^4-3s^2p$, *which (by substituting* $p.\overline{s^2-2p}$ *for its equal* $xy.\overline{x^2+y^2}$) *becomes* $x^4+p.\overline{s^2-2p}+y^4=s^4-3s^2p$, $\therefore x^4+y^4=(s^4-3s^2p-p.\overline{s^2-2p}=)\ s^4-4s^2p+2p^2=$ *the sum of the biquadrates.*

In like manner it may be shewn, that $s^5-5s^3p+5sp^2=$ *the sum of the fifth powers;* $s^6-6s^4p+9s^2p^2-2p^3=$ *the sum of the sixth powers, &c.*

By comparing together these several results, the law of continuation will be manifest; for it appears from the foregoing process, that

The sum of any powers is found by multiplying the sum of the next preceding powers by s, and from this product subtracting the sum of the powers next preceding those multiplied by p.

Thus, the sum of the 4th powers$=s\times$*sum of the cubes*$-p\times$ *sum of the squares.*

The sum of the 5th powers$=s\times$*sum of the 4th powers*$-p\times$ *sum of the cubes.*

The sum of the 6th powers$=s\times$*sum of the 5th powers*$-p\times$ *sum of the 4th powers, &c. &c.*

Hence the sum of the nth powers of x and y will be as follows;

$$x^n+y^n=s^n-n.s^{n-2}p+n.\frac{n-3}{2}.s^{n-4}p^2-n.\frac{n-4}{2}.\frac{n-5}{3}.s^{n-6}p^3$$
$$+n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.s^{n-8}p^4-,\ \&c.$$

13. To investigate the rules of arithmetical progression.

Let $a=$*the least term* } *called also the extremes.*
$z=$*the greatest* }
$n=$*the number of terms*
$d=$*the common difference of the terms*
$s=$*the sum of all the terms.*

Then will $a+\overline{a+d}+\overline{a+2\,d}+\overline{a+3\,d}+$, *&c. to* $\overline{a+\overline{n-1}.d}$ *be an increasing series of terms in arithmetical progression.*

And $z+\overline{z-d}+\overline{z-2\,d}+\overline{z-3\,d}+$, *&c. to* $\overline{z-\overline{n-1}.d}$ *will be a decreasing series in arithmetical progression.*

14. *Now since in the increasing series* $a+\overline{n-1}.d=$*the greatest term, and* $z=$*the greatest term by the notation, therefore* $z=a+\overline{n-1}.d$ (THEOREM 1.) *Whence by transposition, &c.* $a=z-\overline{n-1}.d$ (THEOR. 2.) $d=\frac{z-a}{n-1}$ (THEOR. 3.) *and* $n=\frac{z-a}{d}+1$ (THEOR. 4.) *Whence, of the first term, last term, number of terms, and difference, any three being given, the fourth may be found by one of these four theorems.*

15. Next, in order to find s, and to introduce it into the foregoing theorems, *let either of the above series, and the same series inverted be added together; and since the sum of each series is*$=s$ *by the above notation, the sum of both added together, will evidently be* $2\,s$. *Thus,*

The series . . . . . . $a+\overline{a+d}+\overline{a+2\,d}+\overline{a+3\,d}+$ &c.$=s$.
The series inverted $\overline{a+3\,d}+\overline{a+2\,d}+\overline{a+d}+a\ .\ .\ .\ =s$.
Their sum . . . . . $\overline{2a+3d}+\overline{2a+3d}+\overline{2a+3d}+\overline{2a+3d}=2s$

That is ($\overline{2a+3d}.n$, *or* $\overline{a+a+3d}.n$, *or, since* $a+3d=z$) $\overline{a+z}.n=2s$, *whence* $s=(\frac{\overline{a+z}.n}{2}=)\ \overline{a+z}.\frac{n}{2}$ (THEOR. 5.) *From this equation are derived* $a=\frac{2s}{n}-z$ (THEOR. 6.) $z=\frac{2s}{n}-a$ (THEOR. 7.) *and* $n=\frac{2s}{a+z}$ (THEOR. 8.) *Also by equating the values of* z *in theorems* 1 *and* 7, (*viz.* $a+\overline{n-1}.d=\frac{2s}{n}-a$,) *we obtain* $a=\frac{s}{n}-\frac{n-1}{2}.d$ (THEOR. 9.) $d=(\frac{2s}{n.\overline{n-1}}-\frac{2a}{n-1}=)\ \frac{2}{n}.\frac{s-na}{n-1}$ (THEOR. 10.) $s=\frac{1}{2}n.\overline{2a+\overline{n-1}.d}$ (THEOR. 11.) *and* $n=\frac{\frac{1}{2}d-a+\sqrt{\overline{\frac{1}{2}d-a}|^2+2ds}}{d}$ (THEOR. 12.)

16. *In like manner, by equating the values of* a *in theorems* 2 *and* 6, (*viz.* $z-\overline{n-1}.d=\frac{2s}{n}-z$,) *we derive* $z=\frac{s}{n}+\frac{n-1}{2}.d$ (THEOR. 13.) $d=\frac{2}{n}.\frac{nz-s}{n-1}$ (THEOR. 14.) $s=\frac{1}{2}n.\overline{2z-\overline{n-1}.d}$ (THEOR. 15.) *and* $n=\frac{\frac{1}{2}d+z-\sqrt{\overline{\frac{1}{2}d+z}|^2-2ds}}{d}$ (THEOR. 16.) *and equating the values of* n *in theorems* 4 *and* 8, *we have* $\frac{z-a}{d}+1=\frac{2s}{a+z}$, *whence* $z=\sqrt{\overline{a-\frac{1}{2}d}|^2+2ds}-\frac{1}{2}d$ (THEOR. 17.) $a=\sqrt{\overline{z+\frac{1}{2}d}|^2-2ds}+\frac{1}{2}d$ (THEOR. 18.) $d=\frac{\overline{z+a}.\overline{z-a}}{2s-a-z}$ (THEOR. 19.) $s=\frac{z-a+d}{d}.\frac{z+a}{2}$ (THEOR. 20.)

17. Hence any three of the five quantities a, z, d, n, s, being given, the other two may be found: also if the first term $a=o$, any theorem containing it may be expressed in a simpler manner.

18. The following is a synopsis of the whole doctrine of arithmetical progression, wherein all the theorems above derived are brought into one view.

Theor.	Given	Req.	Solution *when* $a > o$.	Theor.	*When* $a=o$.
I.	a, d, n	z	$z=a+\overline{n-1}.d$	XXI.	$z=\overline{n-1}.d$
XI.		s	$s=\frac{1}{2}.n.\overline{2a+\overline{n-1}.d}$	XXII.	$s=\frac{1}{2}n.\overline{n-1}.d$
III.	a, z, n	d	$d=\frac{z-a}{n-1}$	XXIII.	$d=\frac{z}{n-1}$
V.		s	$s=\overline{a+z}.\frac{n}{2}$	XXIV.	$s=z.\frac{n}{2}$
IV.	a, d, z	n	$n=\frac{z-a}{d}+1$	XXV.	$n=\frac{z}{d}+1$
XX.		s	$s=\frac{z-a+d}{d}.\frac{z+a}{2}$	XXVI.	$s=\frac{z+d}{d}.\frac{z}{2}$
VII.	a, n, s	z	$z=\frac{2s}{n}-a$	XXVII.	$z=\frac{2s}{n}$
X.		d	$d=\frac{2}{n}.\frac{s-na}{n-1}$	XXVIII.	$d=\frac{2}{n}.\frac{s}{n-1}$
VIII.	a, z, s	n	$n=\frac{2s}{a+z}$	XXIX.	$n=\frac{2s}{z}$
XIX.		d	$d=\frac{\overline{z+a}.\overline{z-a}}{2s-a-z}$	XXX.	$d=\frac{z^2}{2s-z}$
XVII.	a, d, s	z	$z=\sqrt{\overline{a-\frac{1}{2}d}\vert^2+2ds}-\frac{1}{2}d$	XXXI.	$z=\sqrt{\frac{1}{4}d^2+2ds}-$
XII.		n	$n=\frac{\frac{1}{2}d-a+\sqrt{\overline{\frac{1}{2}d-a}\vert^2+2ds}}{d}$	XXXII.	$n=\frac{\frac{1}{2}d+\sqrt{\frac{1}{4}d^2+2}}{d}$
II.	z, d, n	a	$a=z-\overline{n-1}.d$		………………
XV.		s	$s=\frac{1}{2}n.\overline{2z-\overline{n-1}.d}$		………………
IX.	d, n, s	a	$a=\frac{s}{n}-\frac{n-1}{2}.d$		………………
XIII.		z	$z=\frac{s}{n}+\frac{n-1}{2}.d$		………………
VI.	z, n, s	a	$a=\frac{2s}{n}-z$		………………
XIV.		d	$d=\frac{2}{n}.\frac{nz-s}{n-1}$		………………
XVIII.	z, d, s	a	$a=\sqrt{\overline{z+\frac{1}{2}d}\vert^2-2ds}+\frac{1}{2}d$		………………
XVI.		n	$n=\frac{\frac{1}{2}d+z-\sqrt{\overline{\frac{1}{2}d+z}\vert^2-2ds}}{d}$		………………

When $d=o$, *then* $a=z=\frac{s}{n}$; $s=na=nz$; $n=\frac{s}{a}=\frac{s}{z}$.

EXAMPLES.—1. In an arithmetical progression, the first term is 3, the number of terms 50, and the common difference 2: what is the last term, and the sum of the series?

Here $a=3$, $n=50$, $d=2$.

Whence, theor. 1. $z=3+\overline{50-1}\times2=101=$ *the last term.*

And, theor. 2. $s=\frac{1}{2}\times50\times\overline{2\times3+\overline{50-1}\times2}=2600=$ *the sum.*

2. Given the first term 3, the last term 101, and the number of terms 50; to find the common difference and the sum of the series?

Here $a=3$, $z=101$, $n=50$.

Whence, theor. 3. $d=\frac{101-3}{50-1}=2=$ *the common difference.*

And theor. 5. $s=\overline{3+101}\times\frac{50}{2}=2600=$ *the sum.*

3. The first term is 3, the common difference 2, and the last term 101; required the number of terms, and the sum?

Here $a=3$, $d=2$, $z=101$.

Wherefore, theor. 4. $n=\frac{101-3}{2}+1=50=$ *the number of terms.*

And, theor. 20. $s=\frac{101-3+2}{2}\times\frac{101+3}{2}=2600=$ *the sum.*

4. The first term is 3, the number of terms 50, and the sum of the series 2600, to find the last term, and difference?

Here $a=3$, $n=50$, $s=2600$.

Then, theor. 7. $z=\frac{2\times2600}{50}-3=101=$ *the last term.*

And, theor. 10. $d=\frac{2}{50}\times\frac{2600-50\times3}{50-1}=2=$ *the common difference.*

5. Given the first term 5, the last term 41, and the sum of the series 299, to find the number of terms, and the common difference? *Ans. by theor.* 8. $n=13$, *and by theor.* 19. $d=3$.

6. Given the first term 4, the common difference 7, and the sum 355, to find the last term, and number of terms? *Ans. by theor.* 17. $z=67$, *and by theor.* 12. $n=10$.

7. The last term is 67, the difference 7, and the number of

terms 10, being given, to find the first term and sum? *Ans. by theor.* 2. $a=4$, *and by theor.* 15. $s=355$.

8. Let the common difference 3, the number of terms 13, and the sum 299 be given, to find the first and last terms? *Ans. by theor.* 9. $a=5$, *and by theor.* 13. $z=41$.

9. Let the last term 67, the number of terms 10, and the sum 355, be given, to find the first term and difference? *Ans. by theor.* 6. $a=4$, *and by theor.* 14. $d=7$.

10. If the last term be 9, the difference 1, and the sum 44, required the first term, and number of terms? *Ans. by theor.* 18. $a=2$, *and by theor.* 16. $n=8$.

11. The first term 0, the last term 15, and the number of terms 6, being given, to determine the difference and sum? *Ans. by theor.* 23. $d=3$, *and by theor.* 24. $s=45$.

12. Bought 100 rabbits, and gave for the first 6*d.* and for the last 34*d.* what did they cost? *Ans.* 8*l.* 6*s.* 8*d.*

13. A labourer earned 3*d.* the first day, 8*d.* the second, 13*d.* the third, and so on, till on the last day he earned 4*s.* 10*d.* how long did he work? *Ans.* 12 *days.*

14. There are 8 equidifferent numbers, the least is 4, and the greatest 32; what are the numbers? *Ans.* 4, 8, 12, 16, 20, 24, 28, *and* 32.

15. A man paid 1000*l.* at 12 equidifferent payments, the first was 10*l.*—what was the second, and the last? *Ans. the second* 23*l.* 6*s.* 8*d. the last* 166*l.* 13*s.* 4*d.*

16. A trader cleared 50*l.* the first year, and for 20 years he cleared regularly every year 5*l.* more than he did the preceding; what did he gain in the last year, and what was the sum of his gains?

17. The sum of a series, consisting of 100 terms, and beginning with a cipher, is 120; required the common difference, and last term?

19. PROBLEMS EXERCISING ARITHMETICAL PROGRESSION.

1. To find three numbers in Arithmetical Progression, the common difference of which is 6, and product 35?

Let the three numbers be $x-6$, x, *and* $x+6$ *respectively. Then by the problem,* ($\overline{x-6}.x.\overline{x+6}=$) $x^3-36x=35$, *or* $x^3-36x-35=0$; *this equation divided by* $x+1$, *gives* ($x^2-x-35=0$, *or*)

$x^2-x=35$; *which resolved, we have* $x=\sqrt{35.25}+.5$, *whence* $x-6=\sqrt{35.25}+5.5$, *and* $x+6=\sqrt{35.25}+6.5$: *the numbers therefore are* .43717, 6.43717, *and* 12.43717, *nearly.*

2. An artist proposed to work as many days at 3 shillings per day, as he had shillings in his pocket; at the end of the time having received his hire, and spent nothing, he finds himself worth 44 shillings; what sum did he begin with?

Let x=*his number of shillings at first, whence also* x=*the number of days he worked: we have therefore here given the first term* x, *the common difference* 3, *and the number of terms* $x+1$, *in an arithmetical progression, to find the last term; now by theor.* 1. ($z=a+\overline{n-1}.d$, *or*) $44=x+\overline{x+1-1}\times 3$, *that is,* $4x=44$, *whence* $x=11$ *shillings=the sum he began with.*

3. To find three numbers in arithmetical progression, such, that their sum may be 12, and the sum of their squares 56?

Let x=*the common difference,* $3s=(12)$ *the sum, then will* s=*the middle number,* $s-x$=*the less extreme, and* $s+x$=*the greater extreme, also let* $a=56$; *then by the problem,* ($\overline{s-x}|^2+s^2+\overline{s+x}|^2=$) $3s^2+2x^2=a$, *whence* $2x^2=a-3s^2$, *and* $x=\sqrt{\dfrac{a-3s^2}{2}}=\sqrt{\dfrac{56-48}{2}}=2$; *therefore* $s=4$, $s-x=2$, *and* $s+x=6$, *that is,* 2, 4, *and* 6, *are the numbers required.*

4. To find four numbers in arithmetical progression, whereof the product of the extremes is 52, and that of the means 70?

Let x=*the less extreme,* y=*the common difference; then will* x, $x+y$, $x+2y$, *and* $x+3y$, *represent the progression. Let* $a=52$, $b=70$, *then by the problem* ($x.\overline{x+3y}=$) $x^2+3xy=a$, *and* ($\overline{x+y}.\overline{x+2y}=$) $x^2+3xy+2y^2=b$; *from the latter equation subtract the former, and* $2y^2=b-a$, *whence* $y=\sqrt{\dfrac{b-a}{2}}=3$; *substitute this value for* y *in the first equation, and it becomes* $x^2+9x=a$; *completing the square, &c. we obtain* $x=\sqrt{a+\dfrac{81}{4}}-\dfrac{9}{2}=4$: *wherefore* 4, 7, 10, *and* 13, *are the numbers required.*

5. The sum of six numbers in arithmetical progression is 48, and if the common difference d be multiplied into the less extreme, the product equals the number of terms; required the terms of the progression?

Let a = *the first term, then* $da=6$, *and* $a=\frac{6}{d}$; *also, since* $s=$ ($\frac{1}{2}n.\overline{2a+n-1.d}=$) $na+\frac{n.\overline{n-1}}{2}.d$ *by theor.* 11. *we have by substitution,* $48=6a+\frac{6\times5}{2}.d$, *that is,* $6a+15d=48$; *whence* $2a+5d=16$, *or (putting* $\frac{6}{d}$ *for* a) $5d^2+12=16d$, *or* $d^2-\frac{16}{5}d=-\frac{12}{5}$; *whence by completing the square, &c.* $d=2$, *therefore* $a=$ ($\frac{6}{2}=$) 3; *consequently the numbers are* 3, 5, 7, 9, 11, *and* 13.

6. The continual product of four numbers in arithmetical progression is 880, and the sum of their squares 214; what are the numbers?

Let $p=880$, $s=214$, $2x$ = *the common difference,* $y-3x$ = *the less extreme; then will* $y-3x$, $y-x$, $y+x$, *and* $y+3x$ = *the terms of the progression: wherefore by the problem,* $\overline{y-3x}.\overline{y-x}.\overline{y+x}.\overline{y+3x}=p$, *and* $\overline{y-3x}|^2+\overline{y-x}|^2+\overline{y+x}|^2+\overline{y+3x}|^2=s$; *these equations reduced, become* $y^4-10y^2x^2+9x^4=p$, *and* $4y^2+20x^2=s$; *from the latter of these* $y^2=\frac{s}{4}-5x^2$, *therefore* $y^4=\frac{s^2}{16}-\frac{5sx^2}{2}+25x^4$; *if these values be substituted for their equals in the former, we have* $\frac{s^2}{16}-\frac{5sx^2}{2}+25x^4-\frac{5sx^2}{2}+50x^4+9x^4=p$, *whence* $x^4-\frac{5sx^2}{84}=\frac{p}{84}-\frac{s^2}{16\times84}$, *or (putting* $a=\frac{5s}{84}$, *and* $\frac{R^2}{4}=\frac{p}{84}-\frac{s^2}{16\times84}$) $x^4-ax^2=\frac{R^2}{4}$; *then by completing the square, &c.* $x=\sqrt{\frac{a+R}{2}}=$ *(by restoring the values of* a *and* R) $1\frac{1}{2}$; *whence* $y=$ ($\sqrt{\frac{s}{4}-5x^2}=$) $6\frac{1}{2}$: *therefore* $y-3x=2$, $y-x=5$, $y+x=8$, *and* $y+3x=11$, *the numbers required.*

20. To find the number of permutations, which can be made with any number of given quantities.

Def. The permutations of quantities are the different orders in which they can be arranged.

Let a and b be two quantities; these will evidently admit of two permutations, viz. ab and ba, which number of permutations may be thus expressed, 1×2.

Let a, b, and c, be three quantities; these admit of six permutations, abc, bac, cab, acb, bca, and cba, viz. $1 \times 2 \times 3$.

Let a, b, c, and d, be four quantities; these admit of 24 permutations; thus,

abcd	*bacd*	*cabd*	*dabc*
abdc	*badc*	*cadb*	*dacb*
acbd	*bcad*	*cbad*	*dbac*
acdb	*bcda*	*cbda*	*dbca*
adbc	*bdac*	*cdab*	*dcab*
adcb	*bdca*	*cdba*	*dcba*

That is, 4 things admit of $1 \times 2 \times 3 \times 4$ *permutations.*

In like manner,

$$\left.\begin{array}{ll} 5 \text{ things admit of} & 1\times2\times3\times4\times5 \\ 6 \ldots\ldots\ldots\ldots & 1\times2\times3\times4\times5\times6 \\ 7 \ldots\ldots\ldots\ldots & 1\times2\times3\times4\times5\times6\times7 \end{array}\right\} \text{permutations.}$$

&c. . . . . &c.

And therefore n things admit of $1 \times 2 \times 3$, *&c. to n, permutations.*

EXAMPLES.—1. How many ways can the musical notes *ut, re, mi, fa, sol, la,* be sung? *Ans.* $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$ *ways.*

2. How many changes can be rung on 12 bells? *Answer,* 479001600.

3. How many permutations can be made with the 24 letters of the alphabet?

21. To find the number of combinations that can be made out of any given number of quantities.

Def. The combinations of quantities, or things, is the taking a less collection out of a greater as often as it can be done, without regarding the order in which the quantities so taken are arranged.

Thus, if a, b, and c, be three quantities, then ab, ac, and bc, are the combinations of these quantities, taken two and two: and here it is necessary to remark, that although ab and ba form two different permutations, yet they form but one combination; in the same manner ac and ca make but one combination, as also bc and cb.

Let there be n things given, namely a, b, c, d, &c. (to n terms,) then if a be placed before each of the rest, $\overline{n-1}$ *permutations*

will be formed; if b be placed before each of the rest, $\overline{n-1}$ permutations will in like manner be formed; and if c, d, e, &c. be placed respectively before each of the rest, $\overline{n-1}$ permutations in each case will arise; consequently, if each of the n things be placed before all the rest, there will be formed in the whole $n.\overline{n-1}$ permutations; that is, there can $n.\overline{n-1}$ permutations be formed of n things taken two at a time.

Hence, if instead of n we suppose $n-1$ things, b, c, d, e, &c. the number of permutations which these afford of the quantities taken two and two, will (by what has been shewn) be $\overline{n-1}.\overline{n-2}$; now if a be prefixed to each of these permutations, there will be $\overline{n-1}.\overline{n-2}$ permutations in which a stands first; in the same manner it appears, that there will be $\overline{n-1}.\overline{n-2}$ permutations in each case when b, c, d, e, &c. respectively stand first; and therefore when each of the n things have stood first, there will be formed in the whole $n.\overline{n-1}.\overline{n-2}$ permutations of n things taken three and three. By similar reasoning it appears that n things taken

$$\left.\begin{array}{ll} 4 \text{ at a time afford} & n.\overline{n-1}.\overline{n-2}.\overline{n-3} \\ 5 \text{ at a time} \ldots\ldots & n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4} \\ r \text{ at a time} \ldots\ldots & n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}\ldots\overline{n-r+1} \end{array}\right\} \text{permutations.}$$

This being premised, we may readily obtain the number of combinations, each consisting of 2, 3, 4, 5, &c. to r things, which can be made out of any given number n; for it appears by the preceding problem, that 2 things admit of 2 permutations, but by the definition they admit of but 1 combination; and therefore any number of things taken 2 at a time, admit of half as many combinations as there are permutations; but the number of permutations in n things, taken two and two, has been shewn to be $n.\overline{n-1}$; therefore the number of combinations in n things, taken two and two, will be $\frac{n.\overline{n-1}}{2}$, or which is the same $\frac{n.\overline{n-1}}{1.2}$.

If three things be taken at a time, then 6 permutations will arise from every 3 things so taken, and but 1 combination; and therefore any number of things taken 3 at a time, admit of one sixth as many combinations, as there are permutations; but the number of permutations in n things taken 3 at a time, has been shewn to be $n.\overline{n-1}.\overline{n-2}$; and therefore the number of combina-

tions in n things, taken 3 at a time, will be $\frac{n.\overline{n-1}.\overline{n-2}}{6}$, *or* $\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$.

By similar reasoning it may be shewn, that the number of combinations in n things, taken

$$\left.\begin{array}{c}4\\5\\6\\r\end{array}\right\} \text{at a time will be} \left\{\begin{array}{l}\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}\\ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}\\ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}}{1.2.3.4.5.6}\\ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}, \text{ \&c. to } \overline{n-r+1}}{1.2.3.4, \text{ \&c. to } r}\end{array}\right.$$

EXAMPLES.—1. How many combinations can be made of 2 letters, out of 10?

Here $n=10$, *whence* $\frac{n.\overline{n-1}}{1.2}=\frac{10\times9}{2}=45$. *Ans.*

2. How many combinations of 5 letters can be made out of the 24 letters of the alphabet?

Here $n=24$, *then* $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3\ 4.5}=10626$. *Ans.*

3. In a ship of war there are 40 officers, and the captain intends to invite 6 of them to dine with him every day; how many parties is it possible to make, so that the same 6 persons may not meet at his table twice?

22. To investigate the rules of simple interest.

Def. 1. The sum lent is called the *principal.*

2. The money paid by the borrower to the lender for the use of the principal, is called *interest.*

3. The interest (or quantity of money to be paid) is previously agreed upon; that is, at a certain sum for the use of 100*l.* for a year: this is called *the rate per cent. per annum* [y].

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[y] Per cent. means *by the hundred,* and per annum, *by the year;* the term 5 *per cent. per annum,* means 5 pounds paid for the use of 100*l.* lent during the space of a year, &c.

Various rates of interest have been given in this country for the use of

4. The principal and interest being added together, the sum is called *the amount.*

Let p=*the principal lent,* r=*the interest of* 1 *pound for a year,* t=*the time during which the principal has been lent,* i= *the interest of* p *pounds for* t *years,* a=*the amount; then will* 1 (*pound*) : r (*interest*) :: p (*pounds*) : pr=*the interest of* p *pounds for a year: and* 1 (*year*) : pr (*interest*) : : t (*years*) : $ptr=i$ (THEOR. 1.)=*the interest of* p *pounds for* t *years, or* t *parts of a year: hence* $p=\frac{i}{tr}$, $t=\frac{i}{pr}$, *and* $r=\frac{i}{pt}$. *If to this interest the principal be added, we shall have* $ptr+p=a$ (THEOR. 2.) *hence by transposition, &c.* $p=\frac{a}{tr+1}$ (THEOR. 3.) $t=\frac{a-p}{pr}$ (THEOR. 4.) *and* $r=\frac{a-p}{pt}$ (THEOR. 5.) *The following is a synopsis of the whole doctrine of simple interest.*

Theor.	Given.	Req.	Solution.
I.	$p, t, r.$	i	$i=ptr.$
II.	$p, t, r.$	a	$a=ptr+p.$
III.	$a, t, r.$	p	$p=\frac{a}{tr+1}.$
IV.	$a, p, r.$	t	$t=\frac{a-p}{pr}.$
V.	$a, p, t.$	r	$r=\frac{a-p}{pt}.$

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money, at different periods, from 5 to 50 per cent. but the law at present is, that not more than 5 per cent. per annum can be taken here, although the legal rate of interest is much higher in some of our colonies.

The interest of money is computed as follows;

In the courts of law ...... in years, quarters, and days.

On South Sea and India bonds ...... calendar months and days.

On Exchequer bills .... quarters of a year and days.

Brokerage, or commission, is an allowance made to brokers and agents in foreign, or other distant places, for buying and selling goods, and performing other money transactions, on my account; it is reckoned at so much per cent. on the money which passes through their hands, and is calculated by the rules of simple interest, the time being always considered as 1. The same rules serve for finding the value of any quantity of stock to be bought or sold, and likewise for finding the price of insurance on houses, ships, goods, &c.

EXAMPLES.—1. Required the simple interest of 765*l.* 10*s.* for 4 years, at 5 per cent. per annum?

Here $p=$(765*l.* 10*s.*$=$) 765.5. $t=4$. $r=(\frac{5}{100}=).05$.

Then $i=ptr$ (*theor.* 1.)$=765.5\times4\times.05=153.1=$153*l.* 2*s.* *Answer.*

2. What is the amount of 75*l.* 10*s.* 6*d.* for $8\frac{1}{2}$ years, at $4\frac{3}{4}$ per cent. per annum?

Here $p=$(75*l.* 10*s.* 6*d.*$=$) 75.525, $t=(8\frac{1}{2}=)$ 8.5, $r=(\frac{4\frac{3}{4}}{100}=)$ .0475: *whence* (*theor.* 2.) $ptr+p=75.525\times8.5\times.0475+75.525$ $=106.01821875=$ 106*l.* 0*s.* 4*d*$\frac{1}{4}$.49$=a$, *the amount.*

3. What sum of money being put out at 3 per cent. simple interest, will amount to 402*l.* 10*s.* in 5 years?

Here $a=$(402*l.* 10*s.*$=$) 402.5, $t=5$, $r=(\frac{3}{100}=).03$: *wherefore* (*theor.* 3.) $\frac{a}{tr+1}=\frac{402.5}{5\times.03+1}=\frac{402.5}{1.15}=$350*l.*$=p$, *the ans.*

4. In what time will 350*l.* amount to 402*l.* 10*s.* at 3 per cent. per annum?

Here $p=350$, $a=402.5$, $r=.03$.

Then (*theor.* 4.) $\frac{a-p}{pr}=\frac{402.5-350}{350\times.03}=\frac{52.5}{10.5}=5$ *years*$=t$, *the answer.*

5. At what rate per cent. will 75*l.* amount to 77*l.* 8*s.* $1\frac{1}{2}$*d.* in $1\frac{1}{2}$ year?

Here $p=75$, $a=$(77*l.* 8*s.* $1\frac{1}{2}$*d.*$=$) 77.40625, $t=(1\frac{1}{2}=)$ 1.5.

Then (*theor.* 5.) $r=\frac{a-p}{pt}=\frac{77.40625-75}{75\times1.5}=.02138=2\frac{1}{10}$ *per cent. nearly,*$=r$, *the answer.*

6. What is the interest of 254*l.* 17*s.* 3*d.* for $2\frac{1}{2}$ years, at 4 per cent. per annum? *Ans.* 25*l.* 9*s.* $8\frac{3}{4}$*d.*

7. What is the amount of 250*l.* in 7 years, at 3 per cent. per annum? *Ans.* 302*l.* 10*s.* 0*d.*

8. What sum being lent for $\frac{3}{4}$ of a year, will amount to 15*s.* $6\frac{3}{4}$*d.* at 5 per cent? *Ans.* 15 *shillings.*

9. In what time will 25*l.* amount to 25*l.* 11*s.* 3*d.* at $4\frac{1}{2}$ per cent. per annum? *Ans. half a year.*

10. At what rate per cent. per annum will 796*l.* 15*s.* amount to 976*l.* 0*s.* $4\frac{1}{2}$*d.* in 5 years? *Ans.* $4\frac{1}{2}$ per cent.

11. Required the interest of 140l. 10s. 6d. for $2\frac{1}{4}$ years, at 5 per cent. per annum?

12. To find the amount of 200l. in 8 years, at $4\frac{1}{2}$ per cent. per annum?

13. Suppose a sum, which has been lent for 120 days at 4 per cent. per annum, amounts to 243l. 3s. $1\frac{1}{4}d$. what is the sum?

14. In what time will 725l. 15s. amount to 731l. 2s. $8\frac{1}{2}d$. at 4 per cent. per annum?

15. At what rate per cent. per annum will 559l. 4s. 0d. amount to 735l. 7s. 0d. in 7 years?

23. To investigate the rules of discount.

Def. 1. When a debt which by agreement between debtor and creditor should be paid *some time hence*, is paid *immediately*, it is usual and just to make an allowance for the early payment; this allowance is called *the discount.*

2. The sum actually paid (that is, the remainder, after the discount has been subtracted from the debt,) is called *the present worth.*

3. The debt is considered as the amount of the present worth, put out at simple interest, at the given rate, and for the given time [a].

Let p = the given debt, r = the interest of 1 pound for a year, t = the time the debt is paid before it is due, in years or parts of a year; then will $1+tr$ = the amount of 1 pound at the rate r, and for the time t: (Art. 22. theor. 2.) then also will the amount of 1 pound be to 1 pound, (or its present worth,) as the given debt, to its present worth; also the amount of 1 pound, is to the interest of 1 pound, as the given debt, to the discount; that is, $1+tr : 1 :: p : \frac{p}{1+tr}$ = the present worth of p pounds paid t time before due, at r per cent. interest; also $1+tr : tr :: p : \frac{ptr}{1+tr}$ = the discount allowed on p pounds, at the said rate, and for the said time.

EXAMPLES.—1. What is the discount, and present worth of 250l. paid 2 years and 75 days before it falls due, at 5 per cent. per annum simple interest?

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[a] In Smart's *Tables of Interest*, there is inserted a table of discounts, by which the discount of any sum of money may be calculated with ease and expedition.

Here $p=250$, $r=.05$, $t=$(2 *y*.75 *d*=) 2.20548 *years*.

Then $\frac{ptr}{1+tr}=\frac{250\times 2.20548\times .05}{1+2.20548\times .05}=\frac{27.5685}{1.110274}=24.83035=$ 24*l*. 16*s*. 7$d\frac{1}{4}$.=*the discount.*

And $\frac{p}{1+tr}=\frac{250}{1+2.20548\times .05}=\frac{250}{1.110274}=225.16964=$ 225*l*. 3*s*. 4$\frac{1}{4}$*d*.=*the present worth.*

2. Required the present worth, and discount, of 487*l*. 12*s*. due 6 months hence, at 3 per cent. per annum? *Ans. pr. worth* 480*l*. 7*s*. 10$\frac{1}{4}$*d*. *disc.* 7*l*. 4*s*. 1$\frac{1}{2}$*d*.

3. Sold goods for 875*l*. 5*s*. 6*d*. to be paid for 5 months hence; what are the present worth and discount at 4$\frac{1}{2}$ per cent. per annum? *Ans. pr. worth* 859*l*. 3*s*. 3$\frac{3}{4}$*d*. *disc.* 16*l*. 2*s*. 2$\frac{1}{4}$*d*.

4. What is the present worth of 150*l*. payable as follows; viz. one third at 4 months, one third at 8 months, and one third at 12 months; at 5 per cent. per annum discount?

5. How much present money can I have for a note of 35*l*. 15*s*. 8*d*. due 13 months hence, at 4$\frac{3}{4}$ per cent. per annum discount?

OF RATIOS.

24. Ratio[a] is the relation which one quantity bears to another in magnitude, the comparison being made by considering how often one of the quantities contains, or is contained in, the other.

Thus, if 12 be compared with 3, we observe that it has a certain relative magnitude with respect to 3, it is 4 times as great as 3, or contains 3 four times; but in comparing it with 6, we discover that it has a different relative magnitude with respect to 6, for it contains 6 but twice.

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[a] *Ratio* is a Latin word implying comparison.

The student must be careful not to confound the idea of ratio with that of proportion, as some through inattention have done: he must bear in mind, that ratio is simply *the comparison of one quantity to another*, both being quantities of the same kind; whereas *proportion is the equality of two ratios*: the former requires two quantities of the same kind to express it, the latter requires at least three quantities, which must be all of the same kind; or four quantities, whereof the two first must be of a kind, and the two last likewise of a kind. See the note on Art. 53, and the note on Art. 127. Part 1. Vol. I.

25. The ratio of two quantities is usually expressed by interposing two dots, placed vertically, between them.

Thus the ratios of a *to* b, *and of* 5 *to* 4, *are written*, $a : b$, *and* $5 : 4$.

26. The former quantity is called the *antecedent*, and the latter the *consequent*.

Thus in the above ratios, a *and* 5 *are the antecedents, and* b *and* 4 *the consequents.*

The antecedent and consequent are called *terms of the ratio.*

27. To determine what multiple, part, or parts the antecedent is of the consequent, (that is, to find how often it contains or is contained in the consequent,) the former must be divided by the latter; and this division is expressed by placing the consequent below the antecedent like a fraction.

Thus the ratio of a *to* b, *or* $a : b$, *is likewise properly expressed thus* $\frac{a}{b}$, *and* $5 : 4$ *thus* $\frac{5}{4}$.

28. Hence, two ratios are equal, when the antecedent of the first ratio is the same multiple, part, or parts of its consequent, that the antecedent of the other ratio is of its consequent; or in other words, when the fraction made by the terms of the former ratio (Art. 27.) is equal to the fraction made by the terms of the latter.

Thus the ratio of $6 : 8$ *is equal to the ratio of* $3 : 4$, *for* $\frac{6}{8} = \frac{3}{4}$.

29. Hence, if both terms of any ratio be multiplied or divided by the same quantity, the ratio is not altered.

Thus if the terms of $3 : 4$ *or* $\frac{3}{4}$ *be both multiplied by any number, suppose* 6, *the result* $\frac{3 \times 6}{4 \times 6} = \frac{18}{24}$, *which fraction is evidently equal to the given fraction* $\frac{3}{4}$; *that is,* $3 : 4$ *is the same as* $18 : 24$; *in like manner, if the terms of the ratio* $a : b$, *or* $\frac{a}{b}$ *be both multiplied by any quantity* n, *the resulting ratio* $an : bn$, *or* $\frac{an}{bn}$ *is the same as* $a : b$, *or* $\frac{a}{b}$; *and the same in general.*

30. Hence, one ratio is greater than another, when the antecedent of the former ratio is a greater multiple, part, or parts of its consequent, than the antecedent of the latter ratio is of its consequent; or, when the fraction constituted by the terms of the first ratio, is greater than that constituted by the terms of the latter.

Thus $6:2$ *is greater than* $8:4$, *for* 6 *contains* 2 *three times, whereas* 8 *contains* 4 *but twice, or* $\frac{6}{2}$ *is greater than* $\frac{8}{4}$.

31. *Having two or more ratios given, to determine which is the greater.*

RULE. Having expressed the given ratios in the form of fractions, (Art. 27.) reduce these fractions to other equivalent ones having a common denominator, (Vol. I. P. 1. Art. 180.) The latter will express the given ratios having a common consequent; wherefore the numerators will express the relative magnitudes of the ratios respectively.

EXAMPLES.—1. Which is the greater ratio, $7:4$, or $8:5$?

These ratios expressed in form of fractions, are $\frac{7}{4}$ *and* $\frac{8}{5}$, *whence* $7\times5=35$, *and* $8\times4=32$, *these are the new numerators; also* $4\times5=20$, *the common denominator.*

Therefore $\frac{7}{4}=\frac{35}{20}$, *and* $\frac{8}{5}=\frac{32}{20}$; *and the former of these being the greater, shews that the ratio of* $7:4$, *is greater than the ratio of* $8:5$.

2. Which is the greater ratio, that of $8:11$, or that of $23:32$?

These ratios expressed like fractions, are $\frac{8}{11}$ *and* $\frac{23}{32}$, *which reduced to other equivalent fractions with a common denominator, become* $\frac{256}{352}$, *and* $\frac{253}{352}$ *respectively; the former of these being the greater, shews that the ratio* $8:11$, *is greater than the ratio* $23:32$.

3. Which is greatest, the ratio of $18:25$, or that of $19:27$? *Ans. the former.*

4. Which is the greatest, and which the least, of the ratios $9:10$, $37:41$, and $75:83$?

32. When the antecedent of a ratio is greater than its consequent, the ratio is called *a ratio of the greater inequality*.

Thus 5 : 3, 11 : 7, *and* 2 : 1, *are ratios of the greater inequality.*

33 [b]. When the antecedent is less than its consequent, the ratio is called *a ratio of the lesser inequality.*

Thus 3 : 5, 7 : 11, *and* 1 : 2, *are ratios of the lesser inequality.*

34. And when the antecedent is equal to its consequent, the ratio is called *a ratio of equality.*

Thus 5 : 5, 1 : 1, *and* $a : a$, *are ratios of equality.*

35. A ratio of the greater inequality is diminished by adding a common quantity to both its terms.

Thus, if 1 *be added to both terms of the ratio* 5 : 3, *it becomes* 6 : 4; *but* $\frac{5}{3}=\frac{20}{12}$, *and* $\frac{6}{4}=\frac{18}{12}$, *the latter of which (being the ratio arising from the addition of* 1 *to the terms of the given ratio) is the least, and therefore the given ratio is diminished: and in general, if* x *be added to both terms of the ratio* 3 : 2, *it becomes* $3+x : 2+x$, *that is* $\frac{3}{2}$ *becomes* $\frac{3+x}{2+x}$; *these fractions reduced to a common denominator as before, become* $\frac{6+3x}{4+2x}$ *and*

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[b] When the antecedent is a multiple of its consequent, the ratio is named a *multiple* ratio; but when the antecedent is an aliquot part of its consequent, the ratio is named a *submultiple* ratio. If the antecedent contains the consequent

$$\left.\begin{array}{ll}\text{twice, as} & 12:6\\ \text{thrice, as} & 12:4\\ \text{four times, as} & 12:3\\ \text{\&c.} & \end{array}\right\}\text{it is called a}\left\{\begin{array}{l}\text{duple,}\\ \text{triple,}\\ \text{quadruple,}\\ \text{\&c.}\end{array}\right\}\text{ratio.}$$

If the antecedent be contained in the consequent

$$\left.\begin{array}{ll}\text{twice, as} & 6:12\\ \text{thrice, as} & 4:12\\ \text{four times, as} & 3:12\\ \text{\&c.} & \end{array}\right\}\text{it is called a}\left\{\begin{array}{l}\text{subduple,}\\ \text{subtriple,}\\ \text{subquadruple,}\\ \text{\&c.}\end{array}\right\}\text{ratio.}$$

There is a great variety of denominations applied to different ratios by the early writers, which is necessary to be understood by those who read the works either of the ancient mathematicians, or of their commentators, and may be seen in Chambers' and Hutton's Dictionary: at present it is usual to name ratios by the least numbers that will express them.

$\frac{6+2x}{4+2x}$ *respectively; and since the latter is evidently the least, it follows that the given ratio is diminished by the addition of* x *to each of its terms.*

36. A ratio of the lesser inequality is increased by the addition of a common quantity to each of its terms.

Thus if 1 *be added to both terms of the ratio* 3 : 5, *it becomes* 4 : 6, *but* $\frac{3}{5}=\frac{18}{30}$, *and* $\frac{4}{6}=\frac{20}{30}$, *the latter of which being the greater, shews that the given ratio is increased: in general, let* 2 : 3 *have any quantity* x *added to both its terms, then the ratio becomes* $2+x:3+x$, *that is* $\frac{2}{3}$ *becomes* $\frac{2+x}{3+x}$; *these reduced to a common denominator, become* $\frac{6+2x}{9+3x}$, *and* $\frac{6+3x}{9+3x}$, *of which the latter being the greater, it shews that the given ratio is increased.*

37. Hence, a ratio of the greater inequality is increased by taking from each of its terms a common quantity less than either.

Thus by taking 1 *from the terms of* 4 : 3, *it becomes* 3 : 2, *but* $\frac{4}{3}=\frac{8}{6}$, *and* $\frac{3}{2}=\frac{9}{6}$, *the latter being the greater, shews that the given ratio is increased.*

38. And a ratio of the lesser inequality is diminished by taking from each of its terms a common quantity less than either.

Thus by taking 2 *from the terms of* 3 : 4, *it becomes* 1 : 2, *but* $\frac{3}{4}=\frac{6}{8}$, *and* $\frac{1}{2}=\frac{4}{8}$, *the latter being the least, shews that the given ratio is diminished.*

39. Hence, a ratio of equality is not altered by adding to, or subtracting from, both its terms any common quantity.

40. If the terms of one ratio be multiplied by the terms of another respectively, namely antecedent by antecedent, and consequent by consequent, the products will constitute a new ratio, which is said to be compounded of the two former; this composition is sometimes called *addition* of ratios.

Thus, if the ratio 3 : 4 *be compounded with the ratio* 2 : 3, *the resulting ratio* ($3\times 2:4\times 3$, *or*) 6 : 12 *is the ratio compounded of the two given ratios* 3 : 4 *and* 2 : 3, *or the sum of the ratios* 3 : 4 *and* 2 : 3.

41. If the ratio $a : b$ be compounded with itself, the resulting ratio $a^2 : b^2$ is the ratio of the squares of a and b, and is said to be *double* the ratio $a : b$, and the ratio $a : b$ is said to be *half* the ratio $a^2 : b^2$; in like manner the ratio $a^3 : b^3$ is said to be *triple* the ratio $a : b$, and $a : b$ *one third* the ratio $a^3 : b^3$; also $a^n : b^n$ is said to be n *times* the ratio of $a : b$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ one n^{th} of the ratio of $a : b$.

41.B. Let $a : 1$ be a given ratio, then $a^2 : 1, a^3 : 1, a^4 : 1, a^n : 1$, are *twice, thrice, four times, n times* the given ratio, where n shews what multiple or part of the ratio $a^n : 1$ the given ratio $a : 1$ is; hence the indices 1, 2, 3, 4, . . . n, are called the measures of the ratios of $a, a^2, a^3, a^4, \ldots a^n$ to 1 respectively, or the logarithms of the quantities $a, a^2, a^3, a^4, \ldots a^n$.

42. If there be several ratios, so that the consequent of the first ratio be the antecedent of the second; the consequent of the second, the antecedent of the third; the consequent of the third, the antecedent of the fourth, &c. then will the ratio compounded of all these ratios, be that of the first antecedent to the last consequent.

For let $a : b, b : c, c : d, d : e$, &c. be any number of given ratios; these compounded by Art. 40. *will be* ($a \times b \times c \times d : b \times c \times d \times e$, *or*) $\frac{a \times b \times c \times d}{b \times c \times d \times e} = \frac{a}{e}$, *or* $a : e$, *the ratio of the first antecedent* a *to the last consequent* e.

43. Hence, in any series of quantities of the same kind, the first will have to the last, the ratio compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, &c. to the last quantity.

44. If two ratios of the greater inequality be compounded together, each ratio is increased.

Thus, let $4 : 3$ *be compounded with* $5 : 2$, *the resulting ratio* ($4 \times 5 : 3 \times 2$ *or*) $\frac{20}{6}$ *is greater than either* $\frac{4}{3}$ *or* $\frac{5}{2}$, *as appears by reducing these fractions to a common denominator. Art.* 31.

45. If two ratios of the lesser inequality be compounded together, each ratio is diminished.

Thus, let $3 : 4$ *be compounded with* $2 : 5$, *the resulting ratio* ($3 \times 2 : 4 \times 5$ *or*) $\frac{6}{20}$, *is less than either of the given ratios* $\frac{3}{4}$ *or* $\frac{2}{5}$, *as appears by reducing these fractions as before.*

46. If a ratio of the greater inequality be compounded with a ratio of the less, the former will be diminished, and the latter increased.

Thus, let $4:3$ *be compounded with* $2:5$, *the resulting ratio* $(4\times 2:3\times 5$ *or*$)$ $\frac{8}{15}$, *is less than the ratio* $\frac{4}{3}$, *but greater than the ratio* $\frac{2}{5}$.

47. From the composition of ratios, the method of their decomposition evidently follows; for since ratios may be represented like fractions, and the sum of two ratios is found by multiplying these fractions representing them together, it is plain that in order to take one ratio from another, we have only to divide the fraction representing the former by that representing the latter. Hence, if the ratio of ($3:4$ or) $\frac{3}{4}$ be compounded with the ratio of ($5:7$ or) $\frac{5}{7}$, we obtain the ratio of ($15:28$ or) $\frac{15}{28}$; now if from this ratio we decompound the former of the given ratios, namely $\frac{3}{4}$, the result will be $(\frac{15}{28}\times\frac{4}{3}=\frac{60}{84}=)\ \frac{5}{7}$, which is the latter of the given ratios; and if from the compounded ratio $\frac{15}{28}$, we decompound the latter given ratio $\frac{5}{7}$, the result will be $(\frac{15}{28}\times\frac{7}{5}=\frac{105}{140}=)\ \frac{3}{4}=$the former given ratio: whence to subtract one ratio from another, this is the rule.

RULE. Let the ratios be represented like fractions. (Art. 27.) Invert the terms of the ratio to be subtracted, and then multiply the correspondent terms of both fractions together; the product reduced to its lowest terms will exhibit the remaining ratio, or that which being compounded with the ratio subtracted, will give the ratio from which it was subtracted.

EXAMPLES.—1. From $5:7$, let $9:8$ be subtracted.

These ratios represented like fractions, are $\frac{5}{7}$ *and* $\frac{9}{8}$.

The latter inverted, becomes $\frac{8}{9}$; *wherefore* $\frac{5}{7}\times\frac{8}{9}=\frac{40}{63}$, *or* 40 : 63, *the difference required.*

2. From 6 : 5, decompound 7 : 10.

Thus $\frac{6}{5}\times\frac{10}{7}=\frac{60}{35}=\frac{12}{7}$, *or* 12 : 7, *the difference required.*

3. From the ratio compounded of the ratios 8 : 7, 3 : 4, and 5 : 9, subtract the ratio compounded of the ratios 1 : 2, 8 : 3, 9 : 7, and 20 : 21.

Thus $\frac{8}{7}\times\frac{3}{4}\times\frac{5}{9}\times\frac{2}{1}\times\frac{3}{8}\times\frac{7}{9}\times\frac{21}{20}=\frac{7}{24}=7:24$, *the difference.*

4. From $a:b$ decompound $x:y$. *Ans.* $ay:bx$.

5. From 11 : 12 decompound 12 : 11. *Ans.* 121 : 144.

6. From 3 : 4 take 3 : 4. *Ans.* 1 : 1.

7. From $a:x$ take $3a:5x$, and from $ax:y^2$ take $y:2ax$.

8. From the ratio compounded of $a:b$, $x:z$, and 5 : 4, take the ratio compounded of $5b:x$, and $2a:3z$.

48. **If the terms of a ratio be nearly equal, or their difference when compared with either of the terms very small, then if this difference be doubled, the result will express double the given ratio; that is, the ratio of the squares of its terms, nearly.**

Let the given ratio be $a+x:a$, *the quantity* x *being very small when compared with* a, *and consequently still smaller when compared with* $a+x$; *then will* ($\overline{a+x}|^2$, *or*) $a^2+2ax+x^2:a^2$ *be the ratio of the squares of the terms* $a+x$ *and* a: *and because* x *is small when compared with* a, $x.x$ (*or* x^2) *is small when compared with* $2a.x$, *and much smaller than* $a.a$; *wherefore if on account of the exceeding smallness of* x^2, *compared with the other quantities, it be rejected, then* (*instead of* $a^2+2ax+x^2:a^2$) *we shall have* $a^2+2ax:a^2$; *that is,* (*by dividing the whole by* a) $a+2x:a$, *for the ratio of the squares of* $a+x:a$, *which was to be shewn.*

EXAMPLES.—1. Required the ratio of the square of 19 to the square of 20?

Here $a=19$, $x=1$, *and* $\frac{a}{a+x}=\frac{19}{20}$, *therefore by the preceding article,* $\frac{a}{a+2x}=\frac{19}{21}$; *that is, the ratio of the square of* 19 *to the*

square of 20 is 19 : 21, nearly. For $\frac{19^2}{20^2}=(\frac{361}{400}=)\frac{7581}{8400}$, *and* $\frac{19}{21}$ $=\frac{7600}{8400}$, *consequently the ratio* $\frac{19}{21}$ *is somewhat too great, but it exceeds the truth by only* $\frac{19}{8400}$, *which is inconsiderable.*

2. Let the ratio of $\overline{80}|^2 : \overline{79}|^2$ be required?

Here $a=79$, $x=1$, *consequently* $\frac{a+x}{a}=\frac{80}{79}$, *and* $\frac{a+2x}{79}=$ $\frac{81}{79}$, *or* $81 : 79=$ *the ratio of* $\overline{80}|^2 : \overline{79}|^2$, *nearly.*

For $\frac{80^2}{79^2}=(\frac{6400}{6241}=)\frac{505600}{493039}$, *and* $\frac{81}{79}=\frac{505521}{493039}$, *which therefore differs from the truth by only* $\frac{79}{493039}$.

3. Let the ratio $\overline{109}|^2 : \overline{111}|^2$ be required? *Ans.* $\frac{109}{113}$.

4. Required the ratio $\overline{1001}|^2 : \overline{1000}|^2$? *Ans.* $\frac{501}{500}$.

5. What are the ratios $\overline{3009}|^2 : \overline{3010}|^2$, and $\overline{10000}|^2 : \overline{10005}|^2$?

49. Hence it appears, that in a ratio of the greater inequality, the above proposed ratio of the squares is somewhat too small; but in a ratio of the less inequality, it is too great.

50. Hence also, because the ratio of the square root of $a+2x : a$ is $a+x : a$ nearly, it follows that if the difference of two quantities be small with respect to either of them, the ratio of their square roots is obtained very nearly by halving the said difference.

EXAMPLES.—1. Given the ratio 120 : 122, required the ratio $\overline{120}|^{\frac{1}{2}} : \overline{122}|^{\frac{1}{2}}$?

Here $a=120$, $2x=2$, $\frac{120}{122}=\frac{a}{a+2x}$, $\because$ $\frac{a}{a+x}=\frac{120}{121}$, *or* 120 : 121 = *the ratio of* $\overline{120}|^{\frac{1}{2}} : \overline{122}|^{\frac{1}{2}}$, *nearly.*

2. Given the ratio 10014 : 10013, to find the ratio of their square roots? *Ans.* 20027 : 20026.

4. Given 9990 : 9996 and 10000 : 10000.5, to find the ratios of their square roots respectively?

51. By similar reasoning it may be shewn, that the ratio of the cubes, of the fourth powers, of the nth powers, is obtained by taking 3, 4, n times the difference respectively, provided 3, 4, or n times the difference is small with respect to either of the terms. And likewise, that the ratio of the 3rd, 4th, or nth roots are obtained nearly by taking $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{n}$ part of the difference respectively.

52. When the terms of a ratio are large numbers, and prime to each other, a ratio may be found in smaller numbers nearly equivalent to the former, by means of what are called continued fractions [e].

Thus, let the given ratio be expressed by $\frac{b}{a}$, *and let b contain a, c times, with a remainder d; let a contain d, e times, with a remainder f; again, let d contain f, g times, with a remainder h, and so on; then by multiplying each divisor by its quotient, and adding the remainder to the product, there arises*

$$\begin{array}{l} a)\ \underline{b}\ (c \\ \quad d)\ \underline{a}\ (e \\ \qquad f)\ \underline{d}\ (g \\ \qquad\quad h)\ \underline{f}\ (k \\ \qquad\qquad l)\ \underline{h}\ (m \\ \qquad\qquad\quad n)\ \underline{l}\ (p \\ \qquad\qquad\qquad q, \&c. \end{array}$$

$$\begin{aligned} b&=ac+d, \\ a&=de+f, \\ d&=fg+h, \\ f&=hk+l, \\ h&=lm+n, \\ l&=np+q, \ \&c. \end{aligned}$$

Hence the given fraction $\frac{b}{a}=(\frac{ac+d}{a}=)\ c+\frac{d}{a}$, *but* $a=de+f$; *this value substituted for a in the preceding equation, we shall have* $\frac{b}{a}=(c+\frac{d}{de+f}=)\ c+\cfrac{1}{e+\cfrac{f}{d}}$; *but since* $d=fg+h$, *by substituting this value for d in the preceding equation, we shall*

[e] The method of finding the approximate value of a ratio in small numbers, has been treated of by Dr. Wallis, in his *Treatise of Algebra*, c 10, 11. and in a tract at the end of Horrox's Works; by Huygens, in *Descript. Autom. Planet. Op. Reliq. p.* 174, *t.* 1; by Mr. Cotes in his *Harmonia Mensurarum*, and by several others.

have $\frac{b}{a}=(c+\cfrac{1}{e+\cfrac{f}{fg+h}}=)\ c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{h}{f}}}$; *but since* $f=hk+l$, *by substituting this value for* f *in the preceding equation, we shall have* $\frac{b}{a}=(c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{h}{hk+l}}}=)\ c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{\frac{k+l}{h}}}}$; *but* $h=lm+n$,

therefore by substituting as before, $\frac{b}{a}=(c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k+\cfrac{l}{lm+n}}}}=)$

$c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k+\cfrac{1}{m+\cfrac{n}{l}}}}}$; *but* $l=np+q$, *therefore* $\frac{b}{a}=$

$(c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k+\cfrac{1}{m+\cfrac{n}{np+q}}}}}=)\ c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k+\cfrac{1}{m+\cfrac{1}{p+\cfrac{q}{n}}}}}}$ *&c. that is,*

$\frac{b}{a}=c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k+\cfrac{1}{m+\cfrac{1}{p}+}}}}$, *&c. a continued fraction.*

Now in this continued fraction, if one term only (viz. c *or* $\frac{c}{1}$*) be taken, it will be an approximation to the ratio* $\frac{b}{a}$ *in small numbers: if two terms, viz.* $c+\frac{1}{e}$ $(=\frac{ce+1}{e})$ *be taken, it will be a*

nearer approximation than the former, to the ratio $\frac{b}{a}$*; but necessarily expressed by a greater number of figures: if three terms be taken, viz.* $c+\cfrac{1}{e+\cfrac{1}{g}}=\left(c+\cfrac{1}{\frac{ge+1}{g}}=c+\frac{g}{ge+1}=\right)\frac{cge+c+g}{ge+1}$, *a nearer approximation to the ratio* $\frac{b}{a}$ *expressed by still more figures; if four terms be taken in, we shall have* $c+\cfrac{1}{e+\cfrac{1}{g+\cfrac{1}{k}}}=$

$$\left(c+\cfrac{1}{e+\cfrac{1}{\frac{gk+1}{k}}}=c+\cfrac{1}{e+\frac{k}{gk+1}}=c+\cfrac{1}{\frac{egk+e+k}{gk+1}}=c+\frac{gk+1}{egk+e+k}\right.$$

$$=\Big)\ \frac{cegk+ce+ck+gk+1}{egk+e+k}.$$

EXAMPLES.—1. Required a series of ratios in smaller numbers, continually approximating to the ratio of 12345 to 67891?

```
12345) 67891 (5
       61725
       ------
        6166) 12345 (2
              12332
              -----
                 13) 6166 (474
                     52
                     --
                      96
                      91
                      --
                       56
                       52
                       --
                        4) 13 (3
                           12
                           --
                            1
```

Here $b=67891$, $a=12345$, $c=5$, $d=6166$, $e=2$, $f=13$, $g=474$, $h=4$, $k=3$, $l=1$.

Then $\frac{c}{1}=\frac{5}{1}$, *an approximation to the given ratio, in the least whole numbers possible.*

Secondly, $\frac{ce+1}{e}=\left(\frac{5\times2+1}{2}=\right)\frac{11}{2}$, *a nearer approximation.*

Thirdly, $\frac{cge+c+g}{ge+1}=(\frac{5\times474\times2+5+474}{474\times2+1}=)\ \frac{5219}{949}$, *a nearer approximation than the former.*

Fourthly, $\frac{cegk+ce+ck+gk+1}{egk+e+k}$

$=(\frac{5\times2\times474\times3+5\times2+5\times3+474\times3+1}{2\times474\times3+2+3}=)\ \frac{15668}{2849}$, *a still nearer approximation than the last.*

2. Required approximate values for the ratio 753171 : 3101000 in more convenient numbers?

OPERATION.

```
753171) 3101000 (4
        3012684
        -------
          88316) 753171 (8
                 706528
                 ------
                  46643) 88316 (1
                         46643
                         -----
                         41673) 46643 (1
                                41673
                                -----
                                 4970) 41673 (8
                                       39760
                                       -----
                                        1913 &c.
```

Here $a=753171$, $b=3101000$, $c=4$, $d=88316$, $e=8$, $f=46643$, $g=1$, $h=41673$, $k=1$, $l=4970$, $m=8$, $n=1913$.

Therefore $\frac{c}{1}=\frac{4}{1}$, *the first approximation.*

$\frac{ce+1}{e}=(\frac{4\times8+1}{8}=)\ \frac{33}{8}$, *the second approximation.*

$\frac{cge+c+g}{ge+1}=(\frac{4\times1\times8+4+1}{1\times8+1}=)\ \frac{37}{9}$, *the third approximation.*

$\frac{cegk+ce+ck+gk+1}{egk+e+k}$

$=(\frac{4\times8\times1\times1+4\times8+4\times1+1\times1+1}{8\times1\times1+8+1}=)\ \frac{70}{17}$, *the fourth approximation, &c. &c.*

3. The ratio of the diameter of a circle to its circumference is nearly as 1000000000 to 3141592653; required approximating values of this ratio in smaller numbers?

Ans. The first $\frac{3}{1}$, *the second* $\frac{22}{7}$, *the third* $\frac{333}{106}$, *the fourth* $\frac{355}{113}$, &c.

4. Required approximate expressions in small numbers for the ratio 7853981635 : 10000000000, being that of the area of a circle, to the square of its diameter, nearly?

Ans. $\frac{1}{1}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{7}{9}$, $\frac{11}{14}$, $\frac{172}{219}$, $\frac{355}{452}$, &c.

5. If the side of a square be 1234000, its diagonal will be 1745139, nearly; required approximations to this ratio in smaller numbers?

OF PROPORTION [d].

53. Four quantities are said to be proportionals, when the first has to the second the same ratio which the third has to the fourth; that is, when the first is the same multiple, part, or parts of the second that the third is of the fourth.

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[d] Ratio is the comparison of magnitudes or quantities; proportion is the equality of ratios; hence there must be two ratios to constitute that equality which is called proportion; that is, there must be three terms at least to express the two ratios necessary to a comparison. Some authors have, with the most unaccountable negligence, confounded and perplexed their inexperienced readers with the definitions they have given of *ratio* and *proportion*. Dr. Hutton, to whose useful labours almost every branch of the mathematics is indebted for elucidation or improvement, in his system of Elementary Mathematics for the use of the Royal Military Academy, thus defines them: "Ratio is the *proportion* which one magnitude bears to another magnitude of the same kind, with respect to quantity;" and immediately after, "Proportion is *the equality of ratios*." Now it has always been held as a necessary maxim in logic, that "in every definition the ideas implied by the terms of the definition, should be more obvious to the mind than the idea of the thing defined," otherwise the definition fails of its purpose; it leaves us just as wise as it found us. Wherefore, supposing the above definitions of ratio and proportion to be adequate and perspicuous, as they ought to be, if we apply this doctrine to them, it will follow from the former, that the idea of proportion is more obvious than that of ratio; and from the latter, that the idea of ratio is more obvious than that of proportion; but the supposition that both these conclusions are true, implies a manifest absurdity, and consequently, that one or both of these definitions must be faulty. It is but justice to suppose, that the learned Doctor must have used the term *proportion*, in the former definition,

54. This proportion, or equality of ratios, is usually expressed by four dots, thus :: interposed between the two ratios.

Thus, $a : b :: c : d$, shews that a has to b the same ratio that c has to d, or that the four quantities, a, b, c, and d, are proportionals, and are usually read, a is to b, as c to d.

55. The first and last terms of the proportion (viz. a and d) are called *the extremes*, and the two middle terms (b and c) *the means.*

56. Since it has been shewn, (Art. 27.) that any ratio is truly expressed by placing its terms in the form of a fraction; therefore, when four quantities are proportionals, that is, when the first has to the second the same ratio which the third has to the fourth, it follows, that the fraction constituted by the terms of the first ratio, will be equal to the fraction constituted by the terms of the other ratio placed in the same order.

Thus, if $a : b :: c : d$, then will $\frac{a}{b}=\frac{c}{d}$, or $\frac{b}{a}=\frac{d}{c}$.

57. If four quantities are proportionals, the product of the extremes is equal to the product of the means.

Let $a : b :: c : d$, then by the preceding article, $\frac{a}{b}=\frac{c}{d}$; multiply the terms of this equation by bd, and ($\frac{a}{b}\times bd=\frac{c}{d}\times bd$, or) $ad=bc$. Euclid 16, 6.

58. Hence, if three quantities are proportionals, the product of the extremes is equal to the square of the mean.

Let $a : c :: c : d$, then $\frac{a}{c}=\frac{c}{d}$, by what has been shewn; multiply both sides by cd, and ($\frac{a}{c}\times cd=\frac{c}{d}\times cd$, or) $ad=c^2$. Euclid 17, 6.

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according to its *vulgar* acceptation, (namely, the comparison of one thing with another,) and in the latter, according to its *mathematical* import. The learner ought to be cautioned to study not to be imposed on by the double meaning of words, and especially to scorn the mean artifice of availing himself on any occasion of the ambiguity of language. A wrangler may confound his opponent by using the same word in two or three different senses; but truth (which is the grand object of science) is discovered only when our reasoning proceeds by means of terms which are strictly limited in their signification.

59. Hence, if three terms of any proportion be given, the fourth may be found:

For since $ad=bc$, *if* a, d, *and* b, *are given, then* $\frac{ad}{b}=c$; *if* a, d, *and* c, *are given,* $\frac{ad}{c}=b$; *if* a, b, *and* c, *are given,* $\frac{bc}{a}=d$; *and if* d, b, *and* c, *are given, then* $\frac{bc}{d}=a$ *.

60. If the product of two quantities be equal to the product of two others, then if the terms of one product be made the means, and the terms of the other product the extremes, the four quantities will be proportionals.

Thus, if $ad=bc$, *divide both sides by* bd, *and* $(\frac{ad}{bd}=\frac{bc}{bd}$, *or*$)$ $\frac{a}{b}=\frac{c}{d}$, *that is,* $a:b::c:d$. Euclid 17, 6.

61. If the first term be to the second, as the third to the fourth, and the third to the fourth as the fifth to the sixth, then will the first be to the second as the fifth to the sixth.

Let $a:b::c:d$, *and* $c:d::e:f$, *then will* $a:b::e:f$; *for since* $\frac{a}{b}=\frac{c}{d}$, *and* $\frac{c}{d}=\frac{e}{f}$, *then is* $\frac{a}{b}=\frac{e}{f}$, *that is,* $a:b::e:f$.

62. Hence, if the same ratio subsists between every two adjacent terms of any rank of quantities, that is, if the terms are in continued proportion, the first term will be to the second as the last but one to the last.

For, let a, b, c, d, e, f, g, h, k, l, *&c. be such, then* $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}=\frac{f}{g}=\frac{g}{h}=\frac{h}{k}=\frac{k}{l}$, *&c. then will* $\frac{a}{b}=\frac{k}{l}$, *or* $a:b::k:l$.

63. If four quantities are proportionals, they are also proportionals when taken inversely.

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* This article furnishes a demonstration of the Rule of Three, except that part of it which respects the reducing of the terms: but the latter is obvious; since in order to compare quantities, it is plain we must bring them to a simple form, and likewise the quantities compared must be of the same denomination, otherwise a comparison cannot be made.

Let $a:b::c:d$, *then will* $b:a::d:c$; *for since* $\frac{a}{b}=\frac{c}{d}$, *let unity be divided by each of these equal fractions, and the quotients* $(1\div\frac{a}{b}=)\ \frac{b}{a}$, *and* $(1\div\frac{c}{d}=)\ \frac{d}{c}$ *will be equal, wherefore* $b:a::d:c$; *this operation and property is usually cited under the name* INVERTENDO. Euclid pr. B. Book 5.

64. If four quantities be proportionals, they are also proportionals when taken alternately.

Let $a:b::c:d$, *then will* $a:c::b:d$; *for* $\frac{a}{b}=\frac{c}{d}$, *wherefore multiplying each of these equals by* $\frac{b}{c}$, *we have* $(\frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{b}{c}$, *or*$)\ \frac{a}{c}=\frac{b}{d}$, *that is,* $a:c::b:d$; *this is named* ALTERNANDO, *or* PERMUTANDO. Euclid 16, 5.

65. If four quantities be proportionals, the sum of the first and second is to the second, as the sum of the third and fourth to the fourth.

Let $a:b::c:d$, *then will* $a+b:b::c+d:d$. *Because* $\frac{a}{b}=\frac{c}{d}$, *let unity be added to each, and* $(\frac{a}{b}+1=\frac{c}{d}+1$, *that is*$)\ \frac{a+b}{b}=\frac{c+d}{d}$, *wherefore* $a+b:b::c+d:d$; *this is named* COMPONENDO. Euclid 18, 5.

66. In like manner, the first is to the sum of the first and second, as the third to the sum of the third and fourth.

For since $\frac{a+b}{b}=\frac{c+d}{d}$, *invertendo* $\frac{b}{a+b}=\frac{d}{c+d}$, *also* (*Art.* 62.) $ad=bc$; *wherefore* $(\frac{b}{a+b}\times ad=\frac{d}{c+d}\times bc$, *or*$)\ \frac{abd}{a+b}=\frac{bcd}{c+d}$; *divide these equals by* bd, *and* $\frac{a}{a+b}=\frac{c}{c+d}$, *or* $a:a+b::c:c+d$.

67. If four quantities be proportionals, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.

Let $a:b::c:d$, *then will* $a-b:b::c-d:d$. *Because* $\frac{a}{b}=$

$\frac{c}{d}$, *let unity be subtracted from each, and* $(\frac{a}{b}-1=\frac{c}{d}-1$, *or*) $\frac{a-b}{b}=\frac{c-d}{d}$, *that is,* $a-b:b::c-d:d$; *this is called* DIVIDENDO. Euclid 17, 5.

68. In like manner, the first is to its excess above the second, as the third to its excess above the fourth.

Because $\frac{a-b}{b}=\frac{c-d}{d}$ *by the preceding, and since* $\frac{b}{a}=\frac{d}{c}$, *therefore* $\frac{a-b}{b}\times\frac{b}{a}=\frac{c-d}{d}\times\frac{d}{c}=\frac{a-b}{a}=\frac{c-d}{c}$, *or* $a-b:a::c-d:c$, *and invertendo* (*Art.* 63.) $a:a-b::c:c-d$; *this is* CONVERTENDO.

69. Hence, because $a-b:a::c-d:c$, the excess of the first above the second is to the first, as the excess of the third above the fourth to the fourth.

70. If four quantities be proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth to their difference.

Let $a:b::c:d$, *then will* $a+b:a-b::c+d:c-d$; *for since* $\frac{a+b}{b}=\frac{c+d}{d}$, (*Art.* 65.) *and* $\frac{a-b}{b}=\frac{c-d}{d}$, (*Art.* 67.) *divide the former equals by the latter, and* $(\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}$, *or*) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, *that is,* $a+b:a-b::c+d:c-d$.

71. Hence, the difference of the first and second is to their sum, as the difference of the third and fourth to their sum.

For since $a+b:a-b::c+d:c-d$, *therefore invertendo* $a-b:a+b::c-d:c+d$.

72. If several quantities be proportionals, as any one of the antecedents is to its consequent, so is the sum of any number of the antecedents, to the sum of their respective consequents.

Let $a:b::c:d::e:f::g:h::k:l::m:n$, &c. *then will* $a:b::a+c+e+g+k+m:b+d+f+h+l+n$. *Because* $a:b::c:d$, *therefore* $ad=bc$, *and* $ab=ba$; *also, because* $a:b::e:f$, *therefore* $af=be$; *in like manner,* $ah=bg$, $al=bk$, *and* $an=bm$: *wherefore* $(ad+af+ah+al+an=bc+be+bg+bk+bm$, *or*) $a\times\overline{d+f+h+l+n}=b\times\overline{c+e+g+k+m}$, *wherefore* $a:b::c+$

$e+g+k+m : d+f+h+l+n$; and the like may be proved, whatever number of antecedents and their respective consequents be taken.

73. If four quantities be proportionals, and if equimultiples or equal parts of the first and second, and equimultiples or equal parts of the third and fourth, be taken, the resulting quantities will likewise be proportionals.

Thus, if $a : b :: c : d$,

Then will

1. $ma : mb :: mc : md$
2. $ma : mb :: nc : nd$
3. $ma : mb :: \frac{r}{n}c : \frac{r}{n}d$
4. $\frac{r}{n}a : \frac{r}{n}b :: mc : md$
5. $\frac{n}{m}a : \frac{n}{m}b :: \frac{r}{s}c : \frac{r}{s}d$, &c.

For in each case, (by multiplying extremes and means,) $ad=bc$, *or* $\frac{a}{b}=\frac{c}{d}$, *or* $a : b :: c : d$.

74. Hence, if two quantities be prime to each other, they are the least in that proportion.

75. If four quantities be proportionals, and the first and third be multiplied or divided by any quantity, and also if the second and fourth be multiplied by the same or any other quantity, the results will be proportionals.

Let $a : b :: c : d$,

Then will

1. $ma : nb :: mc : nd$
2. $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$
3. $ma : \frac{b}{n} :: mc : \frac{d}{n}$
4. $ma : mb :: mc : nd$
5. $\frac{a}{m} : nb :: \frac{c}{m} : nd$, &c.

For in each case, (multiplying extremes and means,) $ad=bc$, *or* $\frac{a}{b}=\frac{c}{d}$, *or* $a : b :: c : d$.

76. Hence, if four quantities be proportionals, their equimultiples, as also their like parts, are proportionals.

77. Hence also, if instead of the first and second terms, or of the first and third, or of the second and fourth, or of the third and fourth, other quantities proportional to them be substituted, the results in each case will be proportionals.

78. In several ranks of proportional quantities, if the corresponding terms be multiplied together, the product will be proportionals.

Thus, let $\left.\begin{array}{l} a : b :: c : d \\ \textit{And } e : f :: g : h \\ \textit{And } k : l :: m : n \end{array}\right\}$ *then will* $aek : bfl :: cgm : dhn$.

For $\frac{a}{b}=\frac{c}{d}$, $\frac{e}{f}=\frac{g}{h}$, *and* $\frac{k}{l}=\frac{m}{n}$, *therefore* $\frac{aek}{bfl}=\frac{cgm}{dhn}$, *or* $aek : bfl :: cgm : dhn$, *and the like may be shewn of any number of ranks.*

79. Hence it follows, that the like powers of proportional quantities (viz. their squares, cubes, &c.) are proportionals.

For, let $a : b :: c : d$

And $a : b :: c : d$

Also $a : b :: c : d$, *&c. then by multiplying two of these ranks together, as in the former article, we have* $a^2 : b^2 :: c^2 : d^2$, *and by multiplying all the three,* $a^3 : b^3 :: c^3 : d^3$; *and the like may be shewn of all higher powers whatever.*

80. Hence also the like roots of proportional quantities are proportionals.

For, let $a : b :: c : d$, *then will* $a^{\frac{1}{2}} : b^{\frac{1}{2}} :: c^{\frac{1}{2}} : d^{\frac{1}{2}}$; *for* $\frac{a}{b}=\frac{c}{d}$, *therefore* $\sqrt{\frac{a}{b}}=\sqrt{\frac{c}{d}}$, *that is,* $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{c^{\frac{1}{2}}}{d^{\frac{1}{2}}}$, *or* $a^{\frac{1}{2}} : b^{\frac{1}{2}} :: c^{\frac{1}{2}} : d^{\frac{1}{2}}$, *and the same may be shewn of any other roots.*

The operation described in the three foregoing articles, is called COMPOUNDING THE PROPORTIONS.

81. If there be any number of quantities, and also as many others, which taken two and two in order are proportionals, namely, the first to the second of the first rank, as the first to the second of the other rank; the second to the third of the first rank, as the second to the third of the other rank, and so on to the last quantity in each; then will the first be to the last of the first rank, as the first to the last of the other rank.

For, let $\left.\begin{array}{l} a:b:c:d:e \\ \textit{And } f:g:h:k:l \end{array}\right\}$ *be such that* $\left\{\begin{array}{l} a:b::f:g \\ b:c::g:h \\ c:d::h:k \\ d:e::k:l \end{array}\right.$

Then will $a:e::f:l$; *for if the above four proportions be compounded,* (*Art.* 78.) *we shall have* $abcd:bcde::fghk:ghkl$, *or* $(\frac{abcd}{bcde}=\frac{fghk}{ghkl}$, *or*$)$ $\frac{a}{e}=\frac{f}{l}$, *wherefore* $a:e::f:l$, *and the like may be demonstrated of any number of ranks.*

This is called EX ÆQUALI IN PROPORTIONE ORDINATA, or simply EX ÆQUO ORDINATO. Euclid 22, 5.

82. If there be any number of quantities, and as many others, which taken two and two in cross order are proportionals, namely, the first to the second of the first rank, as the last but one to the last of the other rank; the second to the third of the first rank, as the last but two to the last but one of the other rank, and so on in cross order; then will the first be to the last of the first rank, as the first to the last of the other rank.

Let $\left.\begin{array}{l} a:b:c:d:e \\ \textit{And } f:g:h:k:l \end{array}\right\}$ *be such that* $\left\{\begin{array}{l} a:b::k:l \\ b:c::h:k \\ c:d::g:h \\ d:e::f:g \end{array}\right.$

Then will $a:e::f:l$; *for compounding the above four proportions,* (*Art.* 78.) *there arises* $abcd:bcde::khgf:lkhg$, *or* $(\frac{abcd}{bcde}=\frac{khgf}{lkhg}$, *that is,*$)$ $\frac{a}{e}=\frac{f}{l}$, *wherefore* $a:e::f:l$, *which was to be shewn; and the like may be proved of any number of ranks.*

This is called EX ÆQUALI IN PROPORTIONE PERTURBATA, or simply, EX ÆQUO PERTURBATO [f]. Euclid 23, 5.

INVERSE, OR RECIPROCAL PROPORTION.

83. The foregoing articles treat of the properties of what is called DIRECT PROPORTION, where the first is to the second as the third is to the fourth; but when the terms are so arranged,

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[f] It must be understood, that what we have delivered on proportion, refers to *commensurable magnitudes only*: it is in substance the same as the 5th book of Euclid's Elements, except that the doctrine there delivered includes both *commensurable* and *incommensurable* magnitudes; Euclid has effected this double object by means of his fifth definition, which although strictly general, has been justly complained of for its ambiguity and clumsiness.

that the first is to the second, as the fourth to the third, it is then named INVERSE PROPORTION, and the four quantities in the order they stand, are said to be INVERSELY PROPORTIONAL.

Thus, $2 : 4 :: 12 : 6$, *and* $9 : 5 :: 10 : 18$, *&c. are inversely proportional.*

84. Hence, an *inverse* proportion may be made *direct,* by changing the order of the terms in either of the ratios which constitute the proportion.

85. The reciprocals of any two quantities will be inversely proportional to the quantities.

Let a and b be two quantities, then will $a : b :: \frac{1}{b} : \frac{1}{a}$, *for multiplying both terms of the latter ratio by ab, we shall have* $a : b :: (\frac{ab}{b} : \frac{ab}{a} ::)\ a : b$, *therefore* $a : b :: \frac{1}{b} : \frac{1}{a}$; *in like manner* $b : a :: \frac{1}{a} : \frac{1}{b}$, *that is, the direct ratio of the quantities is the same as the inverse ratio of their reciprocals; and the inverse ratio of the quantities, the same as the direct of their reciprocals.*

Hence, inverse proportion is likewise frequently called RECIPROCAL PROPORTION.

HARMONICAL PROPORTION.

86. Three quantities are said to be in harmonical or musical proportion, when the first is to the third, as the difference of the first and second to the difference of the second and third; and four terms are said to be in harmonical proportion, when the first is to the fourth, as the difference of the first and second to the difference of the third and fourth.

Thus, if $a : c :: a-b : b-c$, *then are the three quantities, a, b, and c, harmonically proportional.*

And if $a : d :: a-b : c-d$, *then are the four, a, b, c, and d, harmonically proportional.*

87. Hence, if all the terms of any harmonical proportion be either multiplied or divided by any quantity whatever, the results will still be in harmonical proportion.

88. If double the product of any two quantities be divided by their sum, the quotient will be a harmonical mean between the two quantities.

Let a and b be two quantities, then $2ab$ = *double their product, and* $a+b$ = *their sum, wherefore* $\frac{2ab}{a+b}$ *is the harmonical mean required, for (Art. 86.)* $a \,.\, b :: a-\frac{2ab}{a+b} : (\frac{b}{a}\times\overline{a-\frac{2ab}{a+b}}=\frac{ab-b^2}{a+b}=\frac{b^2-ab}{-b-a}=)\ \frac{2ab}{a+b}-b$; *that is, the first is to the third, as the difference between the first and second to the difference between the second and third.*

EXAMPLES.—1. To find a harmonical mean between 2 and 6.

Here $a=2$, $b=6$, *and* $\frac{2ab}{a+b}=\frac{24}{8}=3$, *the mean required; for* $2 : 6 :: (3-2 : 6-3 ::)\ 1 : 3$.

2. Required a harmonical mean between 24 and 12? *Ans.* 16.

3. Required the harmonical mean between 5 and 20? *Ans.* 8.

4. Required the harmonical mean between 10 and 30?

89. If the product of two given quantities be divided by the difference between double the greater and the less, or double the less and the greater, the quotient will be the third harmonical proportional to the two given quantities.

Let a and b be two given quantities, whereof a is the greater; then will $\frac{ab}{2a-b}$ *be the third harmonical proportional to a and b:*

for $a : \frac{ab}{2a-b} :: a-b : (\frac{ab.\overline{a-b}}{a.\overline{2a-b}}=\frac{ab-b^2}{2a-b}=\frac{b^2-ab}{b-2a}=b+\frac{ab}{b-2a}=)$

$b-\frac{ab}{2a-b}$, *the difference between the second and third* [*].

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[*] To what has been said on this subject, the following particulars relating to the comparison, &c. of the three kinds of proportionals, may be added; viz.

1. The reciprocals of an arithmetical progression are in harmonical progression, and the reciprocals of a harmonical progression, are in arithmetical progression.

Thus, a, $a+d$, $a+2d$, $a+3d$, *are arithmetically proportional.*

And, $\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$, $\frac{1}{a+3d}$, *their reciprocals, are harmonically proportional, and the converse.*

EXAMPLES.—1. To find a third harmonical proportional to 48 and 32.

Here $a=48$, $b=32$, *and* $\frac{ab}{2a-b}=\frac{48\times32}{2\times48-32}=\frac{1536}{64}=24$, *the number required; for* 48 : 24 : : (48—32 : 32—24 : :) 16 : 8.

2. Required a third harmonical proportional to 2 and 3? *Ans.* 6.

3. Required the third harmonical proportional to 20 and 8? *Ans.* 5.

4. Required the third harmonical proportional to 10 and 100?

90. Of four harmonical proportionals any three being given, the fourth may be found as follows.

Let a, b, c, and d, be four quantities in harmonical proportion, then since $a : d :: a-b : c-d$, (*Art.* 86.) *by multiplying extremes and means,* $ac-ad=ad-bd$; *from this equation any three of the quantities being given, the remaining one may be found.*

Thus, a, b, and c, being given, we have $d=\frac{ac}{2a-b}$ *one of the extremes; if b, c, and d, be given, then* $a=\frac{bd}{2d-c}$ *the other extreme; if a, b, and d, be given, then* $c=\frac{2ad-bd}{a}$ *one of the means; if a, c, and d, be given, then* $b=\frac{2ad-ac}{d}$ *the other mean.*

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2. If there be taken an arithmetical mean and a harmonical mean between any two quantities, then the four quantities will be geometrically proportional.

Thus, between a and b the harmonical mean is $\frac{2ab}{a+b}$, *and the arithmetical mean* $\frac{a+b}{2}$, *and* $a : \frac{2ab}{a+b} :: \frac{a+b}{2} : b$.

3. The following simple and beautiful comparison of the three kinds of proportionals, is given by Pappus, in his third book of Mathematical Collections.

Let a, b, and c, be the first, second, and third terms; then,

$$\text{In the} \left\{ \begin{array}{ll} \textit{Arithmeticals} & a : a \\ \textit{Geometricals} & a : b \\ \textit{Harmonicals} & a : c \end{array} \right\} :: a-b : b-c.$$

4. There is this remarkable difference between the three kinds of proportion; namely, from any given term there can be raised

A continued arithmetical series, increasing but not decreasing,
A continued harmonical series, decreasing but not increasing,
A continued geometrical series, both increasing and decreasing, } indefinitely.

EXAMPLES.—1. Let there be given 3, 4, and 6, being the first, second, and third terms of a harmonical proportion, to find the fourth?

Here $a=3$, $b=4$, $c=6$, *and* $\frac{ac}{2a-b}=(\frac{3\times 6}{2\times 3-4}=\frac{18}{2}=)\ 9$, *the fourth term required; for* $3 : 9 :: (4-3 : 9-6 ::)\ 1 : 3$.

2. Given the second, third, and fourth terms, viz. 4, 6, and 9, to find the first?

Here $b=4$, $c=6$, $d=9$, *wherefore* $a=\frac{bd}{2d-c}=(\frac{4\times 9}{2\times 9-6}=\frac{36}{12}=)\ 3$, *the first term required.*

3. Given 3, 6, and 9, being the first, third, and fourth terms, to find the second?

Here $a=3$, $c=6$, $d=9$, *and* $b=\frac{2ad-ac}{d}=(\frac{54-18}{9}=)\ 4$, *the second term required.*

4. Given 3, 4, and 9, being the first, second, and fourth, to find the third?

Here $a=3$, $b=4$, $d=9$, *and* $c=\frac{2ad-bd}{a}=(\frac{54-36}{3}=)\ 6$, *the third term, as was required.*

5. Let the first, second, and third terms in harmonical proportion, viz. 36, 48, and 72, be given to find the fourth?

6. Given 24, 36, and 54, or the second, third, and fourth terms, to find the first?

7. Given 27, 36, and 81, being the first, second, and fourth terms, to find the third?

8. Let 48, 96, and 144, being the first, third, and fourth, be given, to find the second?

91. Three quantities are said to be in CONTRA-HARMONICAL PROPORTION, when the third is to the first, as the difference of the first and second to the difference of the second and third.

Thus, let a, b, and c, be three quantities in contra-harmonical proportion, then will $c : a :: a \backsim b : b \backsim c$.

92. The following is a synopsis of the whole doctrine of proportion, as contained in the preceding articles.

Let four quantities a, b, c, and d, be proportionals, then are they also proportionals in all the following forms; viz.

1. Directly .... $a : b :: c : d$.
2. Inversely .... $b : a :: d : c$.
3. Alternately .... $a : c :: b : d$.
4. Alternately and inversely .... $c : a :: d : b$.
5. Compoundedly .... $a : a+b :: c : c+d$.
6. Compoundedly and inversely $a+b : a :: c+d : c$.
7. Compoundedly and alternately $a : c :: a+b : c+d$.
8. Compoundedly, alternately, and inversely .... $c : a :: c+d : a+b$.
9. Dividedly .... $a : a-b :: c : c-d$.
   or, ... $a : b-a :: c : d-c$.
10. Dividedly and alternately .... $a : c :: a-b : c-d$.
    or, ... $a : c :: b-a : d-c$.
11. Mixedly .... $a+b : a-b :: c+d : c-d$.
12. Mixedly and inversely .... $a-b : a+b :: c-d : c+d$.
13. Mixedly and alternately ..... $a+b : c+d :: a-b : c-d$.
14. By multiplication .... $ra : rb :: sc : sd$.
15. By division .... $\frac{a}{r} : \frac{b}{r} :: \frac{c}{s} : \frac{d}{s}$.
16. By involution .... $a^n : b^n :: c^n : d^n$.
17. By evolution .... $a^{\frac{m}{n}} : b^{\frac{m}{n}} :: c^{\frac{m}{n}} : d^{\frac{m}{n}}$.
18. They are inversely proportional when $a : b :: d : c$.
19. They are in harmonical proportion when $a : d :: a \backsim b : c \backsim d$.
20. Three numbers are in contra-harmonical proportion when $c : a :: a \backsim b : c \backsim d$.

The 14th, 15th, 16th, and 17th particulars admit of inversion, alternation, composition, division, &c. in the same manner with the foregoing ones, as is evident from the nature of proportion.

THE COMPARISON OF VARIABLE AND DEPENDANT QUANTITIES[b].

93. A quantity is said to be *variable*, when from its nature and constitution it admits of increase or decrease.

[b] The doctrine of variable and dependant quantities, as laid down in the following articles, should be well understood by all those who intend to read

94. A quantity is said to be *invariable* or *constant*, when its nature is such that it does not change its value.

95. Two variable quantities are said to be *dependant*, when one of them being increased or decreased, the other is increased or decreased respectively, in the same ratio.

Thus, let A and B be two variable quantities, such, that when A is changed into any other value a, B is necessarily changed into a corresponding value b, (in which case A : a :: B : b,) then A and B are said to be mutually dependant.

96. To every proportion four terms are necessary, but in applying the doctrine to practice, although four quantities are always understood, two only are employed. This concise mode of expression is found to possess some advantages above the common method, as it saves trouble, and likewise assists the mind, by enabling it to conceive more readily the relations which the variable and dependant quantities under consideration bear to each other.

97. Of two variable and dependant quantities, each is said to vary directly as the other, or to vary as the other, or simply to be as the other, when one being increased, the other is necessarily increased in the same ratio, or when one is decreased, the other also is decreased in the same ratio.

Thus, if r be any number whatever, and if when A is increased to rA, B is necessarily increased to rB, (that is, when $A : rA :: B : rB$,) *or when A is decreased to* $\frac{A}{r}$, *B is necessarily decreased to* $\frac{B}{r}$, *(that is, when* $A : \frac{A}{r} :: B : \frac{B}{r}$,) *then A is said to vary directly as B; or we say simply, A is directly as B.*

EXAMPLE. A labourer agrees to work a week for a certain sum; now if he work 2 weeks, he receives *twice* that sum, if he work but half a week, he receives but *half* that sum, and so on; in this case, the sum he receives is directly as the time he works.

Sir Isaac Newton's Principia, or any other scientific treatise on Natural Philosophy or Astronomy. See on this subject, *Ludlam's Rudiments*, 5th Edit. p. 235—250. and *Wood's Algebra*, 3d Edit. p. 103—109.

98. One quantity is said to vary *inversely* as another, when the former cannot be increased, but the other is decreased in the same ratio; or the former cannot be decreased, but the other must necessarily be increased in the same ratio; that is, the former cannot be changed, but the reciprocal of the latter is changed in the same ratio.

EXAMPLE. A man walks a certain distance in an hour; now if he walk twice as fast, he will go the given distance in half an hour; but if he walk only half as fast, he will evidently require two hours to complete his journey; in this case his rate of walking is inversely as the time he takes to perform it.

99. The sign $\propto$ placed between two quantities, signifies that they vary as each other.

Thus $A \propto B$ implies that A varies as B, or that A is as B; also $A \propto \frac{1}{B}$ shews that A varies as the reciprocal of B, or that A is inversely as B.

100. One quantity is said to vary as two others jointly, when the former being changed, the product of the two latter must necessarily be changed in the same ratio.

Thus A varies as B and C jointly, that is, $A \propto BC$, when A cannot be changed into a, but the product BC must be changed into bc, or that $A : a :: BC : bc$.

101. In like manner one quantity varies as three others jointly, when the former being changed, the product of the three latter is changed in the same ratio.

Thus $A \propto BCD$, and the like, when more quantities are concerned.

EXAMPLE. The interest of money varies as the product of the principal, rate per cent. and time, or $I \propto PRT$.

102. One quantity is said to vary *directly* as a second, and *inversely* as a third, when the first cannot be changed, but the second multiplied by the reciprocal of the third, (that is, the second divided by the third,) is changed in the same ratio.

Thus A varies directly as B, and inversely as C, or $A \propto \frac{B}{C}$, when $A : a :: \frac{B}{C} : \frac{b}{c}$.

EXAMPLE. A farmer must employ as many reapers, as are *directly* as the number of acres to be reaped, and *inversely* as the number of days he allots for the work, or $R \propto \frac{A}{D}$.

103. If $A \propto B$, and $A \propto C$, then will $A \propto BC$.

For since $B : b :: A : \frac{Ab}{B} = a$, *and* $C : c :: \frac{Ab}{B} : \frac{Abc}{BC} = a =$ *the final value of* A *arising from its successive changes in the ratios of* $B : b$ *and* $C : c$; *wherefore since* $\frac{Abc}{BC} = a$, *or* $Abc = aBC$, $A : a :: BC : bc$, *or* $A \propto BC$.

104. In like manner it may be shewn, that if $A \propto B$, $A \propto C$, and $A \propto D$, then $A \propto BCD$; also if $A \propto B$, and $A \propto \frac{1}{C}$, then $A \propto \frac{B}{C}$; and likewise if $A \propto B$, $A \propto C$, and $A \propto \frac{1}{D}$, then $A \propto \frac{BC}{D}$, the proof of all which is the same as in the former article.

104. B. If $A \propto BC$ and B be constant, then $A \propto C$; if C be constant, then $A \propto B$; if $A \propto \frac{B}{C}$ and C be constant, then $A \propto B$; if B be constant, then $A \propto \frac{1}{C}$.

For since the product BC varies by the increase or decrease of C only, when B is constant, and A varies as that product, therefore when B is invariable, A must evidently vary as C; and when B alone is variable, and C constant, A (varying as the product AB) must in like manner vary as B: after the same manner it may be shewn, that when $A \propto BCD$, *if BC be constant, then* $A \propto D$; *if D be constant, then* $A \propto BC$; *if C be constant, then* $A \propto BD$; *and if B be constant, then* $A \propto CD$; *and in general, if A be as any product or quotient, and if any of the factors be given, A will be as the product or quotient (as the case may be) of all the rest.*

105. If the first quantity vary as the second, the second as the third, the third as the fourth, and so on, then will the first vary as the last.

Let A, B, C, and D, be any number of variable quantities,

and a, b, c and d, corresponding values of them; and let $A \propto B$, $B \propto C$, and $C \propto D$; then will $A \propto D$.

Because $A : a :: B : b$.

And $B : b :: C : c$.

And $C : c :: D : d$, therefore ex æquo (Art. 81.) $A : a :: D : d$, that is, $A \propto D$; and the same may be shewn to be true of any number of variable quantities.

106. If the first be as the second, and the second inversely as the third, then is the first inversely as the third.

Let $A \propto B$, and $B \propto \frac{1}{C}$, then is $A \propto \frac{1}{C}$.

For since $A : a :: B : b$,

And $B : b :: \frac{1}{C} : \frac{1}{c}$, therefore ex æquo $A : a :: \frac{1}{C} : \frac{1}{c}$, that is, $A \propto \frac{1}{C}$.

107. If each of two quantities vary as a third, then will both their sum and difference, and also the square root of their product, vary as the third.

Let $A \propto C$, and $B \propto C$, then will $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

$$\left.\begin{array}{r} \textit{Because } A : a :: C : c, \\ \textit{And } C : c :: B : b, \end{array}\right\} \textit{by hypothesis,}$$

Therefore ex æquali $A : a :: B : b$, and alternando $A : B :: a : b$, wherefore componendo et dividendo $A \pm B : B :: a \pm b : b$, whence alternando $A \pm B : a \pm b :: B : b$; but $B : b :: C : c$, wherefore ex æquali $A \pm B : a \pm b :: C : c$, that is, $\overline{A \pm B} \propto C$, or the sum and the difference of A and B will each be as C.

Again, because $A : a :: C : c$,

And $B : b :: C : c$,

Therefore (Art. 78.) $AB : ab :: C^2 : c^2$,

Whence (Art. 80.) $\sqrt{AB} : \sqrt{ab} :: C : c$, that is, $\sqrt{AB} \propto C$.

108. If one quantity vary as another, it will likewise vary as any multiple or part of the other.

Let m be any constant quantity, and let $A \propto B$, then will $A \propto mB$, and $A \propto \frac{B}{m}$.

Because $A : a :: B : b$, by hypothesis, and

$B : b :: mB : mb$, Art. 73.

Therefore $A : a :: mB : mb$, that is, $A \propto mB$.

109. In the same manner, $A : a :: B : b$.

$$\text{And } B : b :: \frac{B}{m} : \frac{b}{m}.$$

$$\text{Therefore } A : a :: \frac{B}{m} : \frac{b}{m}.$$

$$\text{That is, } A \propto \frac{B}{m}.$$

Since $A \propto B$, A is equal to B multiplied or divided by some constant quantity; for $A : a :: mB : mb :: \frac{B}{m} : \frac{b}{m}$, whence alternando $A : mB :: a : mb ::$

And $A : \frac{B}{m} :: a : \frac{b}{m}$, if m be assumed, so that $A=mB$, or $A=\frac{B}{m}$, then will $a=mb$, or $a=\frac{b}{m}$ respectively.

110. If the corresponding values of A and B be known, then will the value of the constant quantity m be likewise known.

For if a and b be the known corresponding values of A and B, then since $A=mB$, or $A=\frac{B}{m}$, by substituting a and b for A and B, we shall have $a=mb$, or $a=\frac{b}{m}$; whence $m=\frac{a}{b}$, or $m=\frac{b}{a}$: wherefore also (since $A=mB$, or $A=\frac{B}{m}$) $A=\frac{a}{b}\times B$, or $=\frac{b}{a}\times B$.

111. If the product of two quantities be constant, then will the factors be inversely as each other.

Let AB be a constant quantity, then is $A \propto \frac{1}{B}$ and $B \propto \frac{1}{A}$, for AB being constant, it may be considered as 1; that is, $AB \propto 1$, whence $A \propto \frac{1}{B}$, and $B \propto \frac{1}{A}$.

112. Hence, in the constant product ABC, $A \propto \frac{1}{BC}$, $B \propto \frac{1}{AC}$, $C \propto \frac{1}{AB}$, $BC \propto \frac{1}{A}$, $AC \propto \frac{1}{B}$, and $AB \propto \frac{1}{C}$; and the like may be shewn when the product consists of any number of factors.

113. If the quotient of two quantities be constant, then are those quantities directly as each other.

Let $\frac{A}{B} \propto 1$, *then, (multiplying both sides by B,) will* $A \propto B$, *and* $B \propto A$, *and the like may be shewn when the quotient is composed of any number of quantities.*

114. If two quantities vary as each other, their like multiples and also their like parts will vary as each other respectively.

Let $A \propto B$, *and let* m *be any quantity constant or variable, then will* $mA \propto mB$, *and* $\frac{A}{m} \propto \frac{B}{m}$.

For since by hypothesis $A : a :: B : b$, *therefore* $mA : ma :: mB : mb$ *(Art. 73.) that is,* $mA \propto mB$.

Also $\frac{A}{m} : \frac{a}{m} :: \frac{B}{m} : \frac{b}{m}$, *therefore* $\frac{A}{m} \propto \frac{B}{m}$.

115. If two quantities vary as each other, their like powers and like roots will vary as each other respectively.

Let $A \propto B$, *then since* $A : a :: B : b$ *(Art. 95.)* $A^n : a^n :: B^n : b^n$, *and* $A^{\frac{m}{n}} : a^{\frac{m}{n}} :: B^{\frac{m}{n}} : b^{\frac{m}{n}}$, *(Art. 79.) that is,* $A^n \propto B^n$, *and* $A^{\frac{m}{n}} \propto B^{\frac{m}{n}}$.

116. If one quantity vary as two others jointly, then will each of the latter vary as the first *directly*, and as the other *inversely*.

Let $A \propto BC$, *then* $B \propto \frac{A}{C}$, *and* $C \propto \frac{A}{B}$.

For since $BC \propto A$, *divide both by* C, *and* $B \propto \frac{A}{C}$; *divide both by* B, *and* $C \propto \frac{A}{B}$.

117. If the first of four quantities vary as the second, and the third as the fourth, then will the product of the first and third vary as the product of the third and fourth.

Let $A \propto B$, *and* $C \propto D$, *then is* $AC \propto BD$.

For $A : a :: B : b$.

And $C : c :: D : d$.

Therefore (Art. 79.) $AC : ac :: BD : bd$; *or* $AC \propto BD$.

118. If four quantities be proportionals, and one or two of them be constant, to determine how the others vary.

Let $A : B :: C : D$, *then will* $AD = BC$, *and therefore* $AD \propto BC$. *Let* A *be constant, then* $D \propto BC$, *(Art. 104.) let* D

be constant, then $A \propto BC$: *let B be constant, then* $C \propto AD$; *let C be constant, then* $B \propto AD$. *Next, let A and B be both constant, then* $D \propto C$; *let A and C be constant, then* $D \propto B$; *let D and B be constant, then* $A \propto C$; *let D and C be constant, then* $A \propto B$; *let A and D be constant, then B and C will be both constant, or vary inversely as each other, that is,* $B \propto \frac{1}{C}$, *and* $C \propto \frac{1}{B}$; (*Art.* 111.) *in like manner, if B and C be constant, then A and D will both be constant, or vary inversely as each other, namely,* $A \propto \frac{1}{D}$, *and* $D \propto \frac{1}{A}$. *Lastly, if three of the quantities be constant, the fourth will evidently be constant.*

119. To shew the use and great convenience of the conclusions derived in the preceding articles, the following examples are subjoined.

EXAMPLES.—1. Let P=any principal or sum of money lent out at interest, R=the ratio of the rate per cent. T=the time it has been lent at interest, and I=the interest; to determine the relative value of each.

First, supposing all the quantities variable,

Then $I \propto PRT$ (*Art.* 22.) *whence* $P = \frac{I}{RT}$, $R \propto \frac{I}{PT}$, *and* $T \propto \frac{I}{PR}$, (*Art.* 114.) *Let I be given, then* $P \propto \frac{1}{RT}$, $R \propto \frac{1}{PT}$, *and* $T \propto \frac{1}{PR}$, (*Art.* 104.) *let P be given, then* $I \propto RT$, $R \propto \frac{I}{T}$, *and* $T \propto \frac{I}{R}$, (*Art.* 111.) *let R be given, then* $I \propto PT$, $P \propto \frac{I}{T}$, *and* $T \propto \frac{I}{P}$, (*Art.* 111.) *let T be given, then* $I \propto PR$, $P \propto \frac{I}{R}$, *and* $R \propto \frac{I}{P}$; *let I and P be given, then* $R \propto \frac{1}{T}$, *and* $T \propto \frac{1}{R}$; *let I and R be given, then* $P \propto \frac{1}{T}$, *and* $T \propto \frac{1}{P}$; *let I and T be given, then* $P \propto \frac{1}{R}$, *and* $R \propto \frac{1}{P}$; *let P and R be given, then* $I \propto T$; *let P and T be given, then* $I \propto R$. *Lastly, let R and T be given, then* $I \propto P$; *and if any three of the quantities be given, the fourth will be given.*

2. Suppose the quantities of motion in two moving bodies to be in the ratio compounded of the quantities of matter, and the velocities, to determine the other circumstances.

First, let M *= the quantity of motion,* Q *= quantity of matter,* V *= velocity; then* $M \propto QV$ *by hypothesis, wherefore* $Q \propto \frac{M}{V}$; *and if* M *be given,* $Q \propto \frac{1}{V}$; *also* $V \propto \frac{M}{Q}$, *and* M *being given,* $V \propto \frac{1}{Q}$; *if* Q *be given, then* $M \propto V$; *and if* V *be given,* $M \propto Q$.

Secondly, suppose the quantity of matter Q *to be in the compound ratio of the magnitude* m, *and density* D, *or* $Q \propto mD$; *by substituting* mD *for* Q *in the above expressions where* Q *is found, we shall have* $M \propto mDV$, $mD \propto \frac{M}{V}$, $mD \propto \frac{1}{V}$, M *being given;* $V \propto \frac{M}{mD}$, *or* $V \propto \frac{1}{mD}$, M *being given; from these it is plain that a great variety of other expressions may be obtained, and still more, by considering one or more of the quantities invariable.*

Lastly, since the magnitudes of bodies are as the cubes of their homologous lines, (or d^3,*) that is,* $d^3 \propto m$; *if* d^3 *be substituted for* m, *by proceeding as before, we shall obtain at length all the possible relations of the above quantities: but the prosecution of this is left as an exercise for the learner.*

GEOMETRICAL PROGRESSION.

120. To investigate the rules and theorems of Geometrical Progression.

Let a *= the least term,* z *= the greatest term,* } *called also the extremes.*
n *= the number of terms,*
r *= the common ratio,*
s *= the sum of all the terms.*

[i] *Then will* $a + ar + ar^2 + ar^3$, *&c. to* ar^{n-1}, *be an increasing geometrical progression.*

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[i] A progression, consisting of three or four terms only, is usually called *geometrical proportion*, or simply *proportion*. One important property of a geometrical progression is this, namely, the product of the two extreme terms is equal to that of any two terms equally distant from the extremes: hence, in

And $z+\frac{z}{r}+\frac{z}{r^2}+\frac{z}{r^3}$, *&c. to* $\frac{z}{r^{n-1}}$ *will be a decreasing geometrical progression.*

From the former of these we have ar^{n-1}=*the last term of the series, but* z=*the last term by the notation, wherefore* $ar^{n-1}=z$; *from this equation we obtain* $a=\frac{z}{r^{n-1}}$ (THEOR. 1.) $z=ar^{n-1}$ (THEOR. 2.) $r=\overline{\frac{z}{a}}\Big|^{\frac{1}{n-1}}$ (THEOR. 3.) *and since* $1:r::a+ar+ar^2:ar+ar^2+ar^3$, (*Art.* 72.) *that is,* $1:r::s-z:s-a$, *therefore* $s-a=r.\overline{s-z}$, *whence* $r=\frac{s-a}{s-z}$ (THEOR. 4.) $a=s-r.\overline{s-z}$ (THEOR. 5.) $z=\frac{\overline{r-1}.s+a}{r}$ (THEOR. 6.) *and* $s=\frac{rz-a}{r-1}$ (THEOR. 7.) *but since* $z=ar^{n-1}$ *by th.* 2. *substitute this value for* z *in th.* 7, *and* $s=\frac{ar^n-a}{r-1}$ (THEOR. 8.) *whence* $a=\frac{\overline{r-1}.s}{r^n-1}$ (THEOR. 9.) *and since* $r=\overline{\frac{z}{a}}\Big|^{\frac{1}{n-1}}$ (*th.* 3.) *and* $s=\frac{rz-a}{r-1}$ (*th.* 7.) *if for* r *in the latter, its value* $\overline{\frac{z}{a}}\Big|^{\frac{1}{n-1}}$ *be substituted, we shall have* $s=\frac{z.\overline{\frac{z}{a}}\Big|^{\frac{1}{n-1}}-a}{\overline{\frac{z}{a}}\Big|^{\frac{1}{n-1}}-1}$ (THEOR. 10.) *and because* (*th.* 4.) $s-a=sr-zr$, *and* (*th.* 1.) $a=\frac{z}{r^{n-1}}$, *therefore* ($s-a=$) $s-\frac{z}{r^{n-1}}=sr-zr$, *or* $sr-s=$ ($zr-\frac{z}{r^{n-1}}=\frac{zr^n-z}{r^{n-1}}=$) $\frac{\overline{r^n-1}.z}{r^{n-1}}$, *whence* $s=\frac{\overline{r^n-1}.z}{\overline{r-1}.r^{n-1}}$ (THEOR. 11.) *consequently* $z=\frac{\overline{r-1}.r^{n-1}.s}{r^n-1}$ (THEOR. 12.)

The above theorems are all that can be deduced in a general manner, without the aid of logarithms in some cases, and of equations of several dimensions in others. The theorems wanting are four for finding n, two for r, one for a, and one for z: the four theorems for finding the value of n, may be expressed

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four proportionals, the product of the two extremes is equal to the product of the two means; and in three proportionals, the product of the extremes is equal to the square of the mean.

logarithmically; the remaining four cannot be given in a general manner, but their relation to the other quantities may be expressed in an equation, by means of which any particular value will be readily known.

121. We proceed then, first, to deduce the equations from whence the remaining values of r, a, and z, may be found in any particular case; next, we shew how the theorems found are to be turned into their equivalent logarithmic expressions; and lastly, we shall deduce logarithmic theorems for the four expressions of the value of n.

First, because $z=ar^{n-1}$ *(th. 2.) and* $z=\frac{sr-s+a}{r}$ *(th. 6.) therefore* $ar^{n-1}=\frac{sr-s+a}{r}$, *whence* $ar^n=sr-s+a$, *or* $ar^n-sr=a-s$, *or* $r^n-\frac{rs}{a}=\frac{a-s}{a}$ (THEOR. 13.) *which is as near as we can get to the value of r, and which (supposing a, s, and n, given in numbers) if n be greater than 2, will require the solution of a high equation to find its value.*

Secondly, because $s-a=sr-zr$, *(th. 4.) and (th. 1.)* $a=\frac{z}{r^{n-1}}$, *therefore* $(s-a=)\ s-\frac{z}{r^{n-1}}=sr-zr$, *and* $zr^n-z=sr^n-sr^{n-1}$, *or* $\overline{z-s}.r^n-sr^{n-1}=-z$; *whence* $r^n-\frac{s}{z-s}r^{n-1}=-\frac{z}{z-s}$, (THEOR. 14.) *this equation being solved, the value of r will be known.*

Thirdly, since $s-a=sr-zr$, *(th. 4.) and* $r=\left.\frac{z}{a}\right|^{\frac{1}{n-1}}$, *(th. 3.) therefore* $s-a=s.\left.\frac{z}{a}\right|^{\frac{1}{n-1}}-z.\left.\frac{z}{a}\right|^{\frac{1}{n-1}}$, *or* $a.\overline{s-a}|^{n-1}=z.\overline{s-z}|^{n-1}$, (THEOR. 15.) *by the solution of which equation (s, n, and z, being given) a will be found.*

Fourthly, by the same equation, viz. $a.\overline{s-a}|^{n-1}=z.\overline{s-z}|^{n-1}$, (THEOR. 16.) *s, n, and a, being given, z will likewise be known.*

122. It remains now to put the above theorems into a logarithmical form, to place the whole in one point of view, and to deduce the four theorems for finding the value of n: observing that to multiply two factors together, we add their logarithms together; to divide, we subtract the logarithm of the divisor from that of the dividend; to involve or evolve, we multiply

or divide respectively the logarithm of the root or power by its index, as directed in Vol. I. Part 2.

$$\text{Let } \left.\begin{matrix} A \\ Z \\ R \\ S \end{matrix}\right\} \textit{represent the logarithm of} \left\{\begin{matrix} a \\ z \\ r \\ s \end{matrix}\right.$$

And L the logarithm of the quantity to which it is prefixed; then will the following synopsis exhibit the whole doctrine of geometrical progression, as investigated in the preceding articles [k].

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[k] Some of the following logarithmic expressions are extremely inconvenient, particularly theor. 10. The best method of computing the value of s in that theorem, will be, first to find the log. of z, subtract the log. of a from it, add this remainder to the log. of z, and divide the sum by $n-1$; find the natural number corresponding to the quotient, from which subtract a, and find the log. of the remainder. Secondly, from the log. of z, subtract the log. of a, divide the remainder by $n-1$, find the natural number corresponding to the quotient, subtract 1 from it, and subtract the log. of this remainder from that of the former; and the like in other cases.

Theor.	Given.	Req.	Solution by Numbers.	Solution by Logarithms.
II.	a, r, n	z	$z=ar^{n-1}$	$Z=A+R.\overline{n-1}$
VIII.		s	$s=\frac{ar^n-a}{r-1}$	$S=A+L.\overline{B-1}-L.\overline{n-1}$.where B
VII.	a, r, z	s	$s=\frac{rz-a}{r-1}$	$S=L.\overline{rz-a}-L.\overline{r-1}$
XVII.		n	....	$n=\frac{Z-A}{R}+1$
VI.	a, s, r	z	$z=\frac{\overline{r-1}.s+a}{r}$	$Z=L.\overline{\overline{r-1}.s+a}-R$
XIX.		n	....	$n=\frac{L.\overline{\overline{r-1}.s+a}-A}{R}$
IV.	a, z, s	r	$r=\frac{s-a}{s-z}$	$R=L.\overline{s-a}-L.\overline{s-z}$
XVIII.		n	....	$n=\frac{Z-A}{L.\overline{s-a}-L.\overline{s-z}}+1$
XIII.	a, n, s	r	$r^n-\frac{rs}{a}=\frac{a-s}{a}$	....
XVI.		z	$z.\overline{s-z}\rvert^{n-1}=a.\overline{s-a}\rvert^{n-1}$	....
III.	a, n, z	r	$r=\frac{z}{a}\Big\rvert^{\frac{1}{n-1}}$	$R=\frac{Z-A}{n-1}$
X.		s	$s=\frac{z.\frac{z}{a}\Big\rvert^{\frac{1}{n-1}}-a}{\frac{z}{a}\Big\rvert^{\frac{1}{n-1}}-1}$	$S=L.\overline{z.\frac{z}{a}\Big\rvert^{\frac{1}{n-1}}-a}-L.\overline{\frac{z}{a}\Big\rvert^{\frac{1}{n-1}}-1}$
I.	r, n, z	a	$a=\frac{z}{r^{n-1}}$	$A=Z-R.\overline{n-1}$
XI.		s	$s=\frac{\overline{r^n-1}.z}{\overline{r-1}.r^{n-1}}$	$S=L.\overline{r^n-1}+Z-L.\overline{\overline{r-1}+R.\overline{n-1}}$
IX.	r, n, s	a	$a=\frac{\overline{r-1}.s}{r^n-1}$	$A=L.\overline{r-1}+S-R.\overline{r^n-1}$
XII.		z	$z=\frac{\overline{r-1}.r^{n-1}.s}{r^n-1}$	$Z=L.\overline{r-1}+R.\overline{n-1}+S-L.\overline{r^n-}$
V.	r, z, s	a	$a=s-r.\overline{s-z}$	$A=L.\overline{s-r.\overline{s-z}}$
XX.		n	....	$n=\frac{Z-L.\overline{s-r.\overline{s-z}}}{R}+1$
XV.	n, z, s	a	$a.\overline{s-a}\rvert^{n-1}=z.\overline{s-z}\rvert^{n-1}$	....
XIV.		r	$r^n+\frac{s}{s-z}r^{n-1}=-\frac{s}{s-z}$	....

123. To shew how the 17th, 18th, 19th, and 20th theorems are derived.

Since $Z=A+R.\overline{n-1}$ (*th.* 2.) *therefore* $n-1=\frac{Z-A}{R}$, *and* $n=\frac{Z-A}{R}+1$ (THEOR. 17.) *and because* $R=L.\overline{s-a}-L.\overline{s-z}$ (*th.* 4.) *substitute this value for* R *in theor.* 17. *and* $n=\frac{Z-A}{L.\overline{s-a}-L.\overline{s-z}}+1$ (THEOR. 18.) *again, for* Z *in theor.* 17. *substitute its value from theor.* 6. *and* $n=(\frac{L.\overline{\overline{r-1}.s+a}-R-A}{R}+1=)\ \frac{L.\overline{\overline{r-1}.s+a}-A}{R}$ (THEOR. 19.) *Lastly, for* A *in theor.* 17. *substitute its value from theor.* 5. *and* $n=(\frac{Z-A}{R}+1=)$ $\frac{Z-L.\overline{s-\overline{r.s-z}}}{R}+1$. (THEOR. 20.)

EXAMPLES.—1. Given the ratio 2, the number of terms 6, and the last term 96, of a geometrical progression, to find the first term, and the sum of the terms?

Here $r=2$, $n=6$, $z=96$, *whence* (*theor.* 1.) $a=\frac{z}{r^{n-1}}=(\frac{96}{2^5}=)\frac{96}{32}=3$; *and* (*theor.* 11.) $s=\frac{\overline{r^n-1}.z}{\overline{r-1}.r^{n-1}}=(\frac{\overline{2^6-1}\times 96}{1\times 2^5}=\frac{63\times 96}{32}=)$ 189.

By Logarithms.

$Z=$ 1.9822712
$R.\overline{n-1}=0.3010300\times 5=1.5051500$
$z\div r^{n-1}=a=3$ 0.4771212
whence $a=3$.

$L.\overline{r^n-1}=L.\overline{2^6-1}=L.63=1.7993405$
$+Z=L.96=$ 1.9822712
$L.\overline{r^n-1}+Z=$ 3.7816117
$-\begin{cases} L.\overline{r-1}=L.1= \text{ .... } 0.0000000 \\ +R.\overline{n-1}=L.2\times 5= \text{ .... } 1.5051500 \end{cases}$
$s=$ 2.2764617
whence $s=189$.

2. Given the ratio 2, the number of terms 6, and the sum of the terms 189, to find the first and last terms?

Here $r=2$, $n=6$, $s=189$, *and* (*theor.* 9.) $a=\frac{\overline{r-1}.s}{r^n-1}=\left(\frac{1\times189}{2^6-1}=\right)\frac{189}{63}=3$; *also* $z=\frac{\overline{r-1}.r^{n-1}.s}{r^n-1}$ (*theor.* 12.) $=\left(\frac{1\times2^5\times189}{2^6-1}=\right)\frac{32\times189}{63}=96$.

By Logarithms.

$L.\overline{r-1}=$ 0.0000000	$L.\overline{r-1}+R.\overline{n-1}=$ ... 1.5051500
$+S=$ 2.2764617	$+S=$ 2.2764617
$-L.\overline{r^n-1}=$ 1.7993405	$L.\overline{r-1}+R.\overline{n-1}+S=$ 3.7816117
$A=$ 0.4771212	$-L.\overline{r^n-1}=$ 1.7993405
whence $a=3$.	$Z=$ 1.9822712

whence $z=96$.

3. Given the first term 3, the ratio 2, and the last term 96, to find the number, and sum of the terms?

Here $a=3$, $r=2$, $z=96$, *and* (*theor.* 7.) $\frac{rz-a}{r-1}=\left(\frac{2\times96-3}{1}=\right)$ $189=s$.

By Logarithms.

$Z=$ 1.9822712	$L.\overline{rz-a}=L.189=$ 2.2764617
$-A=$......... 0.4771212	$-L.\overline{r-1}=L.1=$ 0.0000000
$+R=$0.3010300)1.5051500(5	$S=$ 2.2764617
therefore $n=5+1=6$, *theor* 17.	*whence* $s=189$.

4. Given the first term 4, the ratio 3, and the sum of the terms 484, to find the last term, and number of terms?

Here $a=4$, $r=3$, $s=484$, *and* (*theor.* 6.) $z=\frac{\overline{r-1}.s+a}{r}=\left(\frac{\overline{3-1}\times484+4}{3}=\right)\frac{972}{3}=324$.

By Logarithms.

$L.\overline{\overline{r-1}.s+a}=L.972=$2.9876663
$-R=$ 0.4771212
$Z=$ ... 2.5105451
whence $z=324$.

$L.\overline{\overline{r-1}.s+a}=L.972=$2.9876663
$-A=$ 0.6020600
$+R=$....0.4771212)2.3856063(5
whence $n=5$, *theor.* 19.

5. Given the first term 2, last term 2048, and sum of the terms 2730, to find the ratio, and number of terms?

Here $a=2$, $z=2048$, $s=2730$, *and* (*theor.* 4.) $r=\frac{s-a}{s-z}=$ $\left(\frac{2730-2}{2730-2048}=\right)\frac{2728}{682}=4.$

By Logarithms.

$L.\overline{s-a}=L.2728=3.4358444$	$Z=3.3113300$
$-L.\overline{s-z}=L.682=2.8337844$	$-A=3.3010300$
$R=\ldots.0.6020600$	$Z-A=3.0103000$
whence $r=4$.	

$$L.\overline{s-a}=L.2728=3.4358444$$
$$-L.\overline{s-z}=L.682=2.8337844$$
$$L.\overline{s-a}-L.\overline{s-z}=0.6020600$$

therefore .6020600)3.0103000(5

whence $n=5+1=6$, *theor.* 18.

6. Given $r=4$, $n=6$, and $s=2730$, to find a and z. *Ans.* $a=2$, $z=2048$.

7. Given $r=2$, $n=6$, and $z=96$, to find a and s. *Ans.* $a=3$, $s=189$.

8. Given the ratio 5, last term 12500, and sum of the terms 15624, to find the first term, and number of terms. *Ans.* $a=4$, $n=6$.

9. Given $a=4$, $n=6$, and $z=12500$, to find r and s. *Answer* $r=5$, $s=15624$.

10. Given $r=3$, $n=4$, and $z=81$, to find a and s.

11. Given $r=6$, $n=5$, and $s=1555$, to find a and z.

12. Given $a=3$, $r=10$, and $n=20$, to find s and z.

124. PROBLEMS IN GEOMETRICAL PROGRESSION.

1. Of three numbers in geometrical progression, the difference of the first and second is 4, and of the second and third 12; required the numbers?

Let x, y, *and* z, *be the numbers.*

Then $y-x=4$, *or* $x=y-4$; $z-y=12$, *or* $z=y+12$.

Wherefore since by the problem $x:y::y:z$, *by substituting the values of* x *and* z *in this analogy, we shall have* $y-4:y::y:y+12$; *wherefore, (by multiplying extremes and means,)* $\overline{y-4}\cdot\overline{y+12}=)\ y^2+8y-48=y^2$, *or* $8y=48$; *wherefore* $y=6$, $x=2$, $z=18$.

2. The product of three numbers in geometrical progression is 1000, and the sum of the first and last 25; required the numbers?

Let x, y, and z, be the numbers; then since $x:y::y:z$, we have $xz=y^2$, (Art. 120. Note,) and ($xyz=xz.y=$) $y^3=1000$, whence $y=10$; also $xz=(y^2=)$ 100, and by the problem $x+z=25$: from the square of this equation subtract four times the preceding, and $x^2-2xz+z^2=225$: extract the square root of this, and $x-z=15$; add this to, and subtract it from, the equation $x+z=25$, and $2x=40$, or $x=20$, also $2z=10$, or $z=5$; whence 5, 10, and 20, are the numbers.

3. To find any number of mean proportionals between two given numbers a and b.

Let $n-2=$ the number of mean proportionals, then will $n=$ the number of terms in the progression: also let $r=$ the ratio, then (theor. 3. Geom. Prog.) $r=\left.\frac{b}{a}\right|^{\frac{1}{n-1}}$; and by logarithms, $\overline{\log. b-\log. a \div n-1}=\log. r$; whence r being found, if the less extreme be continually multiplied, or the greater divided, by r, the results will be the mean proportionals required.

EXAMPLES.—1. To find two mean proportionals between 12 and 4116.

Here $a=12$, $b=4116$, $n=4$, and $r=(\left.\frac{4116}{12}\right|^{\frac{1}{3}}=\overline{343}|^{\frac{1}{3}}=)\,7$; whence $12\times7=84$, the first mean, and $84\times7=588$, the second mean.

2. To find four mean proportionals between 2 and 486. *Ans.* 6, 18, 54, *and* 162.

3. To find five mean proportionals between 1 and 64.

4. There are four numbers in geometrical progression; the sum of the extremes is 9, and the sum of the cubes of the means 72; what are the numbers?

Let x, y, u, and z, be the numbers.

Then by the problem,

$x+z=9$, or $x=9-z$.

$x:y::u:z$, or $xz=uy$, whence $xz=(9-z.z=)\,9z-z^2$.

$x:y::y:u$, or $xu=y^2$ .... ($xuy=$) $x^2z=y^3$.

$y:u::u:z$, or $zy=u^2$ .... ($zyu=$) $xz^2=u^3$.

But ($xz.\overline{x+z}=$) $\overline{9z-z^2}.9=x^2z+xz^2$.

And $(y^3+u^3=)\ 72=x^2z+xz^2$, *and things that are equal to the same are equal; therefore* $\overline{9z-z^2}.9=72$, *or* $9z-z^2=8$, *or* $z^2-9z=-8$; *whence by completing the square, &c.* $z=8$, $x=(9-z=)\ 1$, $y=(\sqrt[3]{x^2z}=)\ 2$, $u=(\sqrt[3]{xz^2}=)\ 4$.

5. Of four numbers in geometrical progression, the product of the two least is 8, and of the two greatest 128; what are the numbers?

Let x, y, u, *and* z, *be the numbers.*

$$\left.\begin{array}{l} \textit{Then } xy=8, \textit{ or } x=\dfrac{8}{y} \\ uz=128, \textit{ or } z=\dfrac{128}{u} \\ xz=uy, \textit{ or } \dfrac{8}{y}.\dfrac{128}{u}=uy \end{array}\right\} \textit{by the problem.}$$

Therefore $(8\times128=)\ 1024=u^2y^2$, *or* $uy=32$, *and* $u=\dfrac{32}{y}$.

But $(x:y::y:u$, *that is,*$)\ \dfrac{8}{y}:y::y:\dfrac{32}{y}$, *where multiplying extremes and means*, $y^2=\dfrac{256}{y^2}$, *or* $y^4=256$; *whence* $y=4$, $x=(\dfrac{8}{y}=)2$, $u=(\dfrac{32}{y}=)\ 8$, $z=(\dfrac{128}{u}=)\ 16$, *the numbers required.*

6. The sum of 3 numbers in geometrical progression is 14, and the greater extreme exceeds the less by 6; what are the numbers? *Ans.* 2, 4, *and* 8.

125. *Def.* Compound Interest is that which is paid for the use, not only of the principal or sum lent, but for both principal and interest, as the latter becomes due at the end of the year, half-year, quarter, or other stated time.

To investigate the rules of Compound Interest.

Let p=*the principal*, r=*the rate per cent.* t=*the time*, $R=(1+r=)$ *the amount of* 1*l. for a year, called the ratio of the rate per cent.* a=*the amount.*

Then since 1 *pound : is to its amount for any given time and rate : : so are any number of pounds : to their amount for the same time and rate; therefore as*

$$1:R::\left\{\begin{array}{l} p\quad\ :pR\ =\textit{the first,} \\ pR\ \ :pR^2=\textit{second,} \\ pR^2:pR^3=\textit{third,} \\ pR^3:pR^4=\textit{fourth,} \\ pR^{t-1}:pR^t=t^{th}, \end{array}\right\} \textit{year's amount.}$$

Whence we have THEOREM 1. $pR^t=a$, THEOR. 2. $\frac{a}{R^t}=p$, THEOR. 3. $\sqrt[t]{\frac{a}{p}}=R$. THEOR. 4. $\frac{\log. a-\log. p}{\log. R}=t$; *the three latter of which follow immediately from the first; the fourth cannot be conveniently exhibited in numbers without the aid of logarithms.*

By means of these four theorems, all questions of compound interest may be solved.

EXAMPLES.—1. What is the amount of 1250*l.* 10*s.* 6*d.* for 5 years, at 4 per cent. per annum, compound interest?

Here $p=$(1250*l.* 10*s.* 6*d.*$=$) 1250.525, $t=5$, $R=1.04$.

Then theor. 1. ($pR^t=$) $1250.525\times\overline{1.04}|^5=1250.525\times1.2166$ $=1521.388715=$ 1521*l.* 7*s.* 9¼*d.*$=a$.

2. What principal will amount to 200*l.* in 3 years, at 4 per cent. per annum?

Here $a=200$, $R=1.04$, $t=3$, *and theor.* 2. $(\frac{a}{R^t}=)\ \frac{200}{\overline{1.04}|^3}=$ $\frac{200}{1.124864}=177.7992=$ 177*l.* 15*s.* 11¾*d.*$=p$.

3. At what rate per cent. per annum will 500*l.* amount to 578*l.* 16*s.* 3*d.* in 3 years?

Here $p=500$, $a=$(578*l.* 16*s.* 3*d.*$=$) 578.8125, $t=3$; *and, theor.* 3. $(\sqrt[t]{\frac{a}{p}}=)\ \sqrt[3]{\frac{578.8125}{500}}=(\frac{1}{5}\ \sqrt[3]{144.7031}$. *See Vol. I. P.* 3. *Art.* 63.$=$) $\frac{1}{5}\times5.25=1.05=R$: *wherefore, (since* $R-1=r$,*) we have* $R-1=.05=r$, *viz.* 5 *per cent. per annum.*

4. In how many years will 225*l.* require to remain at interest, at 5 per cent. per annum, to amount to 260*l.* 9*s.* 3¾*d.*?

Here $p=225$, $R=1.05$, $a=$(260*l.* 9*s.* 3¾*d.*$=$) 260.465625; *whence, theor.* 4. $(\frac{\log. a-\log. p}{\log. R}=\frac{\log. 260.465625-\log. 225}{\log. 1.05}=)$ $\frac{2.4157506-2.3521825}{0.0211893}=\frac{0.0635681}{0.0211893}=3$ *years*$=t$.

5. What sum will 500*l.* amount to in 3 years, at 5 per cent. per annum? *Ans.* 578*l.* 16*s.* 3*d.*

6. What principal will amount to 1521*l.* 7*s.* 9¼*d.* in 5 years, at 4 per cent. per annum? *Ans.* 1250*l.* 10*s.* 6*d.*

7. At what rate per cent. will 721*l.* amount to 1642*l.* 19*s.* $9\frac{1}{4}$*d.* in 21 years? *Ans.* 4 *per cent.*

8. In how many years will 721*l.* be at interest at 4 per cent. to amount to 1642*l.* 19*s.* $9\frac{1}{4}$*d.* *Ans.* 21 *years.*

If the interest be payable half-yearly, make t=the number of half-years, that is=twice the number of years, and r=half the rate per cent. but if the interest be payable quarterly, let t=the number of quarter-years, viz. 4 times the number of years, and r=one-fourth of the rate per cent. and let $R=r+1$ in both cases, as before[k].

126. *To determine some of the most useful properties of numbers.*

Def. 1. One number is said to be a multiple of another, when the former contains the latter some number of times exactly, without remainder.

Thus 12 *is a multiple of* 1, 2, 3, 4, *and* 6.

COR. Hence every whole number is either unity, or a multiple of unity.

2. One number is said to be an aliquot part of another, when the former is contained some number of times exactly in the latter.

Thus 1, 2, 3, 4, *and* 6, *are aliquot parts of* 12, *for* 1 *is* $\frac{1}{12}$, 2 *is* $\frac{1}{6}$, 3 *is* $\frac{1}{4}$, 4 *is* $\frac{1}{3}$, *and* 6 *is* $\frac{1}{2}$ *of* 12.

COR. Hence no number which is greater than half of another number, can be an aliquot part of the latter.

3. One number is said to *measure* another number, when it will divide the latter without remainder.

Thus each of the numbers 1, 2, 4, 5, 10, *and* 20, *measures* 20.

4. One number is said to *be measured* by another, when the latter will divide the former without remainder.

Thus 20 *is measured by* 1, 2, 4, 5, 10, *and* 20.

COR. Hence every aliquot part of a number measures that number, and every number is measured by each of its aliquot parts, and by itself.

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[k] It was at first intended to investigate and apply every rule in arithmetic, but want of room obliges us to omit Equation of Payments, Loss and Gain, Barter, Fellowship, and Exchange; these will be easily understood from the doctrine of proportion, of which we have amply treated.

5. Any number which measures two or more numbers, is called their *common measure*; and the greatest number that will measure them, is called their greatest common measure.

Thus 1, 2, 3, *and* 6, *are the common measures of* 12 *and* 18; *and* 6 *is their greatest common measure.*

Cor. Hence the greatest common measure of several numbers cannot be greater than the least of those numbers; and when the least number is not a common measure, the greatest common measure cannot be greater than half the least. *Def.* 2. *cor.*

6. An even number is that which can be divided into two equal whole numbers.

Thus 6 *is an even number, being divisible into two equal whole numbers,* 3 *and* 3, *&c.*

7. An odd number is that which cannot be divided into two equal whole numbers; or, which differs from an even number by unity. *Thus,* 1, 3, 5, 7, &c. *are odd numbers.*

Cor. Hence any even number may be represented by $2a$, and any odd number by $2a+1$, or $2a-1$.

8. A prime number is that which can be measured by itself and unity only[1].

Thus, 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, &c. *are prime numbers.*

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[1] Hence it appears, that no even number except 2 can be a prime, or that all primes except 2 are odd numbers; but it does not follow that all the odd numbers are primes: every power of an odd number is odd, consequently the powers of all odd numbers greater than 1, after the first power, will be composite numbers.

Several eminent mathematicians, of both ancient and modern times, have made fruitless attempts to discover some general expression for finding the prime numbers: if n be made to represent any of the numbers 1, 2, 3, 4, &c. then will all the values of $6n+1$ and $6n-1$ constitute a series, including all the primes above 3; but this series will have some of its terms composite numbers: thus, let $n=1$, then $6n+1=7$ and $6n-1=5$, both primes; if $n=2$, then $6n+1=13$, and $6n-1=11$, both primes; if $n=3$, then $6n+1=19$, and $6n-1=17$, both primes, &c. Let $n=6$, then $6n+1=37$ a prime, but $6n-1=35$ $(=5\times7)$ a composite number; also if $n=8$, then $6n+1=49$ a composite number, and $6n-1=47$ a prime, &c. For a table of all the prime numbers, and all the odd composite numbers, under 10,000, see *Dr. Hutton's Mathematical Dictionary*, 1795. Vol. II. p. 276, 278.

9. Numbers are said to be prime to each other, when unity is their greatest common measure [m].

Thus, 11 *and* 26 *are prime to each other, for no number greater than* 1 *will divide both without remainder.*

10. A composite number is that which is measured by any number greater than unity.

Thus, 6 *is a composite number, for* 2 *and* 3 *will each measure it.*

COR. Hence every composite number will be measured by two numbers: if one of these numbers be known, the other will be the quotient arising from the division of the composite number, by the known measure.

Thus, $6=3\times2$, *and* $\frac{6}{2}=3$, *also* $\frac{6}{3}=2$.

11. The component parts of any number, are the numbers (each greater than unity) which multiplied together, produce that number exactly.

Thus, 2 *and* 3 *are the component parts of* 6, *for* $2\times3=6$; 3, 4, *and* 5 *are the component parts of* 60, *for* $3\times4\times5=60$, &c.

12. A perfect number [n] is that which is equal to the sum of all its aliquot parts.

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[m] Numbers which are prime *to one another*, are not necessarily *primes* in the sense of def. 8. thus 4 and 15 are *composite* numbers according to def. 10. but they are prime *to each other*, since unity only will divide both. Hence two even numbers cannot be prime to each other.

In the *Scholar's Guide to Arithmetic, 7th Ed. p.* 104. 9. it is asserted, that "If a number cannot be divided by some number less than the square root thereof, that number is a prime." Now this cannot be true; for neither of the square numbers 9, 25, 49, &c. &c. can be measured by any number less than its square root, and yet these numbers are not primes: a slight alteration in the wording will however make it perfectly correct; thus, "If a number *which is not a square,* cannot be divided by some number less than the square root thereof, that number is a prime." This interpretation was undoubtedly intended by the learned author, although his words do not seem to warrant it.

[n] The following table is said to contain all the perfect numbers at present known.

6	8589869056
28	137438691328
496	2305843008139952128
8128	2417851639228158837784576
33550336	9903520314282971830448816128

These numbers were extracted from the Acts of the Petersburg Academy, in several of the volumes of which, Tracts on the subject may be found.

Thus, 6 is a perfect number, for its aliquot parts are $1\,(=\frac{1}{6}$ *of* $6)$ $2\,(=\frac{1}{3}$ *of* $6)$ *and* $3\,(=\frac{1}{2}$ *of* $6)$ *and* $1+2+3=6$.

13. An imperfect number is that which is greater or less than the sum of its aliquot parts; in the former case it is called *an abundant number*, in the latter, a *defective number*.

Thus, 8 and 12 are imperfect numbers; the former (viz. 8) is an abundant number, its aliquot parts being 1, 2 and 4, the sum of which $1+2+4=7$, *is less than the given number 8. The latter (viz. 12) is a defective number, its aliquot parts being 1, 2, 3, 4, and 6, the sum of which, viz. 16, is greater than the given number 12.*

14. A pronic number is that which is equal to the sum of a square number and its root

Thus, 6, 12, 20, 30, &c. are pronic numbers; for $6=(4+\sqrt{4}=)\ 4+2$; $12=(9+\sqrt{9}=)\ 9+3$; $20=(16+\sqrt{16}=)\ 16+4$; $30=(25+\sqrt{25}=)\ 25+5$, *&c.*

Property 1. The sum, difference, or product of any two whole numbers, is a whole number. *This evidently follows from the nature of whole numbers, for it is plain that fractions cannot enter in either case.*

COR. Hence the product of any two proper fractions is a fraction.

2. The sum of any number of even numbers is an even number.

Thus, let $2a$, $2b$, $2c$, *&c. be even numbers. (See def. 7. cor.)*

Then $2a+2b+2c+$, *&c.*$=$*their sum; but this sum is evidently divisible by 2, it is therefore an even number; def. 6.*

COR. Hence if an even number be multiplied by any number whatever, the product will be even.

3. The sum of any even number of odd numbers is an even number.

Thus, (def. 7. cor.) let $2a+1$, $2b+1$, $2c+1$, *and* $2d+1$, *be an even number of odd numbers.*

Then will their sum $2a+2b+2c+2d+1+1+1+1$, *be an even number; for the former part* $2a+2b+2c+2d$ *is even, by def. 6. and the latter consisting of an even number of units is likewise even; wherefore the sum of both will be even, by property 2.*

COR. Hence if an odd number be added to an even, the sum will be odd.

4. The sum of any odd number of odd numbers, is an odd number.

For let $2a+1$, $2b+1$, $2c+1$, *be an odd number of odd numbers, then* $2a+2b+2c+1+1+1=$ *their sum, the former part of which* $2a+2b+2c$, *being divisible by* 2, *(def. 6.) is an even number, and the latter part* $1+1+1$, *consisting of an odd number of units, is odd: now the sum of both, being that of an even number added to an odd, will, by the preceding corollary, be an odd number.*

5. The difference of two even numbers, will be an even number.

For let $2a$ *and* $2b$ *be two even numbers, then since* $2a-2b$ *and* $2b+2a$ *will each be divisible by* 2, *it is plain that the difference of* $2a$ *and* $2b$ *will be even, whichever of them be the greater.*

6. The difference of two odd numbers is even.

For let $2a+1$ *and* $2b+1$ *be two odd numbers, whereof the former is the greater; then since* $\overline{2a+1}-\overline{2b+1}=2a-2b$ *is the proposed difference, which is divisible by* 2, *it is therefore an even number.*

7. The difference of an even number and an odd one will be odd, whichever be the greater.

Let $2a$ *be an even number,* $2b+1$ *an odd number greater than* $2a$, *and* $2c+1$ *an odd number less than* $2a$; *wherefore* ($2b+1-2a=$) $2b-2a+1=$ *the difference, supposing the odd number to be the greater; and* ($2-\overline{2c+1}=$) $2a-2c-1=$ *the difference, supposing the even number the greater. Now each of these differences differs from the even numbers* $2b-2a$, *or* $2a-2c$ *by unity; the difference therefore in both cases is an odd number.*

8. The product of two odd numbers is an odd number.

For let $2a+1$ *and* $2b+1$ *be any two odd numbers, then will* ($\overline{2a+1}.\overline{2b+1}=$) $4ab+2b+2a+1=$ *their product; but the sum of the three first terms is evidently even, being divisible by* 2, *and the whole product exceeds this sum by unity, the product is therefore an odd number. (def. 7.)*

9. If an odd number measure an odd number, the quotient will be odd.

For let $a+1$ *be measured by* $b+1$, *and let the quotient be* q: *thus,* $\frac{a+1}{b+1}=q$; *then will* $\overline{b=1}.q=a+1$; *and since* $b=1$, *and*

$a+1$ are odd, it is plain that q must be odd, otherwise an odd number multiplied by an even number, would produce an odd number, which is impossible. (proper. 2. cor.)

10. If an odd number measure an even number, the quotient will be even[b].

For let $\frac{2a}{2b+1}=q$, *then* $\overline{2b+1}.q=2a$; *and since* $2b+1$ *is*

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[b] Mr. Bonnycastle, in treating on this subject, (Scholar's Guide, 5th Edit. p. 203.) has committed a trifling oversight. Prop. 10. in his book is as follows; "If an odd *or even* number measures an even one, the quotient will be even." The former position is here shewn to be true, but the latter is evidently false, namely, "if an even number measure an even number, the quotient is even." In proof of his assertion he says, "let $\frac{2a}{2b}=q$; then $2b.q=2a$; and since $2a$ and $2b$ are even numbers, q must likewise be an *even* number." This consequence however does not necessarily follow; q may be either *even* or *odd*, for any even number ($2b$) multiplying any *odd* number (q), will evidently produce an *even* number. (See proper. 3.) Hence the quotient of an even number by an even number, may be either *even* or *odd*; thus, $\frac{8}{2}=4$ *an even number*; *but* $\frac{6}{2}=3$ *an odd number*. Mr. Keith has fallen into the same error, or (which is more probable) has copied it from the above work. See his *Complete Practical Arithmetician*, 3d Edition, p. 283. Cor. to Art. 32.

The first named Author is likewise mistaken when he says, (Prop. 11.) "If an odd or an even number measures an even one, it will also measure the half of it." Now the half of any number will evidently measure the whole, and the half measures itself, that is, it is contained *once* in itself; wherefore it follows, according to the tenor of the reasoning there employed, that if one quantity be contained *once* in another, the former quantity measures the latter, but *the whole* is contained *once* in *the whole*, and therefore measures it: but whatever measures the whole measures its half, says Mr. B. wherefore the *whole* must necessarily measure the *half!* This mistake seems to have arisen from a circumstance which might easily have happened—that of confounding the idea of a *measure* with that of an *aliquot part*: had it been said that *every aliquot part of the whole measures the half*, the assertion would have been perfectly accurate. Should the freedom of the above remarks require an apology, I feel it necessary to testify my unreserved admiration of the eminent talents of the learned and respectable authors in question, and to assure them that nothing invidious can possibly be intended: but *truth* is the grand object of the sciences, and he who is engaged in the arduous and important office of instruction, forfeits all claim to fidelity and confidence, if he does not point out error wherever he may happen to find it; and he is scarcely less blameable who omits to do it with becoming candour, and under a sense of his own fallibility.

an odd number, and $2a$ an even one, it follows that q must be even; otherwise the product of two odd numbers would be even, which is impossible. (*proper.* 8.)

11. An even number cannot measure an odd number.

If possible, let $\frac{2a+1}{2b}=q$; *wherefore* $2a+1=2b.q$: *but since* $2b$ *is an even number,* $2b.q$ *is also even,* (*proper.* 2. *cor.*) *that is, an odd number* $(2a+1)$ *is equal to an even one,* $(2b.q,)$ *which is absurd: wherefore an even number, &c.*

12. If one number measure another, it will measure every multiple of the latter.

Let $n=$ *any whole number, and* $\frac{a}{b}=q$, *then will* $\frac{na}{b}=nq$. *But since* q *is by hypothesis a whole number,* nq *must be a whole number,* (*proper.* 1.) *that is,* b *measures* n *times* a.

13. That number which measures the whole, and also a part of another number, will likewise measure the remainder.

For let $\frac{a+b}{c}$ *and* $\frac{a}{c}$ *be each a whole number.*

Then will $(\frac{a+b}{c}-\frac{a}{c}=)\ \frac{b}{c}$ *be a whole number.* (*proper.* 1.)

14. If one number measure two other numbers, it will likewise measure their sum and difference.

Let c *measure both* a *and* b, *then will* $\frac{a}{c}$ *and* $\frac{b}{c}$ *be both whole numbers; wherefore* $(\frac{a}{c}+\frac{b}{c}=)\ \frac{a+b}{c}$, *and* $(\frac{a}{c}-\frac{b}{c}=)$ $\frac{a-b}{c}$, *will also be whole numbers.* (*proper.* 1.)

COR. Hence the common measure of two numbers will likewise be a common measure of the sum and difference of any multiple of the one, and the other.

Thus, if $\frac{a+b}{c}$, *and* $\frac{a-b}{c}$, *be whole numbers, then will* $\frac{na+b}{c}$ *and* $\frac{na-b}{c}$ *be whole numbers.*

15. If the greater of two numbers be divided by the less, and if the divisor be divided by the remainder, and the last divisor by the last remainder continually, until nothing remain,

the last divisor of all will be the greatest common measure of the two given numbers.

Let a and b be two numbers, and let a be contained in b, p times with c remainder; let c be contained in a, q times with d remainder; and let d be contained in c, r times exactly; then will d be the greatest common measure of a and b.

$$\begin{array}{l} a)\,b\,(p \\ \quad c)\,a\,(q \\ \qquad d)\,c\,(r \\ \qquad\quad 0 \end{array}$$

For since ($b=ap+c$, or) $b-ap=c$, and $a-qc=d$, it follows (from proper. 12.) that every quantity which measures a and b, will likewise measure ap, and also $b-ap$ or c, (proper. 13.) in like manner, whatever quantity measures a and c will also measure a and qc, and likewise ($a-qc$, or) d; wherefore any quantity which measures d, must likewise measure c and a and b, but d measures d, therefore it is a common measure of a and b. It likewise appears, that d is the greatest common measure of a and b; for since $rd=c$ and ($\overline{cq+d}=$) $rdq+d=a$, and ($ap+c=$) $rdqp+dp+rd=b$, that is, $\overline{rq+1}.d=a$, and $\overline{rqp+p+r}.d=b$, it follows that d is the greatest common measure of these two values of a and b, or that it is a multiple of all the common measures, except the greatest, of a and b.

Otherwise, since it appears that every common measure of a and b measures d, and d itself measures a and b, it follows that d is the greatest common measure of a and b[p].

16. The sum and the difference of two numbers will each measure the difference of the squares of those numbers.

For since $\overline{a+b}.\overline{a-b}=a^2-b^2$, it follows that $\dfrac{a^2-b^2}{a+b}=a-b$, and $\dfrac{a^2-b^2}{a-b}=a+b$.

17. The sum of any two numbers measures the sum of their cubes, and the difference of any two numbers measures the difference of their cubes.

For $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$; and $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$, as appears by actual division.

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[p] See Wood's Algebra, third Edition, p. 45. The above is a demonstration of the rule in page 148, of the first volume.

COR. Hence if the product of any two numbers be subtracted from the sum of their squares, the remainder measures the sum of their cubes; and if the said product be added to the sum of the squares, the sum measures the difference of their cubes.

18. If any power of one number, measure the same power of another, the former number measures the latter.

For let $\frac{a^n}{b^n}$ *be a whole number produced by* $\frac{a}{b}.\frac{a}{b}.\frac{a}{b}$, *&c. to n terms; then will* $\frac{a}{b}$ *be a whole number; for if not, let it if possible be a fraction, then this fraction being multiplied continually into itself, will at length produce* $(\frac{a^n}{b^n})$ *a whole number, which is absurd; wherefore* $\frac{a}{b}$ *is a whole number, or b measures a.*

COR. Hence if one number measure another, any root or power of the former will measure the like root or power of the latter respectively.

19. If the similar powers of two numbers be multiplied together, the product will be a power of the same kind with that of the factors.

For if a^n *be multiplied by* b^n, *the product* $a^n b^n$ *is likewise an* n^{th} *power, the root of which is ab.*

COR. Hence every power of a square number is a square, every power of a cube number a cube, and in general every power of an n^{th} power is an n^{th} power[q].

20. If any power of one number be divided by the same power of another number, the quotient will be a power of the same kind with that of the said numbers.

Let a^n *and* b^n *be the* n^{th} *powers of a and b; then is* $\frac{a^n}{b^n}$ *also an* n^{th} *power, for its root is* $\frac{a}{b}$.

COR. Hence the quotient of one square by another is a square; the quotient of one cube by another is a cube, &c.

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[q] And it is obvious that all the powers of a prime number (except the first power) will be composite.

21. **If two numbers differ by unity, their sum is equal to the difference of their squares.**

Let a and $a+1$ be any two numbers differing by unity; then will $2a+1$ be their sum, also $(\overline{a+1}|^2-a^2=a^2+2a+1-a^2=)$ $2a+1=$ the difference of their squares, which is the same as their sum.

COR. 1. Hence the differences of 0^2, 1^2, 2^2, 3^2, 4^2, &c.* $(=0, 1, 4, 9, 16,$ &c.) are the odd numbers 1, 3, 5, 7, &c.

COR. 2. Hence the squares of all whole numbers may be found from the series of odd numbers 1, 3, 5, 7, 9, &c. by addition only.

Thus, $1=1^2$; $1+3=(4=)\ 2^2$; $1+3+5=(9=)\ 3^2$; $1+3+5+7=(16=)\ 4^2$; $1+3+5+7+9=(25=)\ 5^2$; *and so on at pleasure.*

22. An odd number which is prime to another number, is likewise prime to double the latter.

For let a be an odd number, and b any other number; then since a, being odd, cannot be measured by any even number, (proper. 11.) it must be measured by an odd one: wherefore if a and $2b$ have a common measure, it must be an odd number; but $2b$ is evidently even, (def. 6.) and if an even number be measured by an odd one, the quotient will be even, (proper. 10.) and since this even quotient can be halved, it is plain that the forementioned odd number, which measures $2b$, will be contained half as many times in b as it is in $2b$, that is, it measures b, whence a and b have a common measure; but they are prime to each other, wherefore a and $2b$ have no common measure.

COR. Hence if an odd number be prime to any other number, it is prime to two, four, eight, sixteen, &c. times the latter.

23. If each of two numbers be prime to a third number, their product is prime to it.

Let a and b be each prime to c, then will ab be prime to c.

Then, since neither a and c, nor b and c, have any common

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* In the Scholar's Guide, p. 204. prop. 19. cor. the 0^2 is by mistake omitted; but without it, the conclusion does not follow.

measure, it is plain that ab and c can have no common measure; wherefore ab is prime to c.

34. If one number be prime to another, every power of the former will be prime to the latter.

Let a be prime to b, then will a^n be prime to b. For since a and b have no common measure, $a.a.a.a$, &c. and b cannot have a common measure; wherefore ($a.a.a.a$, &c.$=$) a^n is prime to b.

35. The sum of two numbers which are prime to each other, is prime to each of the numbers.

Let a be prime to b, then will $a+b$ be prime to a and b. For if not, let c be their common measure; wherefore, since c measures both $a+b$ and a, that is, $\frac{a+b}{c}$ and $\frac{a}{c}$ are whole numbers, by subtracting the latter from the former, the remainder $\frac{b}{c}$ is a whole number. (proper. 1.) In like manner, because $\frac{a+b}{c}$ and $\frac{b}{c}$ are whole numbers, by subtracting the latter from the former, $\frac{a}{c}$ will be also a whole number; wherefore $\frac{a}{c}$ and $\frac{b}{c}$ are both whole numbers, that is, the numbers a and b, which by hypothesis are prime to each other, have a common measure c, which is absurd.

COR. Hence if a part of any number be prime to the whole, the remaining part is prime to the whole.

36. In a series of continued geometrical proportionals beginning at unity, all the odd terms will be squares; the first, fourth, seventh, tenth, &c. terms will be cubes; and the seventh term will be both a square and a cube.

Thus, let $1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9$, &c. be an increasing geometrical series, beginning at 1. Then will $1, r^2, r^4, r^6, r^8$, &c. (that is, all the odd terms) be squares; $1, r^3, r^6, r^9$, (or the 1st, 4th, 7th, and 10th,) will be cubes; also r^6, (or the 7th term,) is both a square and a cube: and the like may be shewn in a decreasing series.

37. Every square number must end in either 1, 4, 5, 6, 9, or 0.

The truth of this will appear by squaring the first ten numbers 1, 2, 3, &c. to 10.

COR. Hence no square can end in 2, 3, 7, or 8.

28. A cube number may end in either of the ten digits.

This will likewise appear by cubing those numbers.

COR. Hence 2, 3, 5, 6, 7, 8, 10, &c. can have no exact square root, nor can 2, 3, 4, 5, 6, 7, 9, 10, &c. have an exact cube root.

29. All the powers of numbers ending in 0, 1, 5, and 6, will end in the same figures respectively; and all powers ending in the above figures, will have their roots ending in the same figures respectively.

Thus $\overline{10}|^{2}=100$, $\overline{10}|^{3}=1000$, $\overline{10}|^{4}=10000$, &c. *ending in* 0.

$\overline{11}|^{2}=121$, $\overline{11}|^{3}=1331$, $\overline{11}|^{4}=14631$, &c. *ending in* 1.

$\overline{5}|^{2}=25$, $\overline{5}|^{3}=125$, $\overline{5}|^{4}=625$, &c. *ending in* 5.

$\overline{6}|^{2}=36$, $\overline{6}|^{3}=216$, $\overline{6}|^{4}=1296$, &c. *ending in* 6.

and the like for the roots of powers ending as above, as is plain.

30. All numbers ending in 4 or 9, will have their even powers end in 6 and 1 respectively; and their odd powers the same as their roots, viz. 4 and 9, respectively.

Thus $\overline{4}|^{2}=16$, $\overline{4}|^{3}=64$, $\overline{4}|^{4}=256$, &c.

$\overline{9}|^{2}=81$, $\overline{9}|^{3}=729$, $\overline{9}|^{4}=6561$, &c.

31. The powers of numbers ending in 2 will end in 4, 8, 6, and 2, alternately; numbers ending in 3 will have their powers ending in 9, 7, 1, and 3, alternately; numbers ending in 7 will have their powers ending in 9, 3, 1, and 7, alternately; and numbers ending in 8 will have their powers ending in 4, 2, 6, and 8, alternately.

This will appear by involving such numbers.

COR. Hence numbers ending in 1 and 9 will have their even powers end in the same figure, viz. 1; numbers ending in 3 and 7 will end their like even powers with the same figure, viz. their squares with 9, their 4th powers with 1, &c.; numbers ending in 2 and 8 will end their even powers alike, viz. their squares with 4, their 4th powers with 6; numbers ending in 4 and 6 will have their even powers end alike, viz. with 6; and in general, the like even powers of any two numbers equally distant from 5, will end in the same figure.

32. The right hand places of any number being ciphers, if the right hand significant figure be odd, the number will be divisible by unity, with as many ciphers subjoined as there are ciphers on the right of the said number; if the right hand signi-

ficant figure be even, it will be divisible by 2, with as many ciphers subjoined.

Thus 1230 is divisible by 10, 3100 by 100, 7000 by 1000, &c.

Also 1240 is divisible by 20, 3200 by 200, 8000 by 2000, &c. and the like is true in all similar cases.

33. Every number ending in 5, is divisible by 5 without remainder.

This is plain, since all such numbers are either 5, or multiples of 5.

COR. Hence, numbers ending in 0 or 5 are divisible by 5.

34. If the two right hand figures of any number be measured by 4, the whole is measured by 4; and if the three right hand figures be measured by 8, the whole is measured by 8.

Thus the two right hand figures of each of the numbers 184, 2148, 37128, 13716, 71104, &c. being divisible by 4, each of these numbers is measured by 4.

Also the three right hand figures of each of the numbers 13328, 27464, 9216, 100800, 2040, &c. being measured by 8, each of the numbers is measured by 8; and the same is true in all similar cases.

35. In any even number, if the sum of its figures be measured by 6, the number itself is measured by 6.

Thus the sum of the figures in the even number 738 is 18; which being measured by 6, the number 738 itself is likewise measured by 6; and the like of all other similar numbers.

36. If the sum of the figures in the first, third, fifth, &c. places in any number, be equal to the sum of those in the second, fourth, sixth, &c. places, the number itself is divisible by 11.

Thus the number 4752 is divisible by 11, because $4+5$ *(the sum of the first and third)* $=7+2$, *(the sum of the second and fourth;) in like manner 1234563 is divisible by 11, for* $1+3+5+3=2+4+6$; *and the same is true of all similar numbers.*

37. Any part of the sum or difference of numbers is found by dividing each of the given numbers separately by the number denoting that part; and any part of their product is found by dividing *one* only of the numbers by the number denoting the part *.

* The properties 32 to 37 inclusive, with some others, are introduced in a

Thus half the sum of $6a+4b-8c$ *is* $3a+2b-4c$.

And half the product of $6a\times4b\times8c$ *is* $3a\times4b\times8c$, *or* $6a\times2b\times8c$, *or* $6a\times4b\times4c$; *each being*$=96\,abc$.

38. Every even square number is measured by 4, and every odd square divided by 4 leaves 1 remainder.

For since the root of an even square must be even, (proper. 8.) *let* $2n$ *be its root; then* $\overline{2n}|^2=4n^2$ *the square, which is evidently divisible by* 4.

Again, since the root of an odd square must be odd, (proper. 11.) *let* $2n+1$ *be such root, then* $\overline{2n+1}|^2=4n^2+4n+1$ *the square; which being divided by* 4, *will evidently leave* 1 *remaining.*

39. If any number, and also the sum of its figures, be each divided by 9, the remainders will be equal.

Let n *be any number composed of the digits* a, b, c, *and* d; *then, according to the established principles of notation,* $1000a+100b+10c+d=n$; *but* $1000a=(\overline{999+1}.a=)\ 999a+a$; $100b=(\overline{99+1}.b=)\ 99b+b$; $10c=(\overline{9+1}.c=)\ 9c+c$: *therefore* $n=(1000a+100b+10c+d=)\ 999a+99b+9c+a+b+c+d$; *consequently* $\frac{n}{9}=111a+11b+c+\frac{a+b+c+d}{9}$, *or the number* n *being divided by* 9 *leaves* $\frac{a+b+c+d}{9}$ *remainder, which is the same as the remainder of the sum of its digits divided by* 9; *as was to be shewn.*

COR. Hence the operations of addition, either of whole numbers or decimals, may be proved by casting out the *nines*; for it is plain that if the excess of nines in two or more numbers be taken, and likewise the excess of nines in these excesses, the last excess will equal the excess of nines in the sum of the given numbers; since the sum of the excesses of the parts (taken separately) is evidently equal to the excess of the whole [t].

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note on p. 155, 156. Vol. I. as useful for readily finding the measures of numbers, and for reducing fractions to their lowest terms.

[t] To shew the method of proving addition by casting out the nines, the following examples are subjoined.

Ex. 1.

$$\left.\begin{array}{r} 3572 \ldots. 8 \\ 6832 \ldots. 1 \\ 7654 \ldots. 4 \\ \underline{8323} \ldots. \underline{7} \\ \underline{26381} \ldots. \underline{2} \end{array}\right\} \textit{Excesses of nines.}$$

Ex. 2.

$$\left.\begin{array}{r} 63.495 \ldots. 0 \\ 35.327 \ldots. 2 \\ 4.7121 \ldots. 6 \\ \underline{123.975} \ldots. \underline{0} \\ \underline{227.5091} \ldots. \underline{8} \end{array}\right\} \textit{Excesses of nines.}$$

40. If each of two numbers be divided by 9, and the product of the remainders also divided by 9, this remainder shall equal the remainder arising from the product of the two given numbers divided by 9.

For let $9A+a$ *and* $9B+b$ *be the two numbers, which being divided by 9, will evidently leave* a *and* b *for remainders; and* $\frac{ab}{9}=$ *the product of these remainders divided by 9.*

Also $\frac{\overline{9A+a}\times\overline{9B+b}}{9}=\left(\frac{81AB+9aB+9Ab+ab}{9}=\right)9AB+aB+Ab+\frac{ab}{9}$; *wherefore* $\frac{ab}{9}$ *is the remainder of the product of the two given numbers divided by 9, and it equals the product of the remainders of the two given numbers divided by 9, as found above; which was to be shewn* [*].

In Ex. 1. the nines being cast out of the top line, the 8 placed opposite remains in excess; in like manner 1, 4, and 7, are respectively the excesses of the second, third, and fourth, lines: now these four excesses being added together, and the nines cast out of the sum, the excess will be 2, and if the nines be cast out of the sum of the numbers proposed, (26381,) the excess is likewise 2, which two excesses agreeing, the work is presumed to be right for the reasons given in property 39. and its corollary. But there are two cases in which this mode of proof does not succeed; the first is when a mistake of 9, or any multiple of 9, has been made in the adding; and the second is when all, or any of the figures have been transposed: in each of these cases, although the work is manifestly wrong, the proof will make it appear right. Subtraction may likewise be proved by the same method, but this will be considered rather as a matter of curiosity than use: in subtracting the excesses, if the lower one be the greater, 9 must be borrowed, as in Ex. 2. below.

	Ex. 1.			Ex. 2.		
From	237165	 6	*Excesses.*	37.45	 1	*Excesses.*
Take	123428	 2		3.1234	 4	
Rem.	113737	 4		34.3266	 6	

In Ex. 1. having cast the nines out of the two given numbers, the lower excess 2 is subtracted from the upper excess 6; then the difference 4 being equal to the excess of nines in (113737) the remainder, shews the work to be right, subject however to the exceptions stated above.

In Ex. 2. the 4 cannot be taken from 1, therefore 9 is borrowed; the rest as in the preceding example.

* The practical application of this property of the number 9, is fully exemplified in the proofs subjoined to the operations of multiplication and division of both whole numbers and decimals. See Vol. I. p. 34—38. 47—49. 215, 219.

41. Any arithmetical progression can be increased *in infinitum*, but not decreased; a harmonical progression can be decreased *in infinitum*, but not increased; but a geometrical progression can be both increased and decreased *in infinitum* [x].

First, let $a+\overline{a+r}+\overline{a+2r}+$, &c. be an arithmetical progression; this series can evidently be increased at pleasure by the constant addition of r: but if you take the series backwards, and decrease its terms successively by r, it will become $\overline{a+r}+a+\overline{a-r}+\overline{a-2r}+$, &c. now when *either* of the quantities r, $2r$, $3r$, becomes equal to a, that term is equal to 0, and the series evidently can proceed no further.

Secondly, let $\frac{1}{a}+\frac{1}{a+r}+\frac{1}{a+2r}+$, &c. be a harmonical series, in which the last term is the least; this can evidently be decreased at pleasure by the constant addition of r to the denominator. Now taking this series backwards, and continually subtracting r from the denominator, it becomes $\frac{1}{a+r}+\frac{1}{a}+\frac{1}{a-r}+\frac{1}{a-2r}+$, &c. but when r, $2r$, $3r$, or some multiple of r, becomes equal to a, it is plain the next term of the series will be negative, or the series terminates, without the possibility of further increase.

Thirdly, let $a+ar+ar^2+ar^3$, be a geometrical series; this series may be increased by constantly multiplying by r, or decreased by constantly dividing by r, as is evident, without the possibility of its terms becoming negative.

The number 3 possesses the same property, but 9 is usually preferred, as being the most convenient for practice: we may add, that the same inconvenience attends the proving of multiplication and division by this method, as that mentioned in the preceding note.

The rule for proving addition by casting out the nines was, according to Mr. Bonnycastle, first published by Dr. Wallis in 1657; but the property of the number 9, on which the rule is founded, was most probably known to the Arabians long before that time: Lucas de Burgo, who wrote in 1494, was well acquainted with this property, and shewed the method of proving the primary operations of arithmetic by it, as is witnessed by Dr. Hutton. *Math. Dict. Vol. I. p.* 66.

[x] This property of the three kinds of progressions was first noticed by Pappus, a Greek Mathematician of the Alexandrian School, who flourished in the latter part of the fourth century, in the third book of his Mathematical Collections.

42. If a harmonical mean and an arithmetical mean be taken between any two numbers, the four terms will be proportionals.

Let a and b be any two numbers, then will $\frac{a+b}{2}$ *be an arithmetical mean, and* $\frac{2ab}{a+b}$ *a harmonical mean between a and b; then will* $a : \frac{2ab}{a+b} :: \frac{a+b}{2} : b$, *for the product of the means* (ab) *is equal to the product of the extremes* (ab), *which is the criterion of proportionality.* (Art. 58.)

43. The square root of a rational quantity cannot be partly *rational,* and partly a *quadratic surd.*

For if possible, let $\sqrt{x}=a+\sqrt{b}$, *of which* $\sqrt{b}$ *is an irreducible surd; square both sides, and* $x=a^2+2a\sqrt{b}+b$, *or,* $2a\sqrt{b}=x-a^2-b$, $\therefore \sqrt{b}=\frac{x-a^2-b}{2a}$, *that is, an irreducible surd equal to a rational quantity, which is absurd; wherefore* $\sqrt{x}$ *cannot equal any quantity of the form of* $a+\sqrt{b}$.

44. If each side of an equation contain rational quantities, and irreducible surds, then will the rational parts be equal to the rational, and the surd parts to the surd.

Let $x+\sqrt{z}=a+\sqrt{b}$, *then will* $x=a$, *and* $\sqrt{z}=\sqrt{b}$.

For if x *be not* $=a$, *let* $x=a+m$, *then* $a+m+\sqrt{z}=a+\sqrt{b}$, *or* $m+\sqrt{z}=\sqrt{b}$, *that is,* $\sqrt{b}$ *is partly rational, and partly surd, which is proved to be impossible in proper.* 43.

45. From the foregoing property we derive an easy method for extracting the square root of a binomial surd, as follows.

EXAMPLE. To find the square root of $m+\sqrt{n}$.

First assume $\sqrt{x}+\sqrt{z}=\sqrt{m+\sqrt{n}}$, *then squaring both sides* $x+2\sqrt{xz}+z=m+\sqrt{n}$; *wherefore* (*proper.* 44.) $x+z=m$, *and* $2\sqrt{xz}=\sqrt{n}$; *these equations squared give* $x^2+2xz+z^2=m^2$, *and* $4xz=n$; *subtract the latter from the former, and* $x^2-2xz+z^2=m^2-n$, $\therefore$ *by evolution* $x-z=\sqrt{m^2-n}$; *but* $x+z=m$, $\therefore x=\frac{m+\sqrt{m^2-n}}{2}$, *and* $z=\frac{m-\sqrt{m^2-n}}{2}$, $\therefore \sqrt{m+\sqrt{n}}=(\sqrt{x}+\sqrt{z}=)$ $\sqrt{\frac{m+\sqrt{m^2-n}}{2}}+\sqrt{\frac{m-\sqrt{m^2-n}}{2}}$, *the root required.*

PART V.

ALGEBRA.

OF EQUATIONS OF SEVERAL DIMENSIONS.

A GENERAL view of the nature, formation, and roots of equations.

1. A simple equation is that which contains the unknown quantity in its first power only.

Thus $ax+b=c$.

2. A quadratic equation is that which contains the second power of the unknown quantity, and no power of it higher than the second.

Thus $ax^2-bx=c$.

3. A cubic equation is that which contains the third, and no higher power of the unknown quantity.

Thus $ax^3-bx^2+cx=d$, *or* $ax^3+bx^2=c$, *or* $ax^3-bx=c$.

4. A biquadratic equation is that which contains the fourth, and no higher power of the unknown quantity.

Thus $ax^4+bx^3-cx^2+dx-e=o$, *&c.*

5. In like manner, an equation of the fifth degree is that which contains the fifth, and no higher power of the unknown quantity; an eqüation of the sixth degree contains the sixth power; one of the seventh degree the seventh power of the unknown quantity, &c. &c.

6. All equations above simple, which contain only one power of the unknown quantity, are called *pure.*

Thus $ax^2=b$ *is a pure quadratic,* $ax^3=b$ *is a pure cubic,* $ax^4=b$ *a pure biquadratic, &c.*

7. All equations containing two or more *different* powers of the unknown quantity, are called *affected* or *adfected* equations.

Thus $ax^2-bx=c$ *is an adfected quadratic;* $ax^3-bx^2=c$, *and* $ax^3+bx=c$ *are adfected cubics;* $x^4-x^2+ax=b$, *and* $ax^4-bx^3=c$, *and* $ax^4-bx^3+cx^2-dx+e=o$, *are adfected biquadratics.*

8. An equation is said to be of as many dimensions, as there are units in the index of the highest power of the unknown quantity contained in it.

Thus a quadratic is said to be an equation of two dimensions; a cubic of three; a biquadratic of four, &c.

9. A complete equation is that which contains all the powers of the unknown quantity, from the highest (by which it is named) downwards.

Thus $ax^2-bx+c=o$, *is a complete quadratic;* $ax^3-bx^2+cx-d=o$, *is a complete cubic;* $x^4-x^3-x^2+x-a=o$, *a complete biquadratic, &c.*

10. A deficient equation is that in which some of the inferior powers of the unknown quantity are wanting.

As $ax^3-bx^2+c=o$, *a deficient cubic;* $ax^4-bx^2+cx-d=o$, *a deficient biquadratic, &c.*

11. An equation is said to be arranged according to its dimensions, when the term containing the highest power of the unknown quantity stands *first* (on the left); that which contains the next highest, *second;* that which contains the next highest, *third;* and so on.

Thus the equation $x^5-ax^4+bx^3-cx^2+dx-e=o$, *is arranged according to its dimensions.*

COR. Hence every complete equation of n dimensions will contain $n+1$ terms.

12. The last term of any equation being always a known quantity, is usually called *the absolute term:* and note, this last or absolute term may be either simple, or compound, consisting of several known quantities connected by the sign + or —; which together are considered as but one term.

13. The roots of an equation are the values of the unknown quantity (expressed in known terms) contained in that equation; hence, to find the roots is the same thing as to resolve the equation.

14. The roots of equations are either possible, or imaginary. Possible roots are such as can be accurately determined, or their values approximated to, by the known principles of Algebra.

Thus $\sqrt{a}$, $\sqrt[3]{a-b}$, $\sqrt[4]{c}$, *&c. are possible roots.*

15. Imaginary or impossible roots are such as come under the form of *an even root of a negative quantity,* which cannot be determined by any known method of analysis.

Thus $\sqrt{-a}$, $\sqrt[4]{-ab}$, $\sqrt[6]{-d}$, *&c. are impossible roots.*

16. The limits of the roots of an equation are two quantities, one of which is greater than the greatest root; and the other, less than the least. The greater of these quantities is called *the superior limit,* and the less, *the inferior limit.* Also the limits of each particular root, are quantities which fall between it and the preceding and following roots.

17. The depression of an equation is the reducing it to another equation, of fewer dimensions than the given one possesses.

18. The transformation of an equation is the changing it into another, differing in the form or magnitude of its roots from the given equation.

OF THE GENERATION OF EQUATIONS OF SEVERAL DIMENSIONS.

19. If several simple equations involving the same unknown quantity be multiplied continually together, the product will form an equation of as many dimensions as there are simple equations employed [y].

Thus, the product of two simple equations is a quadratic; the continued product of three simple equations is a cubic; that of four, a biquadratic; and so on to any number of dimensions.

For, let x be any variable unknown quantity, and let the given quantities a, b, c, d, &c. be its several values, so that $x=a$, $x=b$, $x=c$, $x=d$, &c. these by transposition become $x-a=o$, $x-b=o$, $x-c=o$, $x-d=o$, &c. if the continued product of these simple equations be taken, (viz. $\overline{x-a}.\overline{x-b}.\overline{x-c}.\overline{x-d}$. &c.) it will

[y] This method of generating superior equations by the continual multiplication of inferior ones, was the invention of Mr. Thomas Harriot, a celebrated English mathematician and philosopher, and was first published at London in the year 1631, being ten years after the author's decease, by his friend, Walter Warner, in a folio work, entitled, *Artis Analyticæ Praxis, ad Æquationes Algebraicas nova, expedita, et generali methodo, resolvendas.* By this excellent contrivance the relations of the roots and coefficients, and the whole mystery of equations, are completely developed, and their various relations and properties discovered at a single glance. See on this subject *Sir Isaac Newton's Arithmetica Universalis*, p. 256, 257. *Maclaurin's Algebra*, p. 139. &c. *Hutton's Mathematical Dictionary*, Vol. I. p. 90. *Simpson's Algebra*, p. 131. &c. *Dr. Wallis's Algebra; Professor Vilant's Elements of Mathematical Analysis*, p. 48. and various other writers.

constitute an equation (=o) of as many dimensions as there are factors, or simple equations, employed in its composition: for example.

$$
\begin{array}{rl}
\textit{Let} & x-a=o \\
\textit{Be multip. into} & x-b=o \\
\hline
\textit{The product is} & \left.\begin{array}{r} x^2-a \\ -b \end{array}\right\} x+ab=o, \textit{ a quadratic.} \\
\hline
\textit{Multiplied into} & x-c=o \\
\hline
\textit{The product is} & \left.\begin{array}{r} x^3-a \\ -b \\ -c \end{array}\right\} x^2 \left.\begin{array}{r} +ab \\ +ac \\ +bc \end{array}\right\} x-abc=o, \textit{ a cubic.} \\
\hline
\textit{Multiplied into} & x-d=o \\
\hline
\textit{The product is} & \left.\begin{array}{r} x^4-a \\ -b \\ -c \\ -d \end{array}\right\} x^3 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} x^2 \left.\begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} x+abcd=o, \textit{ a biquadratic.} \\
\hline
& \&c. \qquad\qquad \&c.
\end{array}
$$

From the inspection of these equations it appears, that

20. The product of two simple equations is a quadratic.

21. The continual product of three simple equations, or of one quadratic and one simple equation, is a cubic.

22. The continual product of four simple equations, or of two quadratics, or of one cubic and one simple equation, is a biquadratic; and so on for higher equations [a].

23. The coefficient of the first term or higher power in each equation is unity.

24. The coefficient of the second term in each, is the sum of the roots with their signs changed [a].

Thus, in the quadratic, whose roots are $+a$ and $+b$, the coefficient is $-a-b$; in the cubic, whose roots are $+a$, $+b$, and $+c$, if

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[a] It is in like manner evident, that the roots of the compounded equations will have not only the same roots with its component simple equations, but that its roots will have the same signs as those of the latter.

[a] Hence, if the sum of the affirmative roots be equal to the sum of the negative roots, the coefficient of the second term will be 0; that is, the second term will vanish: and conversely, if in an equation the second term be wanting, the sum of the affirmative roots and the sum of the negative roots are equal.

is $-a-b-c$*; in the biquadratic, whose roots are* $+a, +b, +c$, *and* $+d$, *it is* $-a-b-c-d$, *&c.*

25. The coefficient of the third term in each, is the sum of all the products that can possibly arise by combining the roots, with their proper signs, two and two.

Thus, in the cubic, the coefficient of the third term is $+ab+ac+bc$*; in the biquadratic, it is* $+ab+ac+ad+bc+bd+cd$, *&c.*

26. The coefficient of the fourth term in each, is the sum of all the products that can possibly arise by combining the roots, with their signs changed, three by three.

Thus, in the biquadratic, the coefficient of the fourth term is $-abc-abd-acd-bcd$.

In like manner, in higher equations, the coefficient of the fifth term will be the sum of all the products of the roots, having their proper signs, combined four by four; that of the sixth term, the roots, with their signs changed, five by five, &c.

27. The last, or absolute term, is always the continued product of all the roots, having their signs changed.

Thus, in the quadratic, whose roots are $+a$ *and* $+b$, *the last term is* $+ab$ $(\text{or } -a\times -b)$*; in the cubic, the absolute term is* $-abc$ $(=-a\times -b\times -c)$*; in the biquadratic, the absolute term is* $+abcd$ $(=-a\times -b\times -c\times -d)$, *&c.*

28. The first term is always positive, and some pure power of x.

28. B. The second term is some power of x multiplied into $-a, -b, -c$, &c. and since x is affirmative, and each of these quantities negative, it follows that the second term itself is negative, since $+\times-$ produces $-$.

29. The third term will be positive, for its coefficient being the sum of the products of every two of the negative quantities $-a, -b, -c$, &c. and (since $-\times-$ produces $+$) therefore these sums, multiplied by any power of x, (which is always positive,) will always give a positive result.

30. For like reasons the fourth term will be negative, the fifth positive, the sixth negative, and so on; that is, when the roots are all positive, the signs of the terms of the equation will be alternately positive and negative: and conversely, when the signs of the terms of the equation are alternately $+$ and $-$, all the roots will be positive.

COR. Hence, if the signs of the even terms be changed, the signs of all the roots of the equation will be changed.

31. Let now the roots of the equations, above referred to, be supposed negative; that is, $x=-a$, $x=-b$, $x=-c$, $x=-d$, &c. then by transposition, $x+a=o$, $x+b=o$, $x+c=o$, $x+d=o$, &c. the product of these, or $\overline{x+a}.\overline{x+b}.\overline{x+c}.\overline{x+d}$, &c. will be an equation, having all its terms affirmative; for since all the quantities composing the factors are +, it is plain that the products will all be +.

COR. Hence, when the signs of all the roots (in the above simple equations, having both terms on one side) are —, the signs of all the terms of the equation compounded of them will be +; and conversely, when the signs of all the terms of an equation are +, the signs of all its roots will be —.

32. If equations similar to the foregoing be generated, having some of the roots +, others —, it will appear, that there will be as many changes in the signs of the terms, (from + to —, or from — to +,) as the equation has positive roots; and as many continuations of the same sign, (+ and +, or — and —,) as the equation has negative roots: and conversely, the equation will have as many affirmative roots as it has changes of signs, and as many negative roots as it has continuations of the same sign [b].

COR. It follows from what has been said, that every equation has as many roots as its unknown quantity has dimensions. To be particular; a quadratic has two roots, which are either both affirmative, both negative, or one affirmative and one

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[b] This supposes the roots to be all possible. Every equation will have either an even number of impossible roots, or none: hence a quadratic will have both its roots possible, or both impossible; a cubic one or three possible roots, and two or none impossible; a biquadratic will have either four, two, or none of its roots possible, and none, two, or four, impossible; and the like of higher equations. An impossible root may be considered either as affirmative or negative. The difficulties attending the doctrine of impossible or imaginary roots, have hitherto bid defiance to the skill and address of the learned: a great number of theories and investigations have appeared, it is true; but our knowledge of the origin, nature, properties, &c. of imaginary roots is still very imperfect. The following Authors, among others, have treated on the subject, viz. Cardan, Bombelli, Albert Girard, Wallis, Newton, Maclaurin, James Bernoulli, Emerson, Euler, D'Alembert, Waring, Hutton, Sterling, Playfair, &c.

negative. A cubic has three roots, which are either all affirmative, all negative; two affirmative, and one negative; or one affirmative, and two negative: and the like of higher equations.

33. *If one root of an equation be given, the equation may be depressed one dimension lower; if two roots be given, it may be depressed two dimensions lower, and so on, by the following rule*[c].

RULE. When one root is given, transpose all the terms to one side, whereby the whole will $=o$; transpose in like manner the value of the root, then divide the former expression by the latter, and a new equation will arise $=o$, of one dimension lower than the given equation.

EXAMPLES.—1. Let $x^3-9x^2+26x-24=o$ be an equation, whereof one of the roots is known; namely, $x=3$.

By transposition $x-3=o$, divide the given equation by this quantity.

Thus, $x-3)x^3-9x^2+26x-24(x^2-6x+8=o$, *the resulting equation, which being resolved by the known rule for quadratics, its two remaining roots will be found, viz.* $x=4$, *and* $x=2$.

$$\begin{array}{r} x^3-3x^2 \\ \hline -6x^2+26x \\ -6x^2+18x \\ \hline 8x-24 \\ 8x-24 \\ \hline \end{array}$$

2. Let $x^4+4x^3+19x^2-160x=1400$, whereof one root $=-5$, be given, to depress the equation.

Here by transposition, $x^4+4x^3+19x^2-160x-1400=o$, *and* $x+5=o$; *then, dividing the former by the latter, we have* $\frac{x^4+4x^3+19x^2-160x-1400}{x+5}=x^3-x^2+24x-280=o$, *the resulting equation.*

3. Given $x=3$ in the equation $x^2-5x+6=o$, to depress it. *Ans.* $x-2=o$.

4. If $x-4=o$ be a divisor of the equation $x^3-4x^2-x+4=o$, to depress the equation, and determine its two remaining roots. *Ans. the resulting equation is* $x^2-1=o$, *and its roots* $+1$ *and* -1.

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[c] When the absolute term of an equation $=o$, it is plain that one of the roots is o, and consequently the equation may be divided by the unknown quantity, and reduced one dimension lower. In like manner, if the two last terms be wanting, the equation may be reduced two dimensions lower; if three, three dimensions, &c.

5. To depress the equations $x^3-5x^2+2x+8=o$, and $x^4-23x^2+18x+40=o$, one root of the former being $+4$, and one of the latter -5.

34. If two of the roots be given, $x\pm r=o$, and $x\pm s=o$, the given equation being divided by the product of these, $\overline{x\pm r}.\overline{x\pm s}$, will be depressed thereby two dimensions lower; thus,

6. To depress the equation $x^3-5x^2+2x+8=o$, two of its roots, -1 and $+2$, being given.

Thus, $x+1=o$, *and* $x-2=o$; *then* $\overline{x+1}.\overline{x-2}=x^2-x-2$, *the divisor; wherefore* $\frac{x^3-5x^2+2x+8}{x^2-x-2}=x-4$, *whence* $x-4=o$ *is the resulting equation.*

7. Given $x^3-3x^2-46x-72=o$, having likewise two values of x, viz. -2 and -4, given, to depress the equation. *Answer,* $x-9=o$.

8. Given $x^4-4x^3-19x^2+46x+120=o$, two roots of which are $+4$ and -3, to depress the equation.

35. *To transform an equation into another, the roots of which will be greater, by some given quantity, than the roots of the proposed equation.*

RULE I. Connect the given quantity with any letter, different from that denoting the unknown quantity in the proposed equation, by the sign $-$, and it will form a residual.

II. Substitute this residual and its powers, for the unknown quantity and its powers in the proposed equation, and the result will be a new equation, having its roots greater, by the given quantity, than those of the equation given[d].

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[d] The truth of this rule is clear from the first example, where since $y-3=x$, it is plain that $y=x+3$, or that the equation arising from the substitution of $y-3$ for x will have its roots (or the values of y) greater by 3, than the values of x in the proposed equation: this will be still more evident, if both the given and the resulting equation be solved; the roots of the former will be found to be -7 and $+3$, those of the latter -4 and $+6$. Let it not be thought strange that the negative quantity -7, by being increased by 3, becomes -4, or a less quantity than it was before; for a negative quantity is said to be increased, in proportion as it approaches towards an affirmative value; thus, -3 is said to be greater than -4, -2 than -3, -1 than -2, and 0 than -1: in the present instance, it is plain that -7 added to $+3$ will give -4 for the sum. Hence, if the roots of an equation be increased by a quantity greater than the

EXAMPLES.—1. Given $x^2+4x-21=o$, to transform it into another equation, the roots of which are greater by 3 than those of the given equation.

OPERATION.

Let $y-3=x$, *then*

$$x^2 = (\overline{y-3}|^2 =) y^2 - 6y + 9$$
$$+4x = (\overline{y-3}.4 =) \quad +4y - 12$$
$$-21 = \ldots\ldots\ldots\ldots -21$$
$$x^2+4x-21 = y^2 - 2y - 24 =$$

Wherefore $y^2-2y-24=o$,

is the equation required.

Explanation.

Having substituted $y-3$ for x, I substitute $\overline{y-3}|^2$ for x^2, $\overline{y-3}.4$ for $4x$, and -21 for itself; I then add all the quantities arising from these substitutions together, and make the result $y^2-2y-24=o$, which equation will have its roots greater by 3 than the roots of the equation given in the question.

2. Given the equation $x^3+x^2-10x+4=o$, to transform it into another, the roots of which are greater by 4 than the values of x.

Let $y-4=x$, *then*

$$x^3 = (\overline{y-4}|^3 =) \quad y^3 - 12y^2 + 48y - 64$$
$$+x^2 = (\overline{y-4}|^2 =) \ldots \quad + y^2 - \ 8y + 16$$
$$-10x = (\overline{y-4}.-10 =) \ldots\ldots -10y + 40$$
$$+8 = \ldots\ldots\ldots\ldots\ldots\ldots + 8$$
$$x^3+x^2-10x+8 = \qquad y^3 - 11y^2 + 30y = o$$

This transformed equation is evidently divisible by y (*or* $y+o$, *or* $y-o$); *therefore* o *is one of its roots: by this division it becomes* $y^2-11y+30=o$, *the two roots of which are* $+6$ *and* $+5$; *hence the three roots of the equation* $y^3-11y^2+30y=o$, *being* o .$+6$, *and* $+5$, *those of the proposed equation* $x^3+x^2-10x+8=o$ *are known; for (since* $x=y-4$*) its roots will be* $o-4$, $6-4$, *and* $5-4$; *or* -4, $+2$, *and* $+1$.

COR. Hence, when the roots of an equation are increased by a quantity equal to one of the negative roots, that root is taken away, or becomes o in the transformed equation; and in this case, the transformed equation may be depressed one dimension lower.

3. To increase the roots of the equation $x^3-6x^2+12x-8=o$, by 1.

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greatest negative root, the negative roots will be changed into affirmative ones.

It may be likewise useful to remark, that a deficient equation may be made complete by this rule.

4. To increase the roots of $x^4-4x^3+6x^2-12=o$, by 5.

36. *To transform an equation into another, the roots of which will be less than those of the proposed equation, by some given quantity.*

RULE. Connect the given quantity with some new letter by the sign $+$, and proceed as directed in the preceding rule [*].

EXAMPLES.—1. Transform the equation $x^2-2x-24=o$ into another, the roots of which will be less by 3 than those of the given equation.

OPERATION.

Let $y+3=x$, *then*

$$\begin{aligned} x^2 &= (\overline{y+3}|^2=)\,y^2+6y+9 \\ -2x &= (\overline{y+3}.-2=)-2y-6 \\ -24 &= \ldots\ldots\ldots\ldots -24 \\ \hline x^2-2x-24 &= \quad y^2+4y-21 \end{aligned}$$

Explanation.

Here, substituting $y+3$ for x, $\overline{y+3}|^2$ for x^2, $\overline{y+3}.-2$ for $-2x$, and -24 for itself, the sum of these is $y^2+4y-21=o$, the equation required.

Wherefore $y^2+4y-21=o$, *is the equation required.*

This equation being solved, the roots will be found to be $+3$ *and* -7; *wherefore those of the given equation are* $+3+3$ *and* $-7+3$, *or* $+6$ *and* -4.

2. To transform the equation $x^3-ax^2+bx-c=o$ to another, the roots of which shall be less by e.

Let $y+e=x$, *then*

$$\left.\begin{aligned} x^3 &= (\overline{y+e}|^3=)\,y^3+3y^2e+3ye^2+e^3 \\ -ax^2 &= (\overline{y+e}|^2.-a=)\,-ay^2-2aye-ae^2 \\ +bx &= (\overline{y+e}.b=)\ldots\ldots\ldots by+be \\ -c &= \ldots\ldots\ldots\ldots\ldots -c \end{aligned}\right\} =o, \text{ the equation required.}$$

3. Diminish the roots of $x^3-6x^2+9x-12=o$, by 6.

4. Diminish the roots of $x^4+5x^3-6x^2+7x-8=o$, by 10.

37. *To exterminate the second term of an equation.*

RULE I. Divide the coefficient of the second term, by the index of the highest power of the unknown quantity in the given equation.

II. Change the sign of the quotient, and then connect it with some new letter; this will form a binomial.

[*] The truth of this rule will be plain from ex. 1. for $y+3$ being made equal to x, or $y=x-3$, that is, y less than x, by 3; the roots or values of y in the transformed equation, will be less by 3 than the corresponding values of x in the proposed equation, as is evident.

III. Substitute this binomial and its powers, for the unknown quantity and its powers in the given equation, and there will arise a new equation wanting its second term [f].

EXAMPLES.—1. To transform the equation $x^3+12x^2-8x-9=o$, into an equation wanting its second term.

OPERATION.

$$First\ \frac{12}{3}=+4. \quad Let\ y-4=x.$$

$$\begin{array}{ll}
Then,\ x^3=(\overline{y-4}|^3=) & y^3-12y^2+48y-64. \\
+12x^2=(\overline{y-4}|^2.12=) & +12y^2-96y+192. \\
-8x=(y-4.-8=) & \quad -8y+32. \\
-9 = \ldots\ldots\ldots\ldots & \quad\quad -9. \\
\hline
x^3+12x^2-8x-9= \ldots\ldots & y^3 * -56y+151=o.
\end{array}$$

Explanation.

I first divide the coefficient 12 of the second term by the index 3; the quotient 4 I annex to a new letter y, first changing its sign from + to —, making $y-4$; this quantity and its powers are next substituted for x and its powers, as in the two foregoing rules; then adding the like quantities together, the sum is the equation $y^3 * -56y+151=o$, wanting its second term, as was proposed.

2. To destroy the second term from the equation $x^4-ax^3+bx^2-cx+d=o$.

First, $-\frac{a}{4}$ *is the coefficient of the second term divided by the index of the first.*

Let y *be the new letter, then by the rule,* $y+\frac{a}{4}=x$, *whence*

$$\begin{array}{ll}
x^4= & y^4+y^3a+\frac{3y^2a^2}{8}+\frac{ya^3}{16}+\frac{a^4}{256} \\
-ax^3= & -y^3a-\frac{3y^2a^2}{4}-\frac{3ya^3}{16}-\frac{a^4}{64} \\
+bx^2= & \quad +by^2+\frac{bya}{2}+\frac{ba^2}{16} \\
-cx= & \ldots\ldots\ -cy-\frac{ca}{4} \\
+d= & \ldots\ldots\ldots\ +d \\
\hline
\end{array}$$

$$y^4*-\frac{3y^2a^2}{8}-\frac{3y^2a^2}{4}+by^2-\frac{ya^3}{8}+\frac{bya}{2}-cy+\frac{a^4}{256}-\frac{a^4}{64}+$$

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[f] This rule is necessary to the solution of cubic and biquadratic equations; and the truth of it will appear from an attentive examination of the process in *ex.* 1. The third, fourth, and fifth, &c. terms may be exterminated from any

$\frac{ba^2}{16} - \frac{ca}{4} + d = o$, *which, properly contracted* [g], *becomes* $y^4 + \overline{b - \frac{3a^2}{4}} . y^2 - \overline{\frac{a^3}{8} + \frac{ba}{2} - c} . y - \frac{3a^4 + 16ba^2 - 64ca}{256} = o$, *the equation required.*

3. Given $x^2 - 4x + 8 = o$, to exterminate the second term.

Thus, $\frac{-4}{2} = -2$; *then let* $y + 2 = x$, *and proceed as before.*

4. Given $x^2 + 10x - 100 = o$, to destroy the second term.

Thus, $\frac{10}{2} = +5$; *let* $y - 5 = x$, *and proceed.*

5. To exterminate the second term from $x^3 - 3x^2 + 4x - 5 = o$.

Thus, $\frac{-3}{3} = -1$, *let* $y + 1 = x$; *and proceed.*

6. Let the second term be taken away from the equation $x^4 + 24x^3 - 12x^2 + 4x - 30 = o$ [h].

7. To take away the second term from the equation $x^5 - 50x^4 + 40x^3 - 30x^2 + 20x - 10 = o$.

38. *To multiply the roots of an equation by any given quantity, that is, to transform it into another, the roots of which will be any proposed multiple of those of the given equation.*

RULE I. Take some new letter as before, and divide it by the given multiplier.

II. Substitute the quotient and its powers, for the unknown quantity and its powers, in the given equation, and an equation

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equation, but these transformations being less useful and more difficult than the above, we have in the text omitted the rules: in general, to take away the second term requires the solution of a simple equation; to take away the third term, a quadratic; the fourth term, a cubic; and the n^{th} term requires the solution of an equation of $n-1$ dimensions. *See the note below.*

[g] This contraction consists in the reducing of the fractional coefficients of the same powers of y to a common denominator, and then adding or subtracting, according to the signs; putting the coefficients of the same power of y under the vinculum, &c. &c.

[h] In like manner, to take away the third term from the equation $x^3 - ax^2 + bx - c = o$, we assume $y + e = x$, where e must be taken such that (supposing n = the index of the highest power of x) $n . \frac{n-1}{2} e^2 - \overline{n-1} . ae + b = o$. In which case a quadratic is to be solved; and in general, to take out the m^{th} term, by this method, an equation of $m-1$ dimensions must be solved, as was observed in a preceding note. See *Wood's Algebra*, p. 141.

will thence arise, whose roots are the proposed multiple of those of the given equation.

RULE I. Assume some new letter as before, and place the given quantity under it, for a denominator.

II. Substitute this fraction and its powers, for the unknown quantity and its powers respectively, in the given equation, and a new equation will arise, having its roots respectively equal to the given equation multiplied by the given quantity[i].

EXAMPLES.—1. To transform the equation $x^2+5x-2=o$, into another, the roots of which are 10 times as great as those of the given equation.

Let $\frac{y}{10}=x$.

Then $x^2 \quad =\frac{y^2}{100}$

$+5x = \ldots\ldots +\frac{y}{2}$

$-2 \quad = \ldots\ldots\ldots -2$

Whence $x^2+5x-2=\frac{y^2}{100}+\frac{y}{2}-2=o$, *that is,* $y^2+50y-200=o$, *the equation required*[k].

2. Let the roots of $3x^3-12x^2+15x-21=o$, be multiplied by 3.

Thus, $\frac{y}{3}=x$.

Then $3x^3=(\frac{3y^3}{27}=)\frac{y^3}{9}$

$-12x^2= \ldots\ldots -\frac{4y^2}{3}$

$+15x = \ldots\ldots\ldots +5y$

$-21 \quad = \ldots\ldots\ldots\ldots -21$

Therefore $(\frac{y^3}{9}-\frac{4y^2}{3}+5y-21$, *or*$)\ y^3-12y^2+45y-189=o$, *the equation required.*

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i This rule requires neither proof nor explanation; it is sometimes useful for freeing an equation from fractions and radical quantities.

k Hence it appears, that to multiply the roots of an equation by any quantity, we have only to multiply its terms respectively by those of a geometrical progression, the first term of which is 1, and the ratio the multiplying quan-

4. Let the roots of $x^2-3x+4=o$, be doubled.

5. Let the roots of $x^3+12x^2-20x+50=o$, be multiplied by 100.

39. *To transform any given equation into another, the roots of which are any parts of those of the given equation.*

RULE I. Assume a new letter as before, and let it be multiplied by the number denoting the proposed part.

II. Substitute this quantity and its powers, for the unknown quantity and its powers, in the given equation; the result will be an equation, the roots of which are respectively the parts proposed of those of the given equation[1].

EXAMPLES.—1. Let the roots of $x^2-x-5=o$, be divided by 3.

Assume $3y=x$; *then will*

$$\begin{aligned} x^2 &= 9y^2 \\ -x &= \ldots -3y \\ -5 &= \ldots\ldots -5 \end{aligned}$$

Whence ($9y^2-3y-5=o$, *or*) $y^2-\frac{y}{3}-\frac{5}{9}=o$, *is the equation required.*

2. Let the roots of $x^3+7x^2-29x+2=o$, be divided by 5.

3. Given $x^4-2x^2-3x+4=o$, to divide its roots by 8.

40. *To transform an equation into another, the roots of which are the reciprocals of those of the given equation.*

RULE I. Assume a new letter, and make it equal to the reciprocal of the unknown quantity in the given equation.

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tity. thus, in *ex.* 1. the roots of the equation are to be multiplied by 10; wherefore multiplying the given equation $x^2+5x-2=o$

by the geometrical progression $1 \quad 10 \quad 100$

The product is $x^2+50x-200=o$, as above, where y in the above example answers to x in this; and the like in other cases.

[1] This rule is equally evident with the foregoing; and in like manner, the roots of an equation are divided by any quantity, by dividing its terms by those of a geometrical progression, whose first term is 1, and ratio, the said quantity: Thus, *ex.* 1. to divide the roots of $x^2-x-5=o$ by 3,

Divide its terms respectively by $1 \quad 3 \quad 9$

The quotients are $x^2-\frac{x}{3}-\frac{5}{9}=o$, as above;

where y in that, answers to x in this. It is sometimes necessary to have recourse to this rule, to exterminate surds from an equation.

II. Substitute the reciprocal of this letter and its powers, for the unknown quantity and its powers, in the given equation; the result will be an equation, having its roots the reciprocals of those of the given equation.

EXAMPLES.—1. Let the roots of $x^3-2x^2+3x-4=o$, be transformed into their reciprocals.

Assume $y=\frac{1}{x}$, *that is* $x=\frac{1}{y}$, *then will*

$$x^3=\frac{1}{y^3}$$

$$-2x^2= \;.\; -\frac{2}{y^2}$$

$$+3x= \;.\;.\;.\;.\; +\frac{3}{y}$$

$$-4 \;=\; .\;.\;.\;.\;.\;.\;.\;. -4$$

Whence ($\frac{1}{y^3}-\frac{2}{y^2}+\frac{3}{y}-4=o$, *or multiplying by* y^3, *changing the signs, and dividing by* 4,) $y^3-\frac{3}{4}y^2+\frac{1}{2}y-\frac{1}{4}=o$, *the equation required.*

2. Let the roots of $x^2+10x-25=o$, be changed into their reciprocals.

3. Change the roots of $x^3-ax^2+bx-c=o$, into their reciprocals.

4. Change the roots of $x^n+ax^{n-1}-bx^{n-2}+cx^{n-3}-d=o$, into their reciprocals.

41. *To transform an equation into another, the roots of which are the squares of those of the given equation.*

RULE. Assume a new letter equal to the square of the unknown quantity in the given equation; then by substituting as in the preceding rules an equation will arise, the roots of which are the squares of those of the given equation.

EXAMPLES.—1. Let the roots of the equation $x^2+9x-17=o$, be squared.

Assume $y=x^2$

Then $x^2=y$

$$+9x=+9\sqrt{y}$$

$$-17= \;.\;.\;.\;.\;.\;. -17$$

Whence $y+9\sqrt{y}-17=o$, *the equation required* [m].

[m] The roots of the proposed equation are 1.6 and −10.6: those of the

2. Let the roots of $x^3-x^2+x-7=o$, be squared.

Assume $y=x^2$

Then $x^3=y^{\frac{3}{2}}$

$-x^2=-y$

$+x=+\sqrt{y}$

$-7=\ldots.-7$

Whence $y^{\frac{3}{2}}-y+\sqrt{y}-7=o$, *the equation required.*

3. Square the roots of $x^4+2x^3-3x-12=o$.

4. Square the roots of $x^5-ax^4+bx^3-cx+d=o$.

5. Square the roots of $x^{\frac{2}{3}}-7x^{\frac{1}{3}}-8=o$.

OF THE LIMITS OF THE ROOTS OF EQUATIONS.

42. Let $\overline{x-a}.\overline{x-b}.\overline{x-c}.\overline{x+d}=o$, be an equation, having the root a greater than b, b than c, and c than d [n]; "in which, if a quantity greater than a be substituted for x, (as every factor is, on this supposition, positive,) the result will be positive; if a quantity less than a, but greater than b, be substituted, the result will be negative, because the first factor will be negative, and the rest positive. If a quantity between b and c be substituted, the result will again be positive, because the two first factors are negative, and the rest positive; and so on [o]. Thus,

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transformed equation are 2.56, and 112.36, which are the squares of the former respectively.

[n] "In this series the greater is d, the less is $-d$; and whenever $a, b, c, -d$, &c. are said to be the roots of an equation, taken in order, a is supposed to be the greatest. Also in speaking of the limits of the roots of an equation, we understand the limits of the possible roots." This note, and the article to which it refers, were taken from Mr. Wood's Algebra; see likewise, on this subject, *Maclaurin's Algebra*, part 2. ch. 5. *Wolfius's Algebra*, part 1. sect. 2. ch. 5. *Sir Isaac Newton's Arithmetica Universalis*, p. 258. &c. *Dr. Waring's Meditationes Algebraicæ*, &c.

[o] To illustrate this, let the roots of the equation $x^4-px^3+qx^2-rx+s=o$ be a, b, c, and d; then $x-a=o$, $x-b=o$, $x-c=o$, and $x-d=o$; and let g, which we will suppose less than a, but greater than b, be substituted for x in the latter equations; then will $g-a$ be negative, and the rest, viz. $g-b$, $g-c$, and $g-d$, positive, and consequently their product will be positive; and $g-a$, (a negative quantity,) multiplied into this positive result, will therefore give a negative product: if h, which is less than b, but greater than c, be substituted for x, we have $h-a$ and $h-b$ both negative, and their product positive; but $h-c$ and $h-d$ are both negative, therefore their product is posi-

quantities which are limits to the roots of an equation, (or between which the roots lie,) if substituted for the unknown quantity, give results alternately positive and negative."

43. "Conversely, if two magnitudes, when substituted for the unknown quantity, give results one positive and the other negative, an odd number of roots must lie between these magnitudes: and if as many quantities be found as the equation has dimensions, which give results alternately positive and negative, an odd number of roots will lie between each two succeeding quantities; and it is plain that this odd number cannot exceed unity, since there are no more limiting terms than the equation has dimensions."

44. If when two magnitudes are severally substituted for the unknown quantity, both results have the same sign, either an even number of roots, or no root, lies between the assumed magnitudes.

COR. Hence, any magnitude is greater than the greatest root of the equation, which, being substituted for the unknown quantity, gives a positive result.

45. *To find a limit greater than the greatest root of an equation.*

RULE. Diminish the roots of this equation by the quantity e, (Art. 36.) and if such a value of e can be found, as shall make every term of the transformed equation positive, all its roots will be negative, (Art. 31. Cor.) *consequently e will be greater than the greatest root of the equation.*

EXAMPLES.—1. To find a limit greater than the greatest root of $x^2-5x+6=o$.

$$\begin{aligned} &\textit{Let } x=y+e \\ &\textit{Then will } x^2=y^2+2ye+e^2 \\ &\qquad -5x=\quad -5y-5e \\ &\qquad +6\ =\underline{\ \ldots\ldots\ldots +6} \end{aligned}$$

Whence ($y^2+2ye-5y+e^2-5e+6=o$, *or*) $y^2+\overline{2e-5}\,y$ $+\overline{e.e-5}+6=o$, *is the transformed equation; now it appears by trials, that* 4 *being substituted for* e *in this equation, it will be-*

tive; and these two products multiplied, give likewise a positive product. In like manner it may be shewn, by substituting k, which is less than c, and greater than d, the result will be negative; and substituting m, less than the least root, the result will be positive.

come $y^2+3y+2=o$, *of which all the roots are negative; wherefore* 4 *is greater than the greatest root of the equation* $x^2-5x+6=o$.

2. To find a limit greater than the greatest root of $x^3-12x^2+41x-43=o$.

Let $x=y+e$, *as before.*

$$\begin{aligned} \textit{Then will } x^3 &= y^3+3y^2e+3ye^2+e^3 \\ -12x^2 &= \quad -12y^2-24ye-12e^2 \\ +41x &= \ldots\ldots\ldots +41y\ +41e \\ -43\ &= \ldots\ldots\ldots\ldots\ldots\ldots -43 \end{aligned}$$

Wherefore $(y^3+3y^2e-12y^2+3ye^2-24ye+41y+e^3-12e^2+41e-43=o$, *or*) $y^3+\overline{3e-12}.y^2+\overline{3e^2-24e+41}.y+\overline{e.e^2-12e+41}-43=o$, *is the transformed equation, where (by trials) it is found, that if* 8 *be substituted for e, the terms will be all positive; viz.* $y^3+12y^2+41y+29=o$; *whence* 8 *is greater than the greatest root of the given equation.*

3. Required a limit greater than the greatest root of $x^3-6x^2-25x-12=o$. *Ans.* 9.

4. Find a limit greater than the greatest root of $x^4-5x^3+6x^2-7x+8=o$.

5. To find a limit greater than the greatest root of $x^4+3x^3-5x^2+8x-20=o$.

46. *To find a limit less than the least root of an equation.*

RULE. Change the signs of the even terms, (the second, fourth, sixth, &c.) and proceed as before; then will the limit greater than the greatest root of the transformed equation, with its sign changed, be less than the least root of the given equation. See Cor. to Art. 30. and Art. 45.

EXAMPLES.—1. Let $x^2-7x+8=o$, be given to find a limit less than the least of its roots.

This equation, by changing the sign of its second term, becomes $x^2+7x+8=o$.

Let $x=y+e$.

$$\begin{aligned} \textit{Then } x^2 &= y^2+2ye+e^2 \\ +7x &= \quad +7y+7e \\ +8\ &= \ldots\ldots\ldots\ldots +8 \end{aligned}$$

Whence $(y^2+2ye+7y+e^2+7e+8=o$, *or*) $y^2+\overline{2e+7}.y+\overline{e+7}.e+8=o$, *is the transformed equation; and if* -1 *be substi-*

tuted for e, all its terms will be positive, for the equation becomes $y^2+5y+2=o$; wherefore $+1$ is a limit less than the least root of the equation $x^2-7x+8=o$.

2. To find a limit less than the least root of $x^3+x^2-10x+6=o$.

Changing the signs of the second and fourth terms, the equation becomes $x^3-x^2-10x-6=o$.

Let $x=y+e$, then will

$$\begin{aligned} x^3 &= y^3+3y^2e+3ye^2+e^3 \\ -x^2 &= \quad -\ y^2 \quad -2ye \quad -e^2 \\ -10x &= \ldots\ldots\ -10y \quad -10e \\ -6 &= \ldots\ldots\ldots\ldots\ldots\ldots -6 \end{aligned}$$

Whence $y^3+\overline{3e-1}.y^2+\overline{3e^2-2e-10}.y+\overline{e^3-e^2-10.e-6}=o$, is the transformed equation, in which 4 being substituted for e, it becomes $y^3+11y^2+30y+2=o$; wherefore -4 is less than the least root of the equation $x^3+x^2-10x+6=o$.

3. To find a limit less than the least root of $x^2+12x-20=o$. *Ans.* -14.

4. To find a limit less than the least root of $x^3-4x^2-5x+6=o$.

5. To find a limit less than the least root of $x^4-5x^3-3=o$.

6. To find the limits of the roots of $x^3+x^2-10x+9=o$. *Ans.* $+3$ *and* -5.

7. Required the limits of $x^4-4x^3+8x^2-14x+20=o$?

8. What are the limits of the roots of $x^3-2x^2-5x+7=o$?

9. What are the limits of the roots of $x^4+3x^2-5x+10=o$?

RESOLUTION OF EQUATIONS OF SEVERAL DIMENSIONS.

47. *When the possible roots of an equation are integers, either positive or negative, they may be discovered as follows.*

RULE I. Find all the divisors of the last term, and substitute them successively for the unknown quantity in the proposed equation.

II. When by the substitution of either of these divisors for the root, the resulting equation becomes $=o$, that divisor is a root of the given equation; otherwise it is not.

III. If none of the divisors succeed, the roots are either fractional, irrational, or impossible.

IV. When the last term admits of a great number of divisors, it will be convenient to transform the given equation into another, (Art. 35, 36.) the last term of which will have fewer divisors.

EXAMPLES.—1. Let $x^3-2x^2-5x+6=o$, be given, to find its integral roots by this method.

First, the divisors of the last term 6, *are* $+1, -1, +2, -2, +3, -3, +6$, *and* -6; *now* $+1$ *being substituted for* x *in the given equation, it becomes* $+1-2-5+6=o$; *wherefore* $+1$ *is a root.*

Next, let -1 *be substituted, and the equation becomes* $-1-2+5+6=8$; *wherefore* -1 *is not a root.*

Thirdly, let $+2$ *be substituted, and the equation becomes* $8-8-10+6=-4$; *wherefore* $+2$ *is not a root.*

Fourthly, let -2 *be substituted, and the equation becomes* $-8-8+10+6=o$; *wherefore* -2 *is a root.*

Fifthly, let $+3$ *be substituted, and the equation will then become* $+27-18-15+6=o$; *wherefore* $+3$ *is likewise a root.*

Thus, the three roots of the given equation are $+1, -2$, *and* $+3$; *and it is plain there can be no more than three roots, since the equation arises no higher than the third degree; consequently there is no necessity to try the remaining divisors.*

2. Given $x^4-6x^2-16x+21=o$, to find the roots.

The divisors of the last term 21, *are* $+1, -1, +3, -3, +7, -7, +21$, *and* -21; *these being successively substituted for* x, *we shall have*

Substitutions.	Results.
$+1$	$+1-6-16+21=o$
-1	$+1-6+16+21=32$
$+3$	$+81-54-48+21=o$
-3	$+81-54+48+21=96$
$+7$	$+2401-294-112+21=2016$
-7	$+2401-294+112+21=2240$
$+21$	$+194481-2646-336+21=191520$
-21	$+194481-2646+336+21=192192$

Wherefore $+1$ *and* $+3$ *are the only roots which can be found by this method; the two remaining roots are therefore impossible, being* $-2\pm\sqrt{-3}$.

3. Given $x^4-4x^3-19x^2+106x-120=o$, to find the roots.

Since the last term 120 *has a great number of divisors, it will be proper to transform the equation into another, whose absolute term will have fewer divisors; in order to which, let* $x=y+2$, *then* (*Art.* 36.)

$$\begin{array}{rl} x^4= & y^4+8y^3+24y^2+\ 32y+\ 16 \\ -\ 4x^3= & \quad -4y^3-24y^2-\ 48y-\ 32 \\ -19x^2= & \ldots\ldots-19y^2-\ 76y-\ 76 \\ +106x= & \ldots\ldots\ldots\ldots+106y+212 \\ -120\ = & \underline{\ldots\ldots\ldots\ldots\ldots\ldots\ldots-120} \\ & y^4+4y^3-19y^2+\ 14y=o \end{array}$$

Here the last term vanishing, the number assumed, viz. $+2$, *is one of the roots of the original equation,* (*Art.* 33. *note,*) *and the transformed equation being divisible by* y, *will thereby be reduced one dimension lower: thus,* $y^3+4y^2-19y+14=o$; *the divisors of the last term* 14, *are* $+1,-1,+2,-2,+7,-7,+14,-14$; *each of these being substituted for* y *in the last equation,* $+1,+2$, *and* -7 *are found to succeed, they are therefore the roots of the transformed equation* $y^3+4y^2-19y+14=o$; *wherefore, since* $x=y+2$, *three of the roots of the original equation will be* $(1+2=)\ 3$, $(2+2=)\ 4$, *and* $(-7+2=)-5$, *which with the number* 2 *assumed above, give* $+2,+3,+4$, *and* -5, *for the four roots required.*

4. Given $x^3-3ax^2-4a^2x+12a^3=o$, to find the roots.

The numeral divisors of the last term are $+1,-1,+2,-2,+3,-3,+4,-4,+6,-6,+12$, *and* -12; *and of these,* $+2,-2$, *and* -3 *are found to succeed; wherefore the roots are* $+2a,-2a$, *and* $-3a$.

5. Required the roots of $x^2+x-12=o$? *Ans.* 3, *and* -4.

6. What are the roots of $x^3+4x^2+x-6=o$? *Ans.* $1,-2$, *and* -3.

7. What are the roots of $x^3+2x^2-19x-20=o$? *Ans.* -1, -4, *and* $+5$.

8. Required the roots of $x^3-14x^2+31x+126=o$? *Ans.* $-2,+7$, *and* $+9$.

9. What are the roots of $x^4-15x^2+10x+24=o$? *Ans.* -1, $+2,+3$, *and* -4.

10. Required the roots of $x^3+4x^2-7x-10=o$?

48. SIR ISAAC NEWTON'S METHOD OF DISCOVERING THE ROOTS OF EQUATIONS BY MEANS OF DIVISORS.

RULE I. For the unknown quantity in the given equation, substitute three or more terms of the arithmetical progression 2, 1, 0,—1,—2, &c. and let these terms be placed in a column one under the other.

II. Substitute each number in this column successively for the unknown quantity in the proposed equation; collect all the terms of the equation arising from each substitution into one sum, and let this sum stand opposite the number substituted from whence it arises: these sums will form a second column.

III. Find all the divisors of the sums, and place them in lines opposite their respective sums: these will form a third column.

IV. From among the divisors collect one or more arithmetical progressions, the terms of which differ either by unity, or by some divisor of the coefficient of the highest power of the unknown quantity, observing to take one term only (of each progression) out of each line of the divisors: each of these progressions will form an additional column.

V. Divide that term of the progression thus found, (or of each progression, if there be more than one,) which stands against 0 in the assumed progression, by the common difference of the terms of the former; and if the progression be increasing, prefix the sign + to the quotient; but if it be decreasing, prefix the sign —: this quotient will be a root of the equation.

Hence there will be as many roots found by this method, as there are progressions obtained from the divisors.

EXAMPLES.—1. Given $x^2-2x-24=0$, to find the values of x.

OPERATION.

Substitutions.	*Results.*	*Divisors.*	*Prog^s. derived.*	
2	—24	1, 2, 3, 4, 6, 8, 12, 24	4	6
1	—25	1, 5, 25	5	5
0	—24	1, 2, 3, 4, 6, 8, 12, 24	6	4
—1	—21	1, 3, 7, 21	7	3
—2	—16	1, 2, 4, 8, 16	8	2

Whence the roots are +6 *and* —4.

Explanation.

The left hand column is the assumed progression, the terms of which are substituted successively for x in the given equation: first, by substituting 2

for x, the equation amounts to -24, which is the *result* in this case; this I put in the second column, and its divisors 1, 2, 3, 4, 6, &c. in the third. Secondly, I substitute 1 for x, and the whole equation amounts to -25, which is the second *result*, and its divisors are 1, 5, and 25. Thirdly, by substituting 0 for x, the *result* is -24, and its divisors 1, 2, 3, 4, 6, &c. as in the first case. Fourthly, by substituting -1 for x, the *result* is -21, and its divisors are 1, 3, 7, and 21. Fifthly, by substituting -2, the *result* is -16, the *divisors* of which are 1, 2, 4, 8, and 16. Sixthly, I try to obtain a *progression*, by taking one number out of each line of the divisors: and first I try for an increasing one; the only one that can be found is 4, 5, 6, 7, and 8, viz. 4 out of the first line, 5 out of the second, 6 out of the third, 7 out of the fourth, and 8 out of the fifth; these numbers constitute the fourth column. Seventhly, I try for a decreasing progression, and (proceeding as before) find that 6, 5, 4, 3, and 2, which constitute the fifth column, is the only one that can be obtained. Eighthly, the number 6 and 4, standing opposite the 0 in the assumed progression, divided by the common difference 1, gives 6 and 4 for the roots of the equation. The former being a term of the increasing progression, must have + prefixed to it; the latter being a term of the decreasing progression, must have − prefixed; wherefore the roots are $+6$ and -4.

2. Given $x^3-6x^2-7x+60=0$, to find the roots.

OPERATION.

Substitutions.	*Results.*	*Divisions.*	*Prog^s. derived.*		
2	30	1, 2, 3, 5, 6, 10, 15, 30	2	3	5
1	48	1, 2, 3, 4, 6, 8, 12, 16, &c.	3	4	4
0	60	1, 2, 3, 4, 5, 6, 10, 15, &c.	4	5	3
−1	60	1, 2, 3, 4, 5, 6, 10, 15, &c.	5	6	2
−2	42	1, 2, 3, 6, 7, 14, 21, 42	6	7	1

Roots 4, 5, *and* −3.

Explanation.

Proceeding as before, I obtain three progressions, two increasing, and one decreasing, and the numbers 4, 5, and 3, standing opposite the 0, being divided by 1 the common difference, the quotients are the roots, viz. $+4$ and $+5$ in the increasing progressions, and -3 in the decreasing one.

3. Given $x^3-x^2-10x+6=0$, to find the roots.

Substitutions.	*Results.*	*Divisors.*	*Progressions.*
2	−10	1, 2, 5, 10.	5
1	− 4	1, 2, 4	4
0	+ 6	1, 2, 3, 6	3
−1	+14	1, 2, 7, 14	2
−2	+14	1, 2, 7, 14	1

Here we can derive only one progression, and that a decreasing one; wherefore the only root discovered by this method is -3*: but by means of this root the given equation may be depressed to a quadratic,* (*Art.* 33.) *and the two remaining roots found by the known rule for quadratics; thus, since* $x+3=0$, *dividing the proposed equation by this, we obtain* $\left(\frac{x^3-x^2-10x+6}{x+3}=\right)x^2-4x+$

$2=o$, *the two roots of which are* ($2\pm\sqrt{2}.=$) 3.4142135624 *and* .5857864376.

4. Required the roots of $6x^4-20x^3-12x^2-11x-20=o$?

Substit.	*Results.*	*Divisors.*	*Prog.*
2	-154	1, 2, 7, 11, 14, 22, 77, 154	2
1	-57	1, 3, 19, 57	3
0	-20	1, 2, 4, 5, 10, 20	4
-1	$+5$	1, 5	5
-2	$+210$	1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, &c.	6

Here we obtain only one progression, consequently $+4$ *is the only root found.*

5. Given $x^4+x^3-29x^2-9x+180=o$, to find the roots.

Subst.	*Results.*	*Divisors.*	*Progressions.*			
2	70	1, 2, 5, 7, 10, 14, &c.	1	2	5	7
1	144	1, 2, 3, 4, 6, 8, &c.	2	3	4	6
0	180	1, 2, 3, 4, 5, 6, &c.	3	4	3	5
-1	160	1, 2, 4, 5, 8, 10, &c.	4	5	2	4
-2	90	1, 2, 3, 5, 6, 9, &c.	5	6	1	3

Here are four progressions, two increasing and two decreasing, and the roots are 3, 4, -3, *and* -5.

6. Required the roots of $x^2-x-12=o$? *Ans.* $+4$ *and* -3.

7. Required the roots of $x^3+2x^2-23x-60=o$? *Ans.* $+5$, -4, *and* -3.

8. What are the roots of $2x^3-5x^2+4x-10=o$? *Answer, one root* $+2\frac{1}{2}$.

9. Required the roots of $x^3-3x^2-46x-72=o$? *Ans.* $+9$, -2, *and* -4.

10. To find the roots of $x^3-6x^2+10x-8=o$?

RECURRING EQUATIONS.

49. A recurring equation is one having the sign and coefficient of any term, reckoning from the beginning of the equation, the same with those of the term equally distant from the end; and its roots are of the form a, $\frac{1}{a}$, b, $\frac{1}{b}$, or the reciprocals of one another.

50. If the recurring equation be of an *odd* number of dimensions, $+1$ or -1 is a root; and the equation may be depressed to one of an *even* number of dimensions. (Art. 33.)

Thus, let $x^3-2x^2+1=o$; $+1$ *is evidently one root; therefore,* (*Art.* 32.)

$$\begin{array}{l} x-1)x^3-2x^2+1(x^2-x-1 \\ \quad\;\; \underline{x^3-\;x^2} \\ \quad\;\;\;\; -\;x^2+1 \\ \quad\;\;\;\; \underline{-\;x^2+x} \\ \qquad\qquad -x+1 \\ \qquad\qquad \underline{-x+1} \end{array}$$

This equation $x^2-x-1=o$, *being resolved by the rule for quadratics, its roots will be found to be* $\frac{1\pm\sqrt{5}}{2}$.

COR. Hence, a cubic equation of the form $x^3\pm px^2\pm px+1$ may always be reduced to a quadratic, and its roots found.

51. If the given equation be of even dimensions above a quadratic, its roots may be found by means of an equation of half the number of dimensions.

Thus, by supposing the equation to be the product of the factors $\overline{x-a}.\overline{x-\frac{1}{a}}$, $\overline{x-b}.\overline{x-\frac{1}{b}}$, *&c. by actual multiplication, and putting* $m=a+\frac{1}{a}$, $n=b+\frac{1}{b}$, *&c. we obtain* x^2-mx+1, x^2-nx+1, *&c. wherefore by multiplying these quadratic factors together, and equating the coefficients of each term of the product, with that of the corresponding term of the given equation, the values of m and n will be readily found: and since for every single value of m there will be two values of x, it follows that the equation for finding m will be of but half the number of dimensions necessary for finding the value of x by other methods.*

EXAMPLES.—1. Let $x^4-3x^3+2x^2-3x+1=o$ be the proposed equation.

Assume the product $(\overline{x^2-mx+1}.\overline{x^2-nx+1}=)\;x^4-\overline{m+n}.x^3+\overline{mn+2}.x^2-\overline{m+n}.x+1=$ *the proposed equation: then making the coefficients of like powers of x in this product and the given equation equal, we shall have* $m+n=3$, *and* $mn+2=2$, *or* $mn=o$; *wherefore, if* $n=o$, *then* $m=3$, *and the two equations* $x^2-mx+1=o$, *and* $x^2-nx+1=o$, *become respectively* $x^2-3x+1=o$, *and* $x^2+1=o$; *from the former of these* $x=(\frac{3\pm\sqrt{5}}{2}=)$

2.6180339887, and .3819660112; which two values of x are the reciprocals of each other. From the latter, viz. $x^2+1=o$, *we obtain* $x=\pm\sqrt{-1}$, *or* $+\sqrt{-1}$, *and* $-\sqrt{-1}$, *for the two remaining values of x.*

2. Let $x^3-1=o$ be given, to find the values of x.

Here it is plain that $+1$ *is a root, or* $x-1=o$, *wherefore dividing the given equation by this, we have* $(\frac{x^3-1}{x-1}=)\ x^2+x+1=o$, *the two roots of which are* $\frac{-1+\sqrt{-3}}{2}$, *and* $\frac{-1-\sqrt{-3}}{2}$.

3. Given $x^3+1=o$, to find the values of x. *Ans.* -1, $\frac{1+\sqrt{-3}}{2}$, *and* $\frac{1-\sqrt{-3}}{2}$.

4. Let the equations $x^4-1=o$, $x^4+1=o$, $x^5-1=o$, and $x^6+1=o$, be proposed, to find the values of x in each.

Literal equations, wherein the given quantity and the unknown one are alike affected, may be reduced to others of fewer dimensions, by the following rules.

52. *When the given equation is of even dimensions.*

Rule I. Divide the equation by the equal powers of its two quantities in the middle term.

II. Assume a new equation, by putting some letter equal to the sum of the quotients arising from the division of the given and unknown quantity, alternately, by each other.

III. Substitute in the former equation the values of its terms found by the latter, and an equation will arise of half the dimensions of the given one, from the solution of which the roots of the given equation may be determined.

Examples.—1. Required the roots of $x^4-4ax^3+5a^2x^2-4a^3x+a^4=o$?

First, dividing the whole equation by the equal powers in the middle term, it becomes $(\frac{x^2}{a^2}+\frac{4x}{a}+5-\frac{4a}{x}+\frac{a^2}{x^2}=o$; *or, which is the same,*$)$ $\frac{x^2}{a^2}+\frac{a^2}{x^2}-4.\overline{\frac{x}{a}+\frac{a}{x}}-5=o$. *Let* $\frac{x}{a}+\frac{a}{x}=z$, *then by squaring,* $\frac{x^2}{a^2}+\frac{a^2}{x^2}+2=z^2$, *and by substituting* z^2 *and* z *for their values in the equation* $\frac{x^2}{a^2}+\frac{a^2}{x^2}-4.\overline{\frac{x}{a}+\frac{a}{x}}+5=o$, *it becomes* z^2-4z+3

$=o$, *whence* $z=3$, *or* 1; *but since* $\frac{x}{a}+\frac{a}{x}=z$, *if the former value be taken, then* $\frac{x}{a}+\frac{a}{x}=3$; *whence* $x^2-3\,ax=-a^2$, *which solved, gives* $x=(\frac{a}{2}.\overline{3\pm\sqrt{5}}=)$ $2.618034\,a$, *or* $.381966\,a$. *But if the latter value of z, namely* 1, *be taken, then* $(\frac{x}{a}+\frac{a}{x}=1$, *or*$)$ $x^2-ax=-a^2$, *whence* $x=\frac{a\pm a\sqrt{-3}}{2}$ *are the two remaining roots.*

2. Given $7\,x^6-26\,ax^5-26\,a^5x+7\,a^6=o$, to find the values of x.

This divided by a^3x^3 *becomes* $7.\overline{\frac{x^3}{a^3}+\frac{a^3}{x^3}}-26.\overline{\frac{x^2}{a^2}+\frac{a^2}{x^2}}=o$. *Let* $z=\frac{x}{a}+\frac{a}{x}$, *then* $z^2-2=\frac{x^2}{a^2}+\frac{a^2}{x^2}$, *which multiplied by* $z=\frac{x}{a}+\frac{a}{x}$, *becomes* $z^3-2\,z=(\frac{x^3}{a^3}+\frac{a}{x}+\frac{x}{a}+\frac{a^3}{x^3}=)\frac{x^3}{a^3}+z+\frac{a^3}{x^3}$; *whence* $z^3-3\,z=\frac{x^3}{a^3}+\frac{a^3}{x^3}$.

These values substituted as before, we obtain $7\,z^3-26\,z^2-21\,z+52=o$, *one root of which* (*by Art.* 47.) *is* 4, *and by means of this, the equation may be depressed to the quadratic* $7\,z^2+2\,z-13=o$, (*Art.* 32.) *the two roots of which are* $+1.2273804$, *and* -1.5130947. *Wherefore, since* $z=\frac{x}{a}+\frac{a}{x}$, *or* $x^2-azx=-a^2$, *by the solution of this we obtain* $x=\frac{a.\overline{z\pm\sqrt{z^2-4}}}{2}$, *in which, if the three values of z be successively substituted, the six roots of the given equation will be obtained.*

3. To find the roots of $x^4+6\,ax^3-20\,a^2x^2+6\,a^3x+a^4=o$.

4. To find the roots of $x^4-20\,ax^3+12\,a^2x^2-20\,a^3x+a^4=o$.

5. Required the roots of $x^6-ax^4-a^4x+a^6=o$.

53. *When the given equation is of odd dimensions.*

RULE. Divide the equation by the sum of the known and unknown quantities, and proceed as before.

EXAMPLES.—1. Given $x^5-3\,ax^4+6\,a^2x^3+6\,a^3x^2-3\,a^4x+a^5$, to find the roots.

First, dividing by $x+a$, the quotient is $x^4-4x^3a+10x^2a^2-4xa^3+a^4=o$; wherefore dividing this by x^2a^2, according to the preceding rule, the quotient is $\frac{x^2}{a^2}+\frac{a^2}{x^2}-4.\frac{x}{a}+\frac{a}{x}+10=o$; *let* $z=\frac{x}{a}+\frac{a}{x}$, *then* $z^2=\frac{x^2}{a^2}+\frac{a^2}{x^2}+2$, *and substituting these values as before,* $z^2-4z+6=o$; *whence* $z=2\pm\sqrt{-2}$; *but since* $z=\frac{x}{a}+\frac{a}{x}$, *we have* $x^2-azx=-a^2$; *whence* $x=\frac{az\pm a\sqrt{z^2-4a}}{2}$, *and substituting for z its values found above, we obtain four of the roots, which together with $-a$, (since $x+a=o$,) make up the five roots of the equation.*

2. Given $x^3-ax^2-a^2x+a^3=o$, to find the roots. *Ans.* a, a, *and* $-a$.

3. Required the roots of $x^5+4ax^4-12a^2x^3-12a^3x^2+4a^4x+a^5=o$?

4. To find the roots of $x^7-ax^4-a^4x+a^7=o$.

CARDAN'S RULE FOR CUBIC EQUATIONS.

54. Let $x^3+ax=b$ be any cubic equation, wanting its second term; it is required to find one of its roots, according to Cardan's method[p].

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[p] This rule bears Cardan's name from the circumstance of his having been the first who published it, namely at Milan in 1545, in a work entitled, *Ars Magna*: but it was invented first, in or about the year 1505, by Scipio Ferreus, Professor of Mathematics at Bononia; and afterwards, viz. in 1535, by Nicholas Tartalea, a respectable mathematician of Brescia: from the latter Cardan contrived to extract the secret, which he afterwards published in violation of the most solemn protestations. The rules which Cardan thus obtained were for the three cases $x^3+bx=c$, $x^3=bx+c$, and $x^3+c=bx$; and it must be acknowledged in justice to him, that he greatly improved them, extending them to all forms and varieties of cubic equations, in a manner highly creditable to his abilities as a mathematician. See Tartalea's *Quesiti et Inventioni diverse*, ch. 9. Bossut's *Hist. of the Math.* p. 207. Montucla's *Hist. des Math.* t. 1. p. 591. Dr. Hutton's *Math. Dict.* vol. 1. p. 68—77.

The root obtained by this method is always real, although not always the greatest root of the equation: and it is remarkable, that this rule always exhibits the root under an imaginary form, when all the roots of the equation are real; and under a real form, when two of the roots are imaginary. See Dr. Hutton's Paper on Cubic Equations, in the *Philosoph. Trans.* for 1780.

Assume $y+z=x$, and $3yz=-a$; substitute these values for x and a in the proposed equation, it becomes ($y^3+3y^2z+3yz^2+z^3+a.\overline{y+z}=y^3+z^3+3yz.\overline{y+z}+a.\overline{y+z}=y^3+z^3-a.\overline{y+z}+a.\overline{y+z}=$) $y^3+z^3=b$; from the square of this take four times the cube of $yz=-\frac{a}{3}$, and the result is $y^6-2y^3z^3+z^6=b^2+\frac{4a^3}{27}$, the square root of which is $y^3-z^3=\sqrt{b^2+\frac{4a^3}{27}}$; but $y^3+z^3=b$; wherefore the sum and difference of these two equations being taken, the former is $2y^3=b+\sqrt{b^2+\frac{4a^3}{27}}$, and the latter $2z^3=b-\sqrt{b^2+\frac{4a^3}{27}}$, whence is found $y=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, and $z=\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, whence $x=(y+z=)$ $\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}+\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, which is Cardan's theorem: but the rule may be exhibited in a form rather more convenient for practice; thus, because $z=-\frac{a}{3y}$, we have x

$$=(y+z=y-\frac{a}{3y}=)\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}-\frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}};$$

whence the rule is as follows.

55. RULE I. If the given equation have all its terms, let the second term be taken away by Art. 37.

II. Instead of a and b in either of the above general theorems, substitute the coefficients of the corresponding terms, with their proper signs, in the transformed equation; then, proceeding according to the theorem, the root will be obtained.

If a be negative, and $\frac{1}{27}a^3$ greater than $\frac{1}{4}b^2$, the root cannot be found by this rule[q].

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[q] This is called the *Irreducible Case*; it exhibits the root, although real, under an impossible form: thus the root of the equation $x^3-15x=4$ is 4, but by Cardan's rule it is $\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$, an impossible form.

EXAMPLES.—1. Given $x^3+6x=88$, to find the value of x.

Here the second term is wanting, wherefore $a=6$, $b=88$, and

$$x=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}-\frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}}=$$

$$\sqrt[3]{\frac{88}{2}+\sqrt{\frac{\overline{88}|^2}{4}+\frac{\overline{6}|^3}{27}}}-\frac{\frac{6}{3}}{\sqrt[3]{\frac{88}{2}+\sqrt{\frac{\overline{88}|^2}{4}+\frac{\overline{6}|^3}{27}}}}=$$

Let the cube root of each of these imaginary expressions be extracted, they become $2+\sqrt{-1}+2-\sqrt{-1}$, which being added together, the impossible parts destroy each other, and the sum is 4, agreeably to what has been observed. It is remarkable, that this case never occurs except when the equation has three real roots, as we have before observed.

The irreducible case has exercised the abilities of the greatest algebraists for these three hundred years past, but its solution still remains among the desiderata in science. Dr. Wallis thought he had discovered a general rule, but it was afterwards found to apply only to particular cases. Baron Maseres gave a series, which he deduced by a laborious train of algebraic reasoning from Newton's Binomial Theorem, whereby this case is resolved without the intervention of either negative or impossible quantities. Dr. Hutton has likewise discovered several series applicable to the solution: (see *Philos. Trans.* vol. 68. and 70.) other series for this purpose may be seen in *Clairault's Algebra*, p. 5. Art. 19. *Bossut's Algebra*, Art. 178-9. *Landen's Lucubrations*, *La Caille's Leçons de Math.* Art. 399. &c. *Lorgna's Memoirs of the Italian Academy*, t. 1. p. 707. &c.

The irreducible case may be easily solved by trigonometry; as early as 1579, Bombelli shewed that angles are trisected by the resolution of a cubic equation. Vieta, in 1615, shewed how to resolve cubics and higher equations by angular sections. In 1629, Albert Girard solved the irreducible case by a table of sines, giving a geometrical construction of the problem, and exhibiting the roots by means of the hyperbola and circle. Halley, De Moivre, Emerson, Simpson, Crakelt, Cagnoli, Wales, Maskelyne, Thacker, &c. have employed the same method of sines: and lastly, Mr. Bonnycastle, Professor of the Mathematics at the Royal Military Academy, Woolwich, has communicated additional observations on the irreducible case, and an improved solution by a table of natural sines. *See Hutton's Math. Dict.* vol. 2. p. 743-4.

When one root is obtained by Cardan's rule, the two other roots may be derived not only by depressing the equation, as in ex. 1. but likewise as follows: let $r=$ Cardan's root, and v and $w=$ the two other roots, then will $v+w=-r$, and $vwr=b$, whence $v=-\frac{r}{2}\pm\frac{1}{2}\sqrt{\frac{r^3-4b}{r}}$, and $w=-\frac{r}{2}\mp\frac{1}{2}\sqrt{\frac{r^3-4b}{r}}$.

$\sqrt[3]{44+\sqrt{1936+8}}-\dfrac{2}{\sqrt[3]{44+\sqrt{1936+8}}}=\sqrt[3]{44+44.090815}-\dfrac{2}{\sqrt[3]{44+44.090815}}=4.449-.449=4=$ *the root required.*

If the two remaining roots be required, *depress the given equation,* (*Art.* 33.) *thus* $\left(\dfrac{x^3+6x-88}{x-4}=\right)$ $x^2+4x+22=o$, *of which the roots* (*found by the rule for quadratics, Vol. I. P.* 3. *Art.* 97.) *are* $-2\pm3\sqrt{-2}$.

2. Given $y^3-6y^2+3y-4=o$, to find the value of y.

First, to take away the second term, (*Art.* 37.) *let* $y=\left(x+\dfrac{6}{3}=\right)x+2$.

$$\begin{array}{llllll} \textit{Then } y^3 = & x^3 & +6x^2 & +12x & +8 \\ -6y^2 = & . & -6x^2 & -24x & -24 \\ +3y = & . & . & +3x & +6 \\ -4 = & . & . & . & -4 \\ \hline \end{array}$$

Whence $x^3 \ * \ -9x-14=o$, *or* $x^3-9x=14$.

Here $a=-9$, $b=14$, *and* $x=\sqrt[3]{7+\sqrt{49-27}}-\dfrac{3}{\sqrt[3]{7+\sqrt{49-27}}}=\sqrt[3]{7+\sqrt{22}}-\dfrac{-3}{\sqrt[3]{7+\sqrt{22}}}=\sqrt[3]{7+4.690415}-\dfrac{-3}{\sqrt[3]{7+4.690415}}=\sqrt[3]{11.690415}-\dfrac{-3}{\sqrt[3]{11.690415}}=2.269-\dfrac{-3}{2.269}=2.269+1.322=3.591$, *the root or value of* x; *wherefore* $y=(x+2=)\ 5.591=$ *the root of the proposed equation.*

3. Let $y^3+3y^2+9y=13$ be given, to find y.

Here, putting $y=x-1$, *the equation is transformed* (*Art.* 37.) *into* $x^3+6x=20$; *whence* $a=6$, $b=20$, *and* $x=\sqrt[3]{10+\sqrt{108}}-\dfrac{2}{\sqrt[3]{10+\sqrt{108}}}=\sqrt[3]{20.3923}-\dfrac{2}{\sqrt[3]{20.3923}}=2.732-.732=2$; *wherefore* $y=(x-1=)\ 2-1=1$, *the root required.*

4. Given $x^3-12x=16$, to find x. *Ans.* $x=4$.

5. Given $x^3-6x=-9$, to find x. *Ans.* $x=-3$.

6. Given $y^3+30y=117$, to find y. *Ans.* $y=3$.

7. Given $y^3+24y=250$, to find y. *Ans.* $y=5.05$.

8. Given $y^3-15y^2+81y=243$, to find y. *Ans.* $y=9$.

9. Given $y^3-6y^2+10y-8=o$, to find y. *Ans.* $y=4$.

10. Given $y^3+20y=100$, to find y.

COMPLETING THE CUBE.

55. B. In every complete cubic equation, having its signs either all $+$, or alternately $+$ and $-$, if the coefficient of the third term be equal to three times the square of one third of the coefficient of the second term, the cube may be completed by adding the cube of one third the coefficient of the second term, with its proper sign, to both sides of the equation; and then, by extracting the cube root from both sides, the root of the equation will be found [r].

EXAMPLES.—1. Given $x^3+6x^2+12x=56$, to find the value of x.

Here $\frac{1}{3}$ *of* $6=2$, *and* $12=3\times 2^2$; *wherefore adding* $\overline{2}|^3$ *to both sides, the given equation becomes* $x^3+6x^2+12x+8=(56+8=)\ 64$. *The cube root of this is* $x+2=4$; *wherefore* $x=2$.

2. Given $x^3-12x^2+48x=61$, to find x.

Here $\frac{1}{3}$ *of* $-12=-4$, *and* $3.\overline{-4}|^2=48$; *wherefore* $\overline{-4}|^3=-64$ *is to be added, and the equation becomes* $x^3-12x^2+48x-64=(189-64=)\ 125$. *The cube root of which is* $x-4=5$; *whence* $x=9$.

3. Given $6x^3-90x^2+450x=729.75$, to find x.

First, dividing by 6, *we have* $x^3-15x^2+75x=121.625$. *Also* $\frac{1}{3}$ *of* $-15=-5$, $3.\overline{-5}|^2=+75$, *and* $\overline{-5}|^3=-125$, *to be added; wherefore* $x^3-15x^2+75x-125=(121.625-125=)-3.375$; *and* $x-5=(\sqrt{-3.375}=)-1.5$, *wherefore* $x=(5-1.5=)\ 3.5$.

4. Given $x^3+3x^2+3x=26$, to find x. *Ans.* $x=3$.

5. Given $x^3-18x^2+108x=189$, to find x. *Ans.* $x=-3$.

6. Given $x^3+21x^2+147x=400$, to find x.

7. Given $x^3-21x^2+147x=-64$, to find x.

8. Given $2x^3-x^2+\frac{2x}{27}=\frac{1}{2}$, to find x.

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[r] This rule is evident; for let $\overline{x\pm a}|^3\ (=)\ x^3\pm 3ax^2+3a^2x\pm a^3$ be a complete cube, it is plain that $\pm a$ is $\frac{1}{3}$ the coefficient of the second term, $3.\pm a^2$ = the coefficient of the third term, and the cube of $\pm a$, or $\pm a^3$ the third term; wherefore if $x^3\pm 3ax^2+3a^2x=b$ be given, it is plain that the cube is completed by adding the cube of one third the coefficient of the second term to both sides, making $x^3\pm ax^2+a^2x\pm a^3=b\pm a^3$; then extracting the cube root $x\pm a=\sqrt[3]{b\pm a^3}$, and $x=\mp a+\sqrt[3]{b\pm a^3}$, which is the rule.

The root of a complete cube is found by taking the root of the first term and the root of the last, and connecting them by the sign of the last.

56. DES CARTES' RULE FOR BIQUADRATIC EQUATIONS *.

RULE I. Take away the second term from the given equation, (Art. 37.) and it will be reduced to this form, $x^4+ax^2+bx+c=o$; wherein the coefficients a, b, and c, may represent any quantities whatever, either positive or negative.

II. Assume the product $\overline{x^2+px+q}.\overline{x^2+rx+s}$ equal to the transformed equation $x^4+ax^2+bx+c=o$, and let the two factors be actually multiplied together; then will the product

* Lewis Ferrari, the friend and pupil of the celebrated Cardan, was the first who discovered a rule for the solution of biquadratics; namely, about the year 1540. His rule, which is called the *Italian method*, was first published by Cardan with a demonstration, and likewise its application to a great variety of suitable examples: it proceeds on a very general principle, completing one side of the equation up to a square by the help of multiples, or parts of its own terms, and an assumed unknown quantity; the other side is then made a square, by assuming the product of its first and third terms equal to the square of half the second: then by means of a cubic equation, and other circumstances, the management of which greatly depends on the skill and judgment of the operator, the root is found.

The rule we have given above was invented by that eminent French philosopher and mathematician, René Des Cartes, whose name it bears; and was first published in his Geometry, lib. 3. in 1637, but without any investigation: like Ferrari's method, it requires the intervention of a cubic and two quadratics; both methods are sufficiently laborious, but that of Des Cartes has in some respects the preference.

The reason of the rule is extremely obvious; for it is plain that any biquadratic may be considered as the product of two quadratics; and if the coefficients of the terms of these latter can be found in terms of a, b, c, &c. the coefficients of the transformed biquadratic, (as we have shewn they can by means of a cubic, &c.) then those quadratics being solved, their roots will evidently be those of the transformed biquadratic, from whence the roots of the given equation will be known.

All the roots of a complete biquadratic equation will be real and unequal. First, when $\frac{3}{8}$ of the square of the coefficient of the second term is greater than the product of the coefficients of the first and third terms. Secondly, when $\frac{3}{8}$ the square of the coefficient of the fourth term is greater than the product of the coefficients of the third and fifth terms. Thirdly, when $\frac{4}{9}$ the square of the coefficient of the third term is greater than the product of the coefficients of the second and fourth terms: in all other cases besides these three, the complete biquadratic equation will have imaginary roots.

$$\left.\begin{matrix}x^4 + p\\ + r\end{matrix}\right\} x^3 \left.\begin{matrix}+s\\ +q\\ +pr\end{matrix}\right\} x^2 \left.\begin{matrix}+ps\\ +qr\end{matrix}\right\} x + qs =$$
$$x^4 \quad * \quad +ax^2 \quad + bx \quad + \quad c.$$

III. Make the coefficients of the same power of x on each side this equation equal to each other, in order to find the values of the assumed coefficients p, q, r, and s; then will $p+r=o$, $s+q+pr=a$, $ps+qr=b$, *and* $qs=c$; from the first of these we get $r=-p$, from the second $s+q=(a-pr=$since $r=-p)$ $a+p^2$, and from the third $s-q=\frac{b}{p}$.

IV. From the square of the last but one, subtract the square of the last, and $4qs=a^2+2ap^2+p^4-\frac{b^2}{p^2}$, or (since $qs=c$) $4c=a^2+2ap^2+p^4-\frac{b^2}{p^2}$, which equation reduced, is $p^6+2ap^4+\overline{a^2-4c}.p^2=b^2$, from the solution of which (by Cardan's rule or otherwise) the value of p will be found.

V. Having discovered p, the value of $s=\frac{a}{2}+\frac{p^2}{2}+\frac{b}{2p}$, and that of $q=\frac{a}{2}+\frac{p^2}{2}-\frac{b}{2p}$, will likewise be thence determined; that is, (since $r=-p$,) all the quantities in the two assumed factors $\overline{x^2+px+q}.\overline{x^2+rx+s}$, except the value of x, are known.

VI. Next, find the roots of the two assumed quadratics $x^2+px+q=o$, and $x^2+rx+s=o$, and we shall have, from the former, $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$; and from the latter, $x=(-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-s}$, or since $r=-p=)\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-s}$. Wherefore the four roots of the *transformed* biquadratic equation $x^4+ax^2+bx+c=o$, are $\frac{p}{2}+\sqrt{\frac{p^2}{4}-s}$ .... $\frac{p}{2}-\sqrt{\frac{p^2}{4}-s}$ .... $-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}$, and $-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}$; the roots of the proposed equation.

EXAMPLES.—1. To find the four roots of the biquadratic $z^4-4z^2-8z+32=o$.

First, to take away the second term, (Art. 37.) let $z = x + 1$; *then will*

$$\begin{array}{rl} z^4 = & x^4 + 4x^3 + 6x^2 + 4x + 1 \\ -4z^3 = & \quad -4x^3 - 12x^2 - 12x - 4 \\ -8z = & \ldots\ldots\ldots\ldots - 8x - 8 \\ +32 = & \ldots\ldots\ldots\ldots\ldots\ldots + 32 \\ \hline & x^4 - 6x^2 - 16x + 21 = 0 \end{array}$$

Here, putting $a = -6$, $b = -16$, *and* $c = +21$, *the assumed cubic* ($p^6 + 2ap^4 + a^2 - 4c.p^2 = b^2$) *becomes by substitution* $p^6 - 12p^4 - 48p^2 = 256$. *From this, let the second term be taken away, by putting* $p^2 = y + 4$; *then will*

$$\begin{array}{rl} p^6 = & y^3 + 12y^2 + 48y + 64 \\ -12p^4 = & \quad -12y^2 - 96y - 192 \\ -48p^2 = & \ldots\ldots\ldots - 48y - 192 \\ -256 = & \ldots\ldots\ldots\ldots\ldots - 256 \\ \hline & y^3 - 96y = 576 \end{array}$$

To find the root of this equation by Cardan's rule, (Art. 54, 55.) here $a = -96$, $b = 576$, *and* $\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \dfrac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}} = \sqrt[3]{288 + \sqrt{82944 - 32768}} - \dfrac{-32}{\sqrt[3]{288 + \sqrt{82944 - 32768}}} = 12 = y$; *wherefore* $p = (\sqrt{y+4} =)\ 4$, *whence* $s = (\frac{a}{2} + \frac{p^2}{2} + \frac{b}{2p} = \frac{-6}{2} + \frac{16}{2} + \frac{-16}{8} =)\ 3$; $q = (\frac{a}{2} + \frac{p^2}{2} - \frac{b}{2p} =)\ 7$, $r = (-p =) -4$.

Wherefore the two quadratics to be solved[1], *viz.* $x^2 + px + q = 0$, *and* $x^2 + rx + s = 0$, (*by substituting the above values of* p, q, r, *and* s,) *become* $x^2 + 4x = -7$, *and* $x^2 - 4x = -3$; *the two roots of the former of these are* $x = -2 \pm \sqrt{-3}$; *and of the latter,* $x = 3$;

---

[1] We have the solution of both these quadratics (or rather their answers) in general terms, in the rule; viz. $\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - s}$, and $-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$, in which the values of p, q, and s, being substituted, the roots of the transformed equation will come out as before.

and 1. *Wherefore the four roots of the transformed equation* $x^4-6x^2-16x+21=o$, *are* $-2+\sqrt{-3}$ ... $-2-\sqrt{-3}$ ... 3 *and* 1; *but since* $z=x+1$, *by adding unity to each of these roots, we shall have the four roots of the given equation* $z^4-4z^3-8z+32=o$, *as follows*; $z=-1+\sqrt{-3}$, $z=-1-\sqrt{-3}$, $z=4$, *and* $z=2$, *as was required* [u].

2. Given $z^4-4z^3-3z^2-4z+1=o$, to find the values of z. *Ans.* $z=\frac{5\pm\sqrt{21}}{2}$ *and* $\frac{-1\pm\sqrt{-3}}{2}$.

3. To find the roots of $x^4-3x^2-4x-3=o$. *Ans.* $x=\frac{1\pm\sqrt{3}}{2}$ *and* $\frac{-1\pm\sqrt{-3}}{2}$.

57. EULER'S RULE FOR BIQUADRATIC EQUATIONS [v].

RULE I. Let $x^4-ax^2-bx-c=o$, be a general biquadratic equation wanting its second term, and let $f=\frac{a}{2}$, $g=\frac{a^2}{16}+\frac{c}{4}$, and $h=\frac{b^2}{64}$.

II. With these values of f, g, and h, let the cubic equation $z^3-fz^2+gz-h=o$ be formed, and let its three roots (found by any of the preceding methods) be p, q, and r.

III. Then will the four roots of the proposed biquadratic be as follows, viz.

	When $\frac{1}{8}b$ is positive	When $\frac{1}{8}b$ is negative.
1st root,	$x=\sqrt{p}+\sqrt{q}+\sqrt{r}$	$x=\sqrt{p}+\sqrt{q}-\sqrt{r}$
2nd root,	$x=\sqrt{p}+\sqrt{q}-\sqrt{r}$	$x=\sqrt{p}-\sqrt{q}+\sqrt{r}$
3rd root,	$x=\sqrt{p}-\sqrt{q}+\sqrt{r}$	$x=-\sqrt{p}+\sqrt{q}+\sqrt{r}$
4th root,	$x=\sqrt{p}-\sqrt{q}-\sqrt{r}$	$x=-\sqrt{p}-\sqrt{q}-\sqrt{r}$

[u] This rule applies to that case only in which two of the roots are possible, and two impossible.

[v] The learned and venerable Leonard Euler, joint Professor of Mathematics at the University of Petersburg, was the inventor of this method; which he first published in the 6th volume of the Petersburg Commentaries for the year 1738; and afterwards in his Algebra, translated from the German into French, in 1774, and lately into English.

EXAMPLES.—1. Given $x^4-25x^2+60x-36=o$, to find the four roots.

Here $a=25$, $b=-60$, *and* $c=36$; *wherefore* $f=(\frac{a}{2}=)\ \frac{25}{2}$, $g=(\frac{a^2}{16}+\frac{c}{4}=)\ \frac{769}{16}$, *and* $h=\frac{225}{4}$; *consequently by substituting these values in the cubic equation* $z^3-fz^2+gz-h=o$, *it becomes* $z^3-\frac{25}{2}z^2+\frac{769}{16}z-\frac{225}{4}=o$.

The three roots of this equation being found, will be $z=\frac{9}{4}=p$, $z=4=q$, *and* $z=\frac{25}{4}=r$; *and since* $\frac{1}{8}b$ *is negative, the four roots will be*

$$x=\ \sqrt{p}+\sqrt{q}-\sqrt{r}=\ \sqrt{\frac{9}{4}}+\sqrt{4}-\sqrt{\frac{25}{4}}=\ 1$$

$$x=\ \sqrt{p}-\sqrt{q}+\sqrt{r}=\ \sqrt{\frac{9}{4}}-\sqrt{4}+\sqrt{\frac{25}{4}}=\ 2$$

$$x=-\sqrt{p}+\sqrt{q}+\sqrt{r}=-\sqrt{\frac{9}{4}}+\sqrt{4}+\sqrt{\frac{25}{4}}=\ 3$$

$$x=\ \sqrt{p}-\sqrt{q}-\sqrt{r}=-\sqrt{\frac{9}{4}}-\sqrt{4}-\sqrt{\frac{25}{4}}=-6$$

2. Given $x^4-5x^2+4=o$, to find the roots. *Ans.* $x=+1$, $+2$, -1, *and* -2.

3. Given $x^4-3x^3-36x^2+68x+240=o$, to find the roots. *Ans.* $x=-2$, -5, $+4$, *and* $+6$.

4. Find the roots of $x^4+x^3-29x^2-9x+180=o$. *Ans.* $x=3$, 4, -3, *and* -5.

5. Find the roots of $y^4-4y^3-19y^2+46y+120=o$.

58. SIMPSON'S RULE FOR BIQUADRATIC EQUATIONS[x].

This method supposes the given biquadratic to be equal to the difference of two assumed squares; thus,

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[x] This rule was first given by Mr. Thomas Simpson, Professor of the Mathematics at the Royal Military Academy, Woolwich; and published in the second edition of his Algebra, about the year 1747: it is in some instances preferable to either of the preceding methods, and some trouble is saved by it, as *here* we are not under the necessity of exterminating the second term from the complete biquadratic equation, which in the preceding rules is indispensable.

RULE I. Let $x^4+ax^3+bx^2+cx+d=o$, be the proposed equation, and equal to the difference $\overline{x^2+\frac{1}{2}ax+A}|^2-\overline{Bx+C}|^2$.

II. Square the two latter quantities, making the difference of the squares equal to the proposed equation, and you will have

$$\left.\begin{array}{l} x^4+ax^3+2Ax^2 \ldots\ldots\ldots\ldots \\ \quad +\frac{1}{4}a^2x^2+aAx+A^2 \\ \quad -B^2x^2-2BCx-C^2 \end{array}\right\}=x^4+ax^3+bx^2+cx+d=o.$$

III. Make the coefficient of x in each term on one side of the equation, equal to the coefficient of the same power of x on the other; then will

First, $2A+\frac{1}{4}a^2-B^2=b$, or $2A+\frac{1}{4}a^2-b=B^2$.

Secondly, $aA-2BC=c$, or $aA-c=2BC$.

Thirdly, $A^2-C^2=d$, or $A^2-d=C^2$.

IV. Multiply the first and last of these equations together, and the product (B^2C^2) will evidently be equal to $(\frac{1}{4}4B^2C^2)$ one fourth the square of the second; that is, $2A^3+\overline{\frac{1}{4}a^2-b}.A^2-2dA-d.\overline{\frac{1}{4}a^2-b}=\frac{1}{4}.\overline{a^2A^2-2acA+c^2}$.

V. Let $k=\frac{1}{4}ac-d$, $l=\frac{1}{4}c^2+d.\overline{\frac{1}{4}a^2-b}$; and by this substitution, the preceding equation will become $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=o$.

VI. Find the root or value of A in this cubic equation, by any of the foregoing methods; which being done, B and C will likewise be known, since $B=\sqrt{2A+\frac{1}{4}a^2-b}$, and $C=\frac{aA-c}{2B}$.

VII. And since the proposed quantity $x^4+ax^3+bx^2+cx+d$ is equal to nothing, its equal $\overline{x^2+\frac{1}{2}ax+A}|^2-\overline{Bx+C}|^2$ will likewise be equal to nothing; wherefore it follows, that $\overline{x^2+\frac{1}{2}ax+A}|^2=\overline{Bx+C}|^2$.

VIII. Extract the square root from both sides of this equation, and $x^2+\frac{1}{2}ax+A=\pm Bx\pm C$, whence $x^2+\overline{\frac{1}{2}a\mp B}.x=\pm C-A$; which equation solved, gives $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm\sqrt{\frac{1}{16}a^2\mp\frac{1}{4}aB+\frac{1}{4}B^2\pm C-A}$; wherein all the four roots of the given equation are exhibited, according to the variations of the signs [y].

EXAMPLES.—1. Given $x^4-6x^3-58x^2-114x-11=o$, to find the values of x.

Here $a=-6$, $b=-58$, $c=-114$, and $d=-11$, whence $k=(\frac{1}{4}ac-d=)$ 182, $l=(\frac{1}{4}c^2+d.\overline{\frac{1}{4}a^2-b}=)$ 2512; whence by substituting these values in the cubic equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=o$, it becomes $A^3+29A^2+182A-1256=o$, the root of

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[y] Dr. Hutton remarks, that Mr. Simpson has subjoined an observation to this rule, which has since been proved to be erroneous; namely, that "the value of A, in this equation, will be *commensurate* and *rational*, (and therefore the easier to be discovered,) not only when all the roots of the given equation are *commensurate*, but when they are *irrational*, and even impossible; as will appear from the examples subjoined." This, continues the Doctor, is a strange reason for Simpson to give in proof of a proposition: and it is wonderful that he fell on no examples that disprove it, as the instances in which his assertion holds true, are very few indeed in comparison with those in which it fails. *Math. Dict.* vol. I. p. 211.

When either $A=o$, $B=o$, or $C=o$, the roots of the proposed biquadratic will be obtained by the resolution of a quadratic only. *Simpson's Alg. 6th edit.* p. 155.

Besides the rules by Ferrari, Des Cartes, Euler, and Simpson, two other rules for the solution of biquadratics have been discovered: one by La Fontaine, of the Royal Academy of Sciences at Paris, and inserted in the Memoirs of that learned society for 1747; and the other by Dr. Edward Waring, Lucasian Professor of Mathematics at Cambridge, in a profound work, entitled, *Meditationes Algebraicæ*, published in the year 1770. Attempts have not been wanting to discover methods of resolving equations of the higher orders, but they have hitherto been unsuccessful; no general rule for the solution of affected equations above the fourth power, has yet been discovered.

which (found by Cubics) is $4=A$; *whence* $B=(\sqrt{2A+\frac{1}{4}a^2-b}=)\ 5\sqrt{3}$, $C=(\frac{aA-c}{2B}=)\ 3\sqrt{3}$, *and* $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm\sqrt{\frac{1}{16}a^2\mp\frac{1}{4}aB+\frac{1}{4}B^2\pm C-A}=\pm\frac{5}{2}\cdot\sqrt{3}+\frac{3}{2}\pm\sqrt{17\pm\frac{21}{2}\sqrt{3}}$ $=11.761947$, *or* 3.101693, *or* $+2.830127\pm\sqrt{-1.1865334798}$, *for the four roots; the two latter, expressed by the double sign, are impossible.*

2. Let the roots of $x^4-6x^3+5x^2+2x-10=o$, be found. *Ans.* $x=5, -1, 1+\sqrt{-1}$, *and* $1-\sqrt{-1}$.

3. Given $x^4-12x-17=o$, to find the values of x. *Ans.* $x=2.0567$, *or* $.6425$, *or* $.7071\pm\sqrt{-4\ 7426406}$.

4. Given $x^4-25x^2+60x=-36$, to find the roots. *Answer* $x=3, 2, 1$, *and* -6.

5. Given $x^4-x^3+2x^2-3x+20=o$, to find the roots.

RESOLUTION OF EQUATIONS BY APPROXIMATION [*].

59. The foregoing rules require for the most part great labour and circumspection, and after all, they are applicable

[*] Methods of approximating to the roots of numbers, were employed as early as the time of Lucas de Burgo, who flourished in the 15th century; but the first who are known to have applied the doctrine to the resolution of equations, were Stevinus of Bruges, and Vieta, a celebrated mathematician of Lower Poitou; the former in his Arithmetic, printed at Leyden, in 1585, and in his Algebra, published a little later; and the latter in his *Opera Mathematica*, written about the year 1600, and published by Van Schooten, in 1646. Their methods, although in some respects improved by Oughtred in his *Key to the Mathematics*, 1648, were still very tedious and imperfect: to remedy these defects, Sir Isaac Newton turned his attention to the subject, and it is to his successful application to this branch, that we are principally indebted for a general, easy, and expeditious method of approximating to the roots of all sorts of adfected equations, as may be seen in his tract *De Analysi per Equationes numero terminorum infinitas*, 1711, and elsewhere. Dr. Halley invented two rules for the same purpose, one called his *rational theorem*, and the other, his *irrational theorem*, both of which are still justly esteemed for their utility. This necessary part of Algebra is likewise indebted to the labours of Wallis, Raphson, De Lagni, Thomas Simpson, and others; whose methods have been given by various writers on the subject.

only to equations of particular kinds, all of which taken together, form but a small part of the numerous kinds and endless variety of algebraic problems, which may be proposed. But as we have no general rules whereby the roots of high equations can be found, we must be content to approximate as near to the required root as possible, when it cannot be found exactly.

60. The methods of approximation are general, including equations of every kind and description, applying equally to the foregoing equations, and to all others which do not come under the preceding rules: hence approximation is the most general, easy, and useful method of discovering the possible roots of numeral equations, that can be proposed.

61. It must be observed, that one root only is found by these methods, and that not exactly, but nearly. We begin by making trials of several numbers, which we judge the most likely to answer the conditions of the proposed equation; then, (by a process to be described hereafter,) we find a number nearer than that obtained by trial; we repeat the process, and thereby obtain a number nearer than the last; again we repeat the process, and obtain a number still nearer, and so on, to any assignable degree of exactness.

62. *The simplest method of approximation.*

RULE I. Find by trials a number nearly equal to the root of the proposed equation.

II. Let r=the number thus found, and let z=the difference between r and the root x of the equation: so that if r be less than x, then $r+z=x$; but if r be greater than x, then $r-z=x$.

III. Instead of x in the given equation, substitute its equal $r+z$, or $r-z$, (according as r is less or greater than x,) and a new equation will arise, including only z and known quantities.

IV. Reject every term in this equation which contains any power of z higher than the first, and the value of z will be found by a simple equation.

V. If the sign of the value of z be $+$, this value must be added to the value of r; but if $-$, it must be subtracted, and the result will be nearly equal to the root required.

VI. If this root be not sufficiently near the truth, let the operation be repeated; thus, instead of r in the equation just now resolved, substitute the corrected root, and the second

value of z being added or subtracted according to its sign, a nearer approximation to the root will be had; and if a still nearer approximation be required, the operation may be repeated at pleasure, observing always to substitute the last corrected root for the new value of r.

EXAMPLES.—1. Given $x^2+x=14$, to find x by approximation.

By trials it soon appears that x must be nearly equal to 3; let therefore $r=3$, and $r+z=x$; wherefore substituting this value of x in the given equation, it becomes $\overline{r+z}|^2+r+z=14$, that is, $r^2+2rz+z^2+r+z=14$; whence by transposition, and rejecting z^2, we obtain $2rz+z=14-r^2-r$, and $z=\frac{14-r^2-r}{2r+1}=\frac{14-9-3}{6+1}$ $=\frac{2}{7}=.28$, and $x=(r+z=3+.28=)$ 3.28, nearly.

For a nearer value of x, *let the operation be repeated.*

Thus, let $r=3.28$; and substituting this value for r in the equation $z=\frac{14-r^2-r}{2r+1}$, it becomes $z=(\frac{14-10.7584-3.28}{6.56+1}=$ $\frac{-.0384}{7.56}=)-.00508$, nearly; wherefore $x=(r+z=3.28-$ $.00508=)$ 3.27492, extremely near.*

2. Let $x^3-2x^2+3x=5$ be given, to find x.

It appears by trials, that $x=3$ nearly, wherefore let $r=3$, and $r+z=x$ as before; then will

$$\left.\begin{array}{rl} x^3= & r^3+3r^2z+3rz^2+\ z^3 \\ -2x^2= & -2r^2\ -4rz\ -2z^2 \\ +3x= & \ldots\ldots\ldots 3r\ \ +3z \end{array}\right\}=5.$$

From which, rejecting all the terms which contain z^2 or z^3, we obtain $(r^3+3r^2z-2r^2-4rz+3r+3z=5$, or$)$ $3r^2z-4rz+$

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* Sometimes it happens that the correction consists of several figures; in that case, if a second operation be necessary, it will be convenient not to substitute all the figures for r, but only one figure, or two, such as will nearly express the value of the whole: thus, if x after the first operation be 3.58; for a second operation I will put $r=$(not 3.58, but) 3.6; if at the conclusion of this second process $x=3.648917$, and a third be deemed necessary, I will not employ all these figures, but instead of them put $r=3.65$, and proceed. This method is to be attended to in all cases, as it saves much trouble, and produces scarcely any effect on the approximation.

$3z=5-r^3+2r^2-3r$; *whence* $z=\frac{5-r^3+2r^2-3r}{3r^2-4r+3}=$ $\left(\frac{5-27+18-9}{27-12+3}=\frac{-13}{18}=\right)-.7$; *whence* $x=(3-.7-)\ 2.3$ *nearly.*

For a nearer approximation,

Let $r=2.3$, *this value substituted for r in the preceding equation, we have* $z=\left(\frac{5-12.167+10.58-6.9}{15.87-9.2+3}=\frac{-3.487}{9.67}=\right)-.36$, *whence* $x=(2.3-.36=)\ 1.94$, *still nearer than before; and if* 1.94 *be substituted for r in the equation above alluded to, a third approximation will be had, whereby a nearer value of x will be obtained.*

3. Given $x^3-5x=31$, to find x. *Ans.* $x=8.6032778$.
4. Given $x^3+2x-40=o$, to find x. *Ans.* $x=5.403125$.
5. Given $x^3+x^2+x=90$, to find x. *Ans.* $x=4.10283$.
6. Given $2x^3+4x^2-245x-70=o$, to find x. *Ans.* $x=10.265$.
7. Given $x^4-12x+7=o$, to find x. *Ans.* $x=2.0567$.
8. Given $x^3+10x-20=o$, to find the value of x.

63. *The following method affords a swifter approximation to the unknown quantity than the former rule* [b].

RULE I. Let a number be found by trials nearly equal to the required root, and let $z=$ the difference of the assumed number and the true root, as before.

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[b] This method is given by Mr. Simpson in p. 162. of his Algebra, where he has extended the doctrine beyond what our limits will admit: the above rule is in its simplest form, and triples the number of figures true in the root, at every operation; he calls it an approximation of the *second degree*, ($z=\frac{p}{a}$ being the *first*;) and since $z=\frac{p}{a+bz+cz^2+,\ \&c.}$, if the first value of z (viz. $\frac{p}{a}$) be substituted in the second term of the denominator, and the following terms be rejected, it will become $z=\frac{ap}{a^2+bp}$, an approximation of the *second degree*, the same as the above rule. If for z its second value $q-\frac{bq^2}{a}$ be substituted, then $z=\frac{p}{a+bq-\frac{b^2}{a}-c.q^2}$, an approximation of the *third degree*, which

II. Substitute the assumed quantity $\pm z$, in the given equation, as directed in the preceding rule; and the given equation will be reduced to this form, $az+bz^2+cz^3+$, &c.$=p$.

III. By transposition and division we have $z=\frac{p}{a}-\frac{bz^2}{a}-\frac{cz^3}{a}$, &c. where, if all the terms after the first be rejected, we shall have $z=\frac{p}{a}$; and if q be put for $\frac{p}{a}$, and its square substituted for z^2 in the second term, we shall have $z=q-\frac{bq^2}{a}$.

EXAMPLES.—1. Given $x^3-2x^2+3x=5$, to find x.

Here $x=3$ *nearly; let* $3+z=x$, *then,*

$$\left.\begin{array}{rl} x^3= & 27+27z+9z^2+z^3 \\ -2x^2= & -18-12z-2z^2 \;.\;. \\ +3x= & 9+3z \;.\;.\;.\;.\;. \end{array}\right\}=5,\ \textit{that is,}$$

$$18+18z+7z^2+z^3=5,\ \textit{or}\ 18z+7z^2+z^3=-13.$$

Here $a=18$, $b=7$, $c=1$, $p=-13$, $q=(\frac{p}{a}=\frac{-13}{18}=)-.72$, *and* $q-\frac{bq^2}{a}=(-.72-\frac{7\times .5184}{18}=)-.9216=z$; *wherefore* $x=(3+z=3-.9216=)\ 2.0784$.

For a second approximation,

Let $2+z=x$; *then*

$$\left.\begin{array}{rl} x^3= & 8+12z+6z^2+z^3 \\ -2x^2= & -8-8z-2z^2 \;.\;. \\ +3x= & 6+3z \;.\;.\;.\;.\;. \end{array}\right\}=5,\ \textit{that is,}$$

$$6+7z+4z^2+z^3=5,\ \textit{or}\ 7z+4z^2+z^3=-1.$$

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by making $s=\frac{b}{a}-\frac{c}{b}$, multiplying both terms of the fraction by $1+sq$, and rejecting bs^2q^3 (as very small) from the product, becomes $\frac{\overline{a+sp}.p}{a^2+\overline{b+as}.p}$. By similar methods, and by putting $w=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}$, the approximating rule of the *fourth* degree is $\frac{ap.\overline{a+wp}}{a.a^2+\overline{b+aw}.p+\overline{w-s}.p^2}$, which quintuples the number of figures true at every operation.

Here $a=7$, $b=4$, $c=1$, $p=-1$, *and* $q=(\frac{p}{a}=\frac{-1}{7}=)-.14285$; *wherefore* $q-\frac{bq^2}{a}=(-.14285-\frac{4}{7}\times-\overline{.14285}|^2=)-.15451064=z$.

And $x=(2.0784-.15451064=)$ 1.92388936, *very nearly.*

2. Given $x^2+20x=100$, to find the value of x. *Ans.* $x=4.1421356$.

3. Given $x^3-2x=5$, to find x. *Ans.* $x=2.094551$.

4. Given $x^3-48x^2+200=o$, to find x. *Ans.* $x=47.91287847478$.

5. Given $x^4-38x^3+210x^2+538x+289=o$, to find x. *Answer,* $x=30.53565375285̇27$.

6. Given $x^5+6x^4-10x^3-112x^2-207x-110=o$, to find x. *Ans.* $x=4.4641016151$.

7. Given $2x^2+3x+4=50$, to find the value of x.

64. BERNOULLI'S RULE

Has been sometimes preferred on account of its great simplicity and general application: it is as follows.

RULE I. Find by trials, two numbers as near the true root as possible[c].

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[c] This is perhaps the most easy and general method of resolving equations of every kind, that has ever yet been proposed; it was invented by John Bernoulli, and published in the Leipsic Acts, 1697. The most intricate and difficult forms of equations, however embarrassed and entangled with radical, compound, and mixed quantities, readily submit to this rule without any previous reduction or preparation whatever; and it may be conveniently employed for finding the roots of exponential equations.

The rule is founded on this supposition, that the first error is to the second, as the difference between the true and first assumed number is to the difference between the true and second assumed number: and that it is true according to this supposition, may be thus demonstrated.

Let a and b be the two suppositions; A and B their results produced by similar operations; it is required to find the number from which N is produced by a like operation: in order to which,

Let $N-A=r$, $N-B=s$, and $x=$ the number required; then by hypothesis, $r:s::x-a:x-b$, whence dividendo $r-s:s::b-a:x-b$, that is, $\frac{\overline{b-a}.s}{r-s}=x-b$, which is the rule when both the assumed quantities, a and b, are less than the true root x.

II. Substitute these assumed numbers for the unknown quantity in the given equation, and mark the error which arises from each with the sign $+$, if it be too great, and $-$, if too little.

III. Multiply the difference of the assumed numbers by the least error, and divide the product by the difference of the errors when they have like signs, but by their sum when they have unlike.

IV. Add the quotient to the assumed number belonging to the least error, when that number is too little, but subtract when it is too great; the result will be the root, nearly.

V. The operation may be repeated, if necessary, as in the former rules, either by taking two new assumed numbers, or using one of the former numbers, and assuming a new one.

Examples.—1. Given $10x^4+9x^3+8x^2+7x=1234$, to find x.

Here by trials x appears to be greater than 3; wherefore let 3 and 4 be the two assumed numbers[d].

Next, let A and B be each greater than N, then will $N-A=-r$, and $N-B=-s$; but $-r:-s::+r:+s$, wherefore $r-s:s::a-b:b-x$, or $\frac{\overline{a-b}.s}{r-s}=b-x$, which is the rule when the assumed quantities, a and b, are each greater than x.

Lastly, let one result A be too little, and the other B too great; then will r be positive and s negative. Wherefore $r+s:(-s$, or, which is the same) $+s::a-b:b-x$, that is, $\frac{\overline{a-b}.s}{r+s}=b-x$, which is the rule, when one of the assumed quantities is too great, and the other too small. Q. E. D. All questions in double position are resolved by this method.

[d] The convenience of substituting two numbers which differ by unity is this, it saves the trouble of multiplying the least error by that difference. If the numbers substituted have decimal places, the same method is to be observed: thus, suppose they are 1.34 and 1.35, and the least error 12.5794, in this case the difference of the supposed numbers is .01, and the multiplication is performed by simply removing the decimal mark two places to the left, making the product .125794; and the like in other instances.

First Supposition. or $x=3$.	*Equation.*	*Second Supposition. or $x=4$.*
810	$=10x^4=$	2560
243	$=9x^3=$	576
72	$=8x^2=$	128
21	$=7x=$	28
1146	=*result*=	3292
-88	= *error* =	$+2058$

Difference of the assumed numbers $4+3=1$.

Least error 88. *Sum of the errors* (*they being unlike*) $88+2058=2146$; *wherefore* $\frac{1\times 88}{2146}=\frac{88}{2146}=.041$, *the correction to be added to* 3 *the number from whence the least error arises,* 3 *being too little; wherefore* 3.041 *is the root or value of x, nearly.*

2. Given $\sqrt{1+x}+\sqrt[3]{2+x^2}+\sqrt[4]{3+x^3}=16$, to find x.

From a few trials it appears that x is somewhat greater than 8, *wherefore assuming* 8 *and* 9 *for the values of x, the work will stand thus**.

First Supp. or $x=8$.	*Equation.*	*Second Supp. or $x=9$.*
3.	$=\sqrt{1+x}=$	3.16228
4.04124	$=\sqrt[3]{2+x^2}=$	4.36207
4.76378	$=\sqrt[4]{3+x^3}=$	5.20149
11.80502	=*result*=	12.72584
-4.19498	=*error*=	-3.27416

* The logarithms are of excellent service in all cases of this rule, where roots and powers are required to be found, or where the terms are mixed and complicated: thus in the present instance, supposing $x=8$, then $1+x=9$, the square root of which (viz. 3) immediately occurs; but let $x=9$, then $1+x=10$, to find the square root of which, by the common method, requires rather a long process. I therefore take the logarithm of 10, divide it by 2, (the index of the square,) and the quotient is a logarithm, the natural number corresponding to which is 3.16228, as above. Next, supposing $x=8$, then $\sqrt[3]{2+x_2}=\sqrt[3]{66}$. I find the logarithm of 66, divide it by 3, and the natural number agreeing with the quotient is 4.04124, as above. Let $x=9$, then $\sqrt[3]{2+x^2}=\sqrt[3]{83}$, which by a similar process is found to be 4.36207, as above. Lastly if $x=8$, then $\sqrt[4]{3+x^3}=\sqrt[4]{515}$; if $x=9$, then $\sqrt[4]{3+x^3}=\sqrt[4]{732}$, the roots of both which are found by a similar operation, and are as above, viz. 4.76378 and 5.20149. See Vol. I. Part 2. Art. 38.

Diff. of assumed numbers$=1$, *least error* 3.27416, *diff. of the errors (having like signs)* $4.19498-3.27416=.92082$; *wherefore* $\frac{3.27416}{.927416}=3.5309$, *the correction to be added; consequently* 12.5309 *is the value of x nearly.*

For a second approximation,

Let the numbers 11 *and* 12 *be assumed, then*

First Supp. or $x=11$.	*Equation.*	*Second Supp. or* $x=12$.
3.38525	$=\sqrt{1+x}=$	3.60555
4.9732	$=\sqrt[3]{2+x^2}=$	5.26563
6.0435	$=\sqrt[4]{3+x^3}=$	6.4502
14.40195	$=$*result*$=$	15.32138
-1.59805	$=$*error*$=$	$-.67862$

Least error .67862, *diff. of errors* $(1.59805-.67862=)$ .91943; *whence* $\frac{.67862}{.91943}=.73809$, *the correction to be added to* 12; *wherefore* $x=12.73809$, *very nearly.*

3. Given $x^{\frac{8}{9}}-\frac{4}{5}x^{\frac{5}{7}}+x\sqrt[3]{x^2+2x\sqrt{x^2+x}}-\frac{x+1}{x\sqrt{x-1}}=45$, to find the value of x.

Here x will be found by trials to be nearly equal to 10, *wherefore let* 10 *and* 11 *be two assumed numbers; then,*

First Supp. or $x=10$.	*Equation.*	*Second Supp. or* $x=11$.
7.74264	$=x^{\frac{8}{9}}=$	8.42718
-4.14358	$=-\frac{4}{5}x^{\frac{5}{7}}=$	-4.43549
67.6616	$=+x\sqrt[3]{x^2+2x\sqrt{x^2+x}}=$	79.2363
$-.36666$	$=-\frac{x+1}{x\sqrt{x-1}}=$	$-.34497$
70.894	$=$*result*$=$	82.88302
$+25.894$	$=$*error*$=$	$+37.88302$

Least error 25.894, *diff. of errors* $(37.88302-25.894=)$ 11.98902; *wherefore* $\frac{25.894}{11.98902}=2.1598$, *to be substracted from* 10: *consequently* $x=(10-2.1598=)$ 7.8402 *nearly; and if*

greater exactness be required, the operation may be repeated at pleasure, as in the second example.

4. Given $x^2+3x=20$, to find the value of x. *Ans.* $x=3.13939$.

5. If $x^3+x^2+x=20$, what is the value of x? *Ans.* $x=2.32174$.

6. Let $2x^3+3x^2+4x=100$ be given, to find x. *Ans.* $x=3.0896$.

7. Given $\frac{1}{4}x^3-12x^2-50=o$, to find x. *Answer,* $x=11.9782196186948$.

8. Given $\frac{x^5}{2}+3x^4-5x^3-56x^2-103\frac{1}{2}x=55$, to find x. *Ans.* $x=2.2320508075$.

9. Given $\sqrt{1+x^2}+\sqrt{2+x^2}+\sqrt{3+x^2}=10$, to find x. *Ans.* $x=3.0209475$.

10. Given $\frac{100x}{\sqrt{16+5x+x^2}}+\frac{x\sqrt{5+x^2}}{5}=170$, to find x.

EXPONENTIAL EQUATIONS.

By the foregoing rule, the roots of Exponential Equations may be approximated to, with the assistance of logarithms.

65. An exponential equation is that in which the indices, as well as some of the quantities themselves, are unknown quantities to be determined.

EXAMPLES.—1. Given $x^x=1000$, to find the value of x.

It appears by trials that x is greater than 4, *but less than* 5. *Let* 4.4 *and* 4.5 *be the numbers proposed.*

Then since $x\times$ *log. of* $x=$ *log. of* 1000, *that is,*

First, (4.4 × *log. of* 4.4=) $4.4\times0.6434527=2.83119188$

But the log. of $1000=3.00000000$

Error -0.16880812

Secondly, (4.5 × *log. of* 4.5=) $4.5\times0.6532125=2.93945625$

Log. of $1000=3.00000000$

Error -0.06054375

Subtract this error from the former, and the diff. is 0.10826437

Then $4.5-4.4=.1=$ *diff. of numbers found by trial, and* .06054375, *least error; therefore* $\frac{.1\times.06054375}{.10826437}=.055922$, *the*

correction; wherefore $x=(4.5+.055922=)$ 4.55922, *the answer, very nearly; for* $\overline{4.55922}|^{4.55922}=$ *(by logarithms)* 1009.315, *which result exceeds the truth by* 9.315.

To repeat the operation,

Let 4.55 *and* 4.56 *be the assumed numbers.*

$$\begin{array}{rr}
\textit{Then}\ (4.55\times \log. 4.55)=4.55\times 0.6580114= & 2.99395187 \\
\textit{Log. of}\ 1000= & 3.00000000 \\
\textit{Error} - & \overline{0.00604813} \\
\textit{Also}\ (4.56\times \log. 4.56=)\ 4.56\times 0.6589648= & 3.00487948 \\
\textit{Log. of}\ 1000= & 3.00000000 \\
\textit{Error (least)}\ + & \overline{0.00487945}
\end{array}$$

Then $0.00604813+0.00487945=.01092758$, *sum of the errors.*

Therefore $\dfrac{.001\times .01092758}{.00487945}=\dfrac{.00001092758}{.00487945}=.00224$, *correction.*

Wherefore $x=4.56-.00224=4.55776$, *nearly.*

For $\overline{4.55776}|^{4.55776}=1005.6$, *which is too great by* 5.6; *and for a still nearer approximation, the operation may again be repeated; thus, let* 4.556 *and* 4.557 *be proposed, and proceed as before.*

2. Given $x^x=100$, to find x. *Ans.* $x=3.597285$.

3. Given $z^z=7827577897$, to find z. *Ans.* $z=11.295859$.

4. Given $x^x=123456789$, to find x. *Ans.* $x=8.6400268$.

5. Given $y^x=3000$, and $x^y=5000$, to find x and y. *Ans.* $x=4.691445$, *and* $y=5.510132$.

6. Given $x^x=400$, to find x. *Ans.* $x=2.3244318$.

66. *Two or more equations, involving as many unknown quantities, may be resolved by approximation, as follows.*

RULE I. Reduce all the equations to one, (by either of the methods for reducing equations containing two or more unknown quantities, Vol. I. Part 3. Art. 90—95.) this equation will contain only one unknown quantity.

II. Find the value of this unknown quantity by one of the preceding rules; from whence that of the others may be obtained.

EXAMPLES.—1. Given $x+y+z=22$, $2x-3y+5z=40$, and $3x+4y-2z^3=-100$, to find x, y, and z.

From eq. 1. $x=22-y-z$; *substitute this value of* x *in the second and third, and* $(44-2y-2z-3y+5z=40$, *or*$)$ $5y-3z=4$; *also* $(66-3y-3z+4y-2z^2=-100$, *or*$)$ $2z^2+3z-y=166$; *let now the value of* y $(=\frac{3z+4}{5})$ *in the last but one be substituted in the last, and it becomes* $(2z^2+3z-\frac{3z+4}{5}=166$, *or*$)$ $10z^2+12z=834$.

Now it appears from trial, that z *is greater than* 4, *but less than* 5; *let these two numbers therefore be substituted for* z, *then by the last rule,*

1st Supp. or $z=4$.	Equation.	2nd Supp. or $z=5$.
640	$=10z^2=$	1250
48	$=+12z=$	60
688	$=$ *result* $=$	1310
-146	$=$ *error* $=$	$+476$

Then $(\frac{146}{146+476}=\frac{146}{622}=).2347$, *wherefore* $z=4.2347$.

For a nearer approximation. *Let* 4.2 *and* 4.3 *be put for* z, *and*

1st Supp.	Equation.	2nd Supp.
740.88	$=10z^2=$	795.07
50.4	$=+12z=$	51.6
791.28	$=$*result*$=$	846.67
-42.72	$=$*error*$=$	$+12.67$

Then $(\frac{12.67\times .1}{42.72+12.67}=\frac{1.267}{55.39}=).022874$, *the correction.*

Wherefore $z=(4.3-.022874=)$ 4.277126, *very nearly.*

Whence $y=(\frac{3z+4}{5}=)$ 3.366275, *and* $x=(22-y-z=)$ 14.356599, *nearly.*

2. Given $z-x=10$, $xy+xz=900$, *and* $xyz=3000$, to find x, y, and z.

From eq. 1. $z=10+x$; *substitute this value for* z *in the second, and it becomes* $xy+10x+x^2=900$, *and* $y=\frac{900-10x-x^2}{x}$; *write this value for* y, *and* $10+x$ *for* z *in the third, and it will become* $(9000+800x-20x^2-x^3=3000$, *or*$)$ $x^3+20x^2-800x=6000$.

Here by trials x is found to be greater than 23, but less than 24; then using these two numbers as suppositions, and proceeding as before, $x=23.923443456$, $y=3.696558933$, and $z=33.923443456$, nearly.

3. Given $x^2+y=157$, and $y^2-x=6$, to find x and y. *Ans.* $x=12.34$, $y=4.321$.

4. Given $x+xy=80$, and $x^2y-y^2=495$, to find x and y. *Ans.* $x=8$, $y=9$.

5. Given $x^3+y^2=12$, and $x^2+y^3=8$, to find x and y.

6. Given $x+yz=20$, $y+xz=22$, and $z+xy=28$, to find x, y, and z.

67. *Dr. HUTTON'S RULE for extracting the roots of numbers by approximation.*

Rule I. Let N= the number of which any root is required to be extracted, $\frac{1}{n}$= the index of the proposed root, r= the number found by trials, which is nearly equal to the root, namely, $r^n=N$ *nearly*, and let x= the root, or $x^n=N$ *exactly*.

II. Then will $x=\dfrac{\overline{n+1}.N+\overline{n-1}.r^n}{\overline{n+1}.r^n+\overline{n-1}.N}\times r$, nearly[f].

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[f] The rule is thus demonstrated; let N= the given number, the root of which it is proposed to evolve; $\frac{1}{n}$= the index of the root, r= the nearest rational root, v= the difference between r and the exact root, $x=r+v$= the exact root; then since $N|^{\frac{1}{n}}=r+v$, we shall have $N=\overline{r+v}|^n=r^n+nr^{n-1}v+n.\frac{n-1}{2}r^{n-2}v^2+$, &c. (Vol. I. P. 3. Art. 54.) and by transposition and division, $\frac{N-r^n}{nr^{n-1}}=v+\frac{n-1}{2}.\frac{v^2}{r}$, &c. in which, rejecting $\frac{n-1}{2}.\frac{v^2}{r}$ on account of its smallness, v may be considered as $=\frac{N-r^n}{nr^{n-1}}$. But from the first equation, $N-r^n=nr^{n-1}v+n.\frac{n-1}{2}r^{n-2}v^2+$, &c. $=(nr^{n-1}+n.\frac{n-1}{2}r^{n-2}v)\times v$, in which, if the former value of v (viz. $\frac{N-r^n}{nr^{n-1}}$) be substituted, we shall have $N-r^n=(nr^{n-1}+\frac{n-1}{2}.\frac{N-r^n}{r})\times v=\frac{2nr^n+\overline{n-1}.N-nr^n+r^n}{2r}\times v=\frac{\overline{n+1}.r^n+\overline{n-1}.N}{2r}\times v$; consequently $v=\frac{(N-r^n).2r}{\overline{n+1}.r^n+\overline{n-1}.N}$, and $x=(r+v=)$

III. To find a nearer value, let this value of x be substituted for r in the above theorem, and the result will approach nearer the root than the former.

IV. In like manner, by continually substituting the last value of x for r, the root may be found to any degree of exactness.

EXAMPLES.—1. Let $x^4=19$ be given, to find the value of x.

Here $N=19$, $n=4$, *and the nearest whole number to the fourth root of* 19 *is* 2; *let therefore* $r=2$, *then will* $r^n=16$, *and* $x=$ $\frac{\overline{n+1}.N+\overline{n-1}.r^n}{\overline{n+1}.r^n-\overline{n-1}.N}\times r=(\frac{5\times19+3\times16}{5\times16+3\times19}\times 2=)\frac{286}{137}=2.08$, *nearly.*

To repeat the process for a nearer approximation.

Let $r=2.08$, *then* $r^n=(\overline{2.08}|^4=)$ 18.71773696; *these numbers being substituted in the theorem, we shall have* $x=$ $(\frac{5\times19+3\times18.71773696}{5\times18.71773696+3\times19}\times 2.08=)\frac{151.15321088}{150.5886848}\times 2.08=$ 2.0877975, *extremely near; and if a nearer value of* x *be required, this number must be substituted for* r, *and repeat the operation.*

2. Given $x^3=510$, to find x. *Ans.* $x=7.999$, &c.
3. Given $x^5=7900$, to find x. *Ans.* $x=6.019014897$.
4. Extract the sixth root of 262140. *Ans.* $x=3.9999$, &c.
5. Required the sixth root of 21035.8? *Ans.* $x=5.254037$.
6. Extract the sixth root of 272.

68. PROBLEMS PRODUCING EQUATIONS OF THREE OR MORE DIMENSIONS.

1. What number is that, which being subtracted from twice its cube, the remainder is 679? *Ans.* 7.

2. What number is that, which if its square be subtracted from its cube, the remainder will exceed ten times the given number by 1100? *Ans.* 11.

$r+\frac{(N-r^n).2r}{\overline{n+1}.r^n+\overline{n-1}.N}=\frac{\overline{n+1}.N+\overline{n-1}.r^n}{\overline{n+1}.r^n+\overline{n-1}.N}.r$, which is the rule. This is the investigation of the rule in Vol. I. page 260: the theorem was first given by Dr. Hutton, in the first Volume of his Mathematical Tracts; it includes all the rational formulæ of Halley and De Lagni, and is perhaps more convenient for memory and operation than any other rule that has been discovered.

3. What number is that, which being added to its square, the sum will be less by 56 than $\frac{1}{4}$ its cube? *Ans.* 8.

4. There is a number, thrice the square of which exceeds twice the cube by .972; required the number? *Ans.* $\frac{9}{10}$.

5. If to a number its square and cube be added, four times the sum will equal $\frac{43}{54}$ of the fourth power; required the number? *Ans.* 6.

6. If the sum of the cube and square of a number be multiplied by ten times that number, the product shall exceed twice the sum of the first, second, third, and fourth powers by 180; what is the number? *Ans.* 2.

7. Required two numbers, of which the product multiplied by the greater produces 18, and their difference multiplied by the less, 2? *Ans.* 3 *and* 2.

8. The days being 16 hours long, a person who was asked the time of day, replied, "If to the cube of the hours passed since sun-rise you add 40, and from the square of the hours to come before sun-set you subtract 40, the results will be equal:" required the hour of the day? *Ans.* 8 *in the morning.*

9. To find two mean proportionals between 1 and 2. *Ans.* 1.25992, *and* 1.5874.

10. The ages of a man and his wife are such, that the sum of their square roots is 11, and the difference of their cubes 31031; what are their ages? *Ans.* 36 *and* 25.

11. If the cube root of a father's age be added to the square root of his son's, the sum will be 8; and if twice the cube root of half the son's age be added to the square root of the father's, the sum will be 12; what is the age of each? *Ans. the father's* 64, *the son's* 16.

13. There are in a statuary's shop three cubical blocks of marble, the side of the second exceeds that of the first by 3 inches; and the side of the third exceeds that of the second by 2 inches; moreover, the solid content of all the three together is 1136 cubic inches; required the side of each? *Ans.* 4, 7, *and* 9 *inches.*

PART VI.

ALGEBRA.

THE INDETERMINATE ANALYSIS [a].

1. A PROBLEM is said to be *indeterminate*, or *unlimited*, when the number of unknown quantities to be found is greater than the number of conditions, or equations proposed [b].

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[a] For some account of the subject, see the note on Diophantine problems.

[b] If the number of *quæsita* exceed the number of *data*, the problem is unlimited. If the *quæsita* be equal in number to the *data*, the problem is limited. If the *data* exceed the *quæsita*, the excess is either deducible from the other conditions, or inconsistent with them; in the former case the excess is redundant, and therefore unnecessary; in the latter it renders the problem absurd, and its solution impossible. To give an example of each.

1. *Let $x+y=6$ be given, to find the values of x and y.*

Here we have but *one* condition proposed, and *two* quantities required to be found, the problem is therefore *unlimited;* for (admitting whole numbers only) x may $=1$, then $y=5$; if $x=2$, then $y=4$; if $x=3$, then $y=3$; if $x=4$, then $y=2$; if $x=5$, then $y=1$.

2. *Let $x+y=6$, and $x-y=4$, be given.*

Here we have *two* conditions proposed, and *two* quantities to be found, whence the problem is *limited;* (see Vol. I. P. 3. Art. 89.) for $x=5$, $y=1$: and no other numbers can possibly be found, that will fulfil the conditions.

3. *Let $x+y=6$, $x-y=4$, and $xy=5$, be given.*

Here is a redundancy, *three* conditions are laid down, and but *two* quantities to be found. By the preceding example $x=5$, $y=1$; wherefore $xy=5\times1=5$, or the latter condition ($xy=5$) is deducible from the two former.

4. *Let $x+y=6$, $x-y=4$, and $xy=12$, be given.*

Here is not only a redundancy, but an *inconsistency;* for the greatest product that can possibly be made of any two parts of 6, is 9, that is, $xy=9$; it cannot then be divided into two parts, x and y, so that $xy=12$; wherefore the latter condition is inconsistent with the two former, and renders the problem impossible. There is a mistake in the appendix to *Ludlam's Rudiments*, 5th edit. p. 338. Art. 107. by which the subject is altogether perverted.

2. An indeterminate problem will frequently admit of innumerable answers, if fractions, negative quantities, and surds be admitted; but if the answers be restricted to positive whole numbers, the number of answers will in many cases be limited.

3. The *indeterminate analysis* is the method of resolving indeterminate problems; it depends on the following self-evident principles, viz.

"The sum, difference, and product of two whole numbers, are likewise whole numbers."

"If a number measure the whole, and likewise a part of another number, it will measure the remaining part."

4. *In the given equation $ax=by+c$, to find the values of x and y in positive whole numbers.*

RULE I. Let W stand for the words *whole number*, then (since x and y are by hypothesis whole numbers) the above equation $ax=by+c$ reduced, will be $x=\frac{by+c}{a}=W$.

II. If $\frac{by+c}{a}$ be an improper fraction, reduce it to its equivalent mixed quantity; (see Vol. I. p. 380. ex. 9, 10.) that is, let $\frac{by+c}{a}=m+\frac{dy+f}{a}$: from which rejecting m, we have $\frac{dy+f}{a}=W$ by Art. 3.

III. Take the difference of $\frac{dy+f}{a}$ or any of its multiples, and y or any of its multiples, viz. $\frac{ay}{a}$, $\frac{2\,ay}{a}$, $\frac{3\,ay}{a}$, &c. in order to reduce the coefficient of y to unity, or as near unity as possible, and the remainder will be $=W$.

IV. Take the difference of this remainder and any of the foregoing fractions, or any other whole number nearly equal to it, then will the remainder $=W$.

V. Proceed in this manner, till the coefficient of y becomes unity, or $\frac{y+g}{a}=W$.

VI. Let $\frac{y+g}{a}=p$, then will $y=ap-g$; and if any whole num-

ber whatever be substituted for p, the value of y will be known; whence x $(=\frac{by+c}{a})$ will likewise be known.

EXAMPLES.—1. Given $4x=5y-10$, to find the values of x and y in whole numbers.

First, $x=\frac{5y-10}{4}=W$; *but* $\frac{5y-10}{4}=y-2+\frac{y-2}{4}$, *whence* (*rejecting* $y-2$) $\frac{y-2}{4}=W=p$, *therefore* $y-2=4p$, *and* $y=4p+2$; *let* $p=o$, *then* $y=2$, *whence* $x=(\frac{5y-10}{4}=\frac{10-10}{4}=)\ o$.

Secondly, let p *be taken* $=1$, *then* $y=(4p+2=)\ 6$, *and* $x=(\frac{5y-10}{4}=\frac{20}{4}=)\ 5$.

Thirdly, let $p=2$, *then* $y=(4y+2=)\ 10$, *and* $x=(\frac{5y-10}{4}=\frac{40}{4}=)\ 10$.

Fourthly, let $p=3$, *then* $y=(4p+2=)\ 14$, *and* $x=(\frac{5y-10}{4}=)\ 15$.

Fifthly, let $p=4$, *then* $y=18$, *and* $x=20$.

Sixthly, let $p=5$, *then* $y=22$, *and* $x=25$. *&c. &c.*

Hence it appears, that the values of x (viz. 0, 5, 10, 15, 20, 25, &c.) differ by the coefficient (5) of y; and the values of y (viz. 2, 6, 10, 14, 18, 22, &c.) by the coefficient (4) of x; and it is plain, that this will be the case universally in every equation of the form $ax=by-c$, viz. the successive values of x will differ by b, and those of y by a.

2. Given $17x=13y-14$, to find x and y in positive whole numbers.

First, $x=\frac{13y-14}{17}=W$, *also* $\frac{17y}{17}=W$; *wherefore* (*Art.* 3.) $\frac{17y}{17}-\frac{13y-14}{17}=\frac{4y+14}{17}=W$; *likewise* $(\frac{4y+14}{17}\times 4=)\ \frac{16y+56}{17}=W$, *that is,* $\frac{16y+5}{17}+3=W$, *whence* $\frac{16y+5}{17}=W$; *and* $(\frac{17y}{17}$

$-\frac{16y+5}{17}=)\ \frac{y-5}{17}=W=p$, *whence* $y=17p+5$; *let* $p=o$, *then* $y=5$, *and* $x=(\frac{13y-14}{17}=\frac{65-14}{17}=)\ 3$.

And by continually adding 13 to the value of x, and 17 to the value of y, we obtain the following values, viz.

$x=3, 16, 29, 42, 55, 68, 81, 94, 107$, &c.

$y=5, 22, 39, 56, 73, 90, 107, 124, 141$, &c.

3. Let $4x+7y=23$, be given, to find x and y.

First, $x=(\frac{23-7y}{4}=)\ 5-y+\frac{3-3y}{4}$, *whence rejecting* $5-y$, *we have* $\frac{3-3y}{4}=W$, *wherefore* $(\frac{4y}{4}+\frac{3-3y}{4}=)\ \frac{y+3}{4}=W=p$; *consequently* $y+3=4p$, *and* $y=4p-3$; *let* $p=1$, *then* $y=(4p-3=)\ 1$, *and* $x=(\frac{23-7y}{4}=\frac{16}{4}=)\ 4$; *which are the only affirmative answers the question admits of.*

4. Given $19x+14y=1000$, to find x and y.

First, $x=(\frac{1000-14y}{19}=)\ 52+\frac{12-14y}{19}$; *rejecting* 52, *we have* $\frac{12-14y}{19}=W$, *consequently* $(\frac{19y}{19}+\frac{12-14y}{19}=)\ \frac{5y+12}{19}=W$, *also* $(\frac{5y+12}{19}\times 4=\frac{20y+48}{19}=)\ \frac{20y+10}{19}+2=W$, *whence* $\frac{20y+10}{19}=W$; *wherefore* $(\frac{20y+10}{19}-\frac{19y}{19}=)\ \frac{y+10}{19}=W=p$, *or* $y=19p-10$; *let* $p=1$, *then* $y=9$, *and* $x=(\frac{1000-14y}{19}=)\ 46$.

Let $p=2$, *then* $y=28$, *and* $x=32$.

Let $p=3$, *then* $y=47$, *and* $x=18$.

Let $p=4$, *then* $y=66$, *and* $x=4$.

These are all the affirmative values of x *and* y; *for if* p *be taken*$=5$, *then will* $y=85$, *and* $x=-10$, *a negative quantity.*

The above values will be obtained by adding the coefficient (19) of x, to the preceding value of y; and subtracting the coefficient (14) of y, from the corresponding value of x; and the same is universally true of every equation of the form of $ax+by=c$.

5. Given $13x=21y-3$, to find the least values of x and y in whole numbers. *Ans.* $x=3$, $y=2$.

6. Given $41x=42y-52$, to find x and y. *Ans.* $x=10$, $y=11$.

7. Given $8x+9y=25$, to find x and y. *Ans.* $x=2$, $y=1$.

8. How many positive values of x and y in whole numbers can be found from the equation $9x=2000-13y$? *Ans.* 17 *values of each.*

9. Given $13x=14y+36$, to find x and y.

10. Given $101x=4321-177y$, to find x and y[e].

5. *To find a whole number, which being divided by given numbers, shall leave given remainders.*

RULE I. Let x=the number required; a, b, c, &c.=the given divisors; f, g, h, &c.=the given remainders; then will $\frac{x-f}{a}=W$, $\frac{x-g}{b}=W$, $\frac{x-h}{c}=W$, &c.

II. Make the first fraction$=p$, find the value of x from it, and substitute this value for x in the second fraction.

III. Find the least value of p in the second fraction, (Art. 4.) in terms of r, and thence x in terms of r.

IV. Substitute this last value for x in the third fraction, whence find the least value of r in terms of s, and thence the value of x in terms of s.

V. Substitute this value in the fourth fraction, &c. and proceed in this manner to the last fraction, from whence the value of x will be known.

EXAMPLES.—1. What number is that, which being divided by 3, will leave 2 remainder, and being divided by 2, will leave 1 remainder?

Let x=*the number, then will* $\frac{x-2}{3}=W$, *and* $\frac{x-1}{2}=W$; *let* $\frac{x-2}{3}=p$, *then will* $x=3p+2$; *substitute this value for* x *in the fraction* $\frac{x-1}{2}$, *and it becomes* $\frac{3p+1}{2}=W$: *but* $\frac{2p}{2}=W$, *wherefore* $(\frac{3p+1}{2}-\frac{2p}{2}=)\frac{p+1}{2}=W=r$, *whence* $p=2r-1$; *let*

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[e] By similar methods indeterminate equations, involving three or more unknown quantities, may be resolved.

r be taken $=1$, then $p=(2r-1=2-1=)\ 1$, and $x=(3p+2=)$ 5, the number required.

2. What is the least number which can be divided by 2, 3, 5, 7, and 11, and leave 1, 2, 3, 4, and 5, for the respective remainders?

Let $x=$ the number, then $\frac{x-1}{2}=W$, $\frac{x-2}{3}=W$, $\frac{x-3}{5}=W$, $\frac{x-4}{7}=W$, and $\frac{x-5}{11}=W$, by the problem. Let $\frac{x-1}{2}=p$, then $x=2p+1$; substitute this value for x in the fraction $\frac{x-2}{3}$, and it becomes $\frac{2p-1}{3}=W$; but $\frac{3p}{3}=W$, wherefore $(\frac{3p}{3}-\frac{2p-1}{3}=)$ $\frac{p+1}{3}=W=r$, and $p=3r-1$, wherefore $x=(2p+1=)\ 6r-1$; substitute this value for x in the third fraction $\frac{x-3}{5}$, and it becomes $\frac{6r-4}{5}=W$; but $\frac{5r}{5}=W$, wherefore $(\frac{6r-4}{5}-\frac{5r}{5}=)\ \frac{r-4}{5}$ $=W=s$, and $r=5s+4$, consequently $x=(6r-1=)\ 30s+23$; this value being substituted for x in the fourth fraction $\frac{x-4}{7}$, it becomes $\frac{30s+19}{7}=4s+2+\frac{2s+5}{7}=W$, whence (rejecting $4s+2$) $\frac{2s+5}{7}=W$; also $(\frac{2s+5}{7}\times 3=\frac{6s+15}{7}=)\ \frac{6s+1}{7}+2$, wherefore (rejecting the 2) $\frac{6s+1}{7}=W$; but $\frac{7s}{7}=W$, consequently $(\frac{7s}{7}-\frac{6s+1}{7}=)\ \frac{s-1}{7}=W=t$, whence $s=7t+1$, and $x=(30s+23=)$ $210t+53$; this value substituted for x in the fifth fraction $\frac{x-5}{11}$, it becomes $\frac{210t+48}{11}=19t+4+\frac{t+4}{11}$, from whence rejecting $19t+4$, we have $\frac{t+4}{11}=W=u$, whence $t=11u-4$; let $u=1$, then $t=(11u-4=)\ 7$, and $x=(210t+53=)\ 1523$.

3. Required the least whole number, which being divided by

3, will leave 2 remainder; but if divided by 4, will leave 3 remainder? *Ans.* 11.

4. Required the least whole number, which being divided by 6, 5, and 4, will leave 5, 2, and 1, for the respective remainders? *Ans.* 17.

5. To find the least whole number, which being divided by 3, 5, 7, and 2, there shall remain 2, 4, 6, and 0, respectively. *Ans.* 104.

6. Required the least whole number, which being divided by 16, 17, 18, 19, and 20, will leave the remainders 6, 7, 8, 9, and 10, respectively?

6. Any equation involving two different powers only of the unknown quantity, may be reduced by substitution to the form of an indeterminate equation, involving two variable quantities. Hence, all commensurate quadratic equations, commensurate cubics wanting one term, commensurate biquadratics wanting two terms, &c. may be resolved by this method. It will be proper for the convenience of reference, to premise the following table of roots and powers [d].

Roots	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Squares	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144.
Cubes	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000,	1331,	1728.

EXAMPLES.—1. Let $x^2+4x=32$ be given, to find x.

First, by transposition and division, $\frac{32-x^2}{x}=\frac{4}{1}$. *Secondly, it is plain, that whatever equimultiples of* 4 *and* 1 *be taken, the fractions whose terms are constituted of these equimultiples respectively will be equal to* $\frac{4}{1}$ *and to one another, that is,* $\frac{4}{1}=\frac{8}{2}$ $=\frac{12}{3}=\frac{16}{4}=\frac{20}{5}$, *&c. Wherefore, thirdly, if the quantity* $\frac{32-x^2}{x}$ *be made equal to either of these fractions, which (after transposing the known quantity* 32) *will give the resulting numerator equal to the square of the denominator, that denominator will be the value*

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[d] See on this subject, Dodson's *Mathematical Repository*, Vol. I. Emerson's *Algebra*, Simpson's *Algebra* and *Select Exercises*, Vilant's *Elements of Mathematical Analysis*, &c.

of x in the proposed equation; that is, $\frac{32-x^2}{x}=\frac{4}{1}=\frac{8}{2}=\frac{12}{3}=\frac{16}{4}$ $=\frac{20}{5}$, *&c. here it is plain, that if the fraction* $\frac{16}{4}$ *be taken, we shall have* $32-x^2=16$, *or* $x^2=(32-16=)\ 16$, *whence* $x=4$.

2. Given $x^2+6x=40$, to find x.

By transposition and division, as before, we have $\frac{40-x^2}{x}=\frac{6}{1}=\frac{12}{2}=\frac{18}{3}=\frac{24}{4}$; *and if the latter fraction be taken, we shall have* $\begin{cases} 40-24=16, \\ \textit{and} \quad x=4, \textit{ the answer.} \end{cases}$

3. Given $x^2+3x=88$, to find x.

Here $\frac{88-x^2}{x}=\frac{3}{1}=\frac{6}{2}=\frac{9}{3}=\frac{12}{4}=\frac{15}{5}=\frac{18}{6}=\frac{21}{7}=\frac{24}{8}$, &c.

And $\begin{cases} 88-24\ =64, \\ \textit{whence } x=8, \textit{ the answer.} \end{cases}$

4. Given $x^2-5x=14$, to find x.

Here $\frac{x^2-14}{x}=\frac{5}{1}=\frac{10}{2}=\frac{15}{3}=\frac{20}{4}=\frac{25}{5}=\frac{30}{6}=\frac{35}{7}$, &c.

Wherefore $\begin{cases} x^2=(35+14=)49, \\ \textit{and } x=7, \textit{ the answer.} \end{cases}$

5. Given $x^2-\frac{1}{4}x=118\frac{1}{4}$, to find x.

Here $\frac{x^2-118\frac{1}{4}}{x}=\frac{\frac{1}{4}}{1}=\frac{\frac{1}{2}}{2}=\frac{\frac{3}{4}}{3}=\frac{1}{4}=\frac{1\frac{1}{4}}{5}=\frac{1\frac{1}{2}}{6}=\frac{1\frac{3}{4}}{7}=\frac{2}{8}=\frac{2\frac{1}{4}}{9}$ $=\frac{2\frac{1}{2}}{10}=\frac{2\frac{3}{4}}{11}$, &c. *Wherefore* $\begin{cases} x^2=(2\frac{3}{4}+118\frac{1}{4}=)121, \\ \textit{and } x=11, \textit{ the answer.} \end{cases}$

6. Given $x^2-5x-6=o$, to find x.

Here $\frac{x^2+6}{x}=\frac{5}{1}=\frac{10}{2}=\frac{15}{3}$.

Wherefore $\begin{cases} x^2=(10-6=)4, \\ \textit{and } x=2, \end{cases}$ *or* $\begin{cases} x^2=(15-6=)9, \\ \textit{and } x=3. \end{cases}$

Consequently $x=+2$, *or* $+3$.

7. Given $y^6+4y^3=96$, to find y.

Let $v=y^3$, *then will* $v^2+4v=96$, *and* $\frac{96-v^2}{v}=\frac{4}{1}=\frac{8}{2}=\frac{12}{3}=\frac{16}{4}=\frac{20}{5}=\frac{24}{6}=\frac{28}{7}=\frac{32}{8}$; *whence* $\begin{cases} 96-v^2=(96-32=)64, \\ \textit{and } v=8, \textit{ the answer.} \end{cases}$

But $v=y^3$, *whence* $y=\sqrt[3]{v}=(\sqrt[3]{8}=)\ 2$.

Or thus,

Because $v^2+4v=96$, *therefore* $\frac{v^2}{24-v}=\frac{4}{1}=\frac{16}{4}=\frac{36}{9}=\frac{64}{16}$.

Whence $\begin{cases} v^2=64, \textit{ or } v=8, \\ 24-v=16, \textit{ or } v=8\ ; \textit{ whence } y=2, \textit{ as before.} \end{cases}$

8. Given $y^3-7y=36$, to find y.

Here $\frac{y^3-36}{y}=\frac{7}{1}=\frac{14}{2}=\frac{21}{3}=\frac{28}{4}$.

Whence $\begin{cases} y^3=(36+28=)\ 64, \\ \textit{and } y=4, \textit{ the answer.} \end{cases}$

9. Given $z^3-1\frac{3}{4}z=-\frac{3}{4}$, to find z.

Here $\frac{z^3+\frac{3}{4}}{z}=\frac{\frac{7}{4}}{1}$, *whence* $z=1$.

10. Given $9z^2-z^3=100$, to find z.

Here $\frac{100+z^3}{z^2}=\frac{9}{1}=\frac{36}{4}=\frac{81}{9}=\frac{144}{16}=\frac{225}{25}$.

Whence $\begin{cases} z^3=(225-100=)125, \\ \textit{and } z=(\sqrt[3]{125}=)\ 5. \end{cases}$

11. Given $x^2+2x=8$, to find x. *Ans.* $x=2$.

12. Given $x^2-5x=6$, to find x. *Ans.* $x=6$.

13. Given $x^2+20=9x$, to find x. *Ans.* $x=5$, *or* 4.

14. Given $y^3+70=39y$, to find y. *Ans.* $y=5$.

15. Given $z^3-21z+20=o$, to find z. *Ans.* $z=4$.

16. Given $60-x^3=11x$, to find x. *Ans.* $x=3$.

7. INDETERMINATE PROBLEMS*.

1. How must tea, at 7 shillings per pound, be mixed with tea at 4 shillings per pound, so that the mixture may be worth 6 shillings per pound?

Let x = *the number of pounds at 7 shillings, then* $7x$ = *their value;* y = *the number at 4 shillings, then* $4y$ = *their value.*

Whence by the problem $7x+4y=(6\overline{x+y}=)6x+6y$, *or* $x=2y$, *or* $1\times x=2\times y$ ∴ $x:y::2:1$ ∴ *there must be twice as much in the mixture at 7 shillings, as there is at 4 shillings.*

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* These problems are of the kind which belong to the rule of Alligation.

2. Twenty poor persons received among them 20 pence; the men had $4d.$ each, the women $\frac{1}{2}d.$ each, and the children $\frac{1}{4}d.$ each; what number of men, women, and children, were relieved?

Let x=the number of men, y=the number of women, z=the number of children; then by the problem $x+y+z=20$, and $(4x+\frac{1}{2}y+\frac{1}{4}z=20$, or$)$ $16x+2y+z=80$: subtract the first equation from this, and $15x+y=60$, or $y=(60-15x=)\overline{4-x}.15$, or $\frac{y}{4-x}$ $=\frac{15}{1}=\frac{30}{2}=\frac{45}{3}$, &c. but by the problem $y < 20$ $\because$ $y=15$; and since $4-x=1$ $\because$ $x=3$, and $z=(20-x-y=)20-18=2$.

3. How many ways can $100l.$ be paid in guineas and crown-pieces?

Let x=the number of guineas, y=the number of crowns.

Then by the problem $21x+5y=(100\times 20=)2000$.

Whence $x=(\frac{2000-5y}{21}=)95+\frac{5-5y}{21}$, $\because$ $\frac{5-5y}{21}=W$, $\because$ $(\frac{5-5y}{21}\times 4=)$ $\frac{20-20y}{21}=W$; also $\frac{21y}{21}=W$, $\because$ $(\frac{20-20y}{21}+\frac{21y}{21}=)\frac{20+y}{21}=W=p$, $\because$ $y=21p-20$; let $p=1$, then $y=1$ crown, and $x=(\frac{2000-5y}{21}=)$ 95 guineas: and if (21) the coefficient of x, be continually added to the value of y, and (5) the coefficient of y, continually subtracted from that of x, the corresponding values of x and y will be as follows, viz.

x=95, 90, 85, 80, 75, 70, 65, 60, 55, 50, 45, 40, 35, 30, 25, 20, 15, 10, 5, 0.

y=1, 22, 43, 64, 85, 106, 127, 148, 169, 190, 211, 232, 253, 274, 295, 316, 337, 358, 379, 400. *Ans.* 19 *ways.*

4. To divide the number 19 into three parts, such that seven times the first part, four times the second, and twice the third, being added together, the sum will be 90.

Let the parts be x, y, and z; then by the problem $x+y+z=19$, and $7x+4y+2z=90$; from the first $x=19-y-z$, this value being substituted for x in the second, it becomes $(133-7y-7z+4y+2z=)133-3y-5z=90$; or $(3y=43-5z$, or$)$ $y=\frac{43-5z}{3}$.

$=14-z+\frac{1-2z}{3}$, $\because \frac{1-2z}{3}=W$; *also* $\frac{3z}{3}=W$, $\because (\frac{1-2z}{3}+\frac{3z}{3}$ $=)\frac{1+z}{3}=W=p \because z=3p-1$; *if* p *be taken*$=1$, *then* $z=2$, $y=$ $(\frac{43-5z}{3}=)11$, *and* $x=(19-y-z=)\,6$; *if* $p=2$, *then will* $z=5$, $y=6$, *and* $x=8$; *if* $p=3$, *then* $z=8$, $y=1$, *and* $x=10$: *these are all the possible values in whole numbers.*

5. How many ways is it possible to pay 100*l.* in guineas at 21 shillings each, and pistoles at 17 shillings each? *Ans.* 6.

6. If 27 times A.'s age be added to 16 times B.'s, the sum will be 1600; what is the age of each? *Ans. A.'s* 48, *B.'s* 19.

7. A Higler's boy, sent on a market day
With eggs, fell down and smash'd them by the way;
The news reach'd home, and Master, in a rage,
Vow'd him a whipping, bridewell, or the cage:
"'Tis through your negligence the eggs are lost,
"So pay me if you please the sum they cost."
The boy, since nought avail his tears and prayers,
Fetches his leathern bag of cash down stairs;
The cash a year's hard earnings had put in,
But much he wish'd to sleep in a whole skin.
"How many were there, Master?" In a doubt,
The Higler calls his wife to help him out;
Says she, "I counted them by twos, threes, fours,
"Fives, sixes, sev'ns, before he left these doors;
"And one, two, three, four, five, and nought, remain'd
"Respectively, nor more can be explain'd."
At nine a groat, ingenious Tyros, say,
What sum will for the sad disaster pay?

Ans. 4*s.* 4$d\frac{2}{3}$.

8. Is it possible to pay 100*l.* with guineas and moidores only? *Ans. It is impossible.*

9. A, who owes B a shilling, has nothing but guineas about him, and B has nothing but louis d'ors at 17 shillings each; how, under these circumstances, is the shilling to be paid? *Ans. A must give B* 13 *guineas, and receive* 16 *louis d'ors change.*

10. With guineas and moidores the fewest, which way
Three hundred and fifty-one pounds can I pay?

And when paid ev'ry way 'twill admit, the amount
Of the whole is required?—Take paper and count.

8. DIOPHANTINE PROBLEMS.

Unlimited problems relating to square and cube numbers, right angled triangles, &c. were first and chiefly treated of by Diophantus of Alexandria, and from that circumstance they are usually named *Diophantine Problems* [f].

These problems, if not duly ordered, will frequently bring out answers in irrational quantities; but with proper management this inconvenience may in many cases be avoided, and the answers obtained in commensurable numbers.

The intricate nature and almost endless variety of problems of this kind, render it impossible to lay down a general rule for their solution, or to give rules for an innumerable variety of particular cases which may occur. The following rules will, perhaps, be found among the best and most generally applicable of any that have been proposed.

RULE I. Substitute one or more letters for the required root of the given square, cube, &c. so that, when involved, either the given number, or the highest power of the unknown quantity, may be exterminated from the given equation.

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[f] Diophantus has been considered by some writers as the inventor of Algebra; others have ascribed to him the invention of unlimited problems: but the difficult nature of the latter, and the masterly and elegant solutions he has given to most of them, plainly indicate that both opinions are erroneous.

Diophantus flourished, according to some, before the Christian æra; some place him in the second century after Christ, others in the fourth, and others in the eighth or ninth. His Arithmetica, (out of which have been extracted most of the curious problems of the kind at present extant,) consisted originally of thirteen books, six of which, with the imperfect seventh, were published at Basil in 1575, by Xylander; this fragment is the only work on Algebra, which has descended to us from the ancients: the remaining books have never been discovered. *See Vol. I. p.* 327.

Of those who have written on, and successfully cultivated, the Diophantine Algebra, the chief are, Bachet de Meseriac, Braucker, Bernoulli, Bonnycastle, De Billy, Euler, Fermat, Kersey, Ozanam, Prestet, Saunderson, Vieta, and Wolfius.

II. If, after this operation, the unknown quantity be of but one dimension, reduce the equation, and the answer will be found.

III. But if the unknown quantity be still a square, cube, &c. substitute some new letter or letters for the root, and proceed as before directed.

IV. Repeat the operation until the unknown quantity is reduced to one dimension; its value will then readily be found, from whence the values of all the other quantities will likewise be known.

1. To divide a given square number into two parts, so that each may be a square number.

ANALYSIS. *Let* a^2*=the given square number,* x^2*=one of the parts; then will* a^2-x^2*=the other part, which, by the problem, must likewise be a square. Let* $rx-a$*=the side of the latter square, then will* $(\overline{rx-a}|^2=)\ r^2x^2-2\,arx+a^2=a^2-x^2$*, whence* $x=\frac{2\,ar}{r^2+1}$*=the side of the first square, and* $rx-a=(\frac{2\,ar^2}{r^2+1}-a=)\ \frac{ar^2-a}{r^2+1}$*=the side of the second square; wherefore* $\overline{\frac{2\,ar}{r^2+1}}\Big|^2$ *and* $\overline{\frac{ar^2-a}{r^2+1}}\Big|^2$ *are the parts required; where a and r may be any numbers taken at pleasure, provided r be greater or less than unity*[*]. *Q. E. I.*

SYNTHESIS. *First,* $\overline{\frac{2\,ar}{r^2+1}}\Big|^2+\overline{\frac{ar^2-a}{r^2+1}}\Big|^2=(\frac{4\,a^2r^2}{r^4+2\,r^2+1}+\frac{a^2r^4-2\,a^2r^2+a^2}{r^4+2\,r^2+1}=\frac{a^2r^4+2\,a^2r^2+a^2}{r^4+2\,r^2+1}=)\ \frac{\overline{r^4+2\,r^2+1}.a^2}{r^4+2\,r^2+1}=a^2$*, which is the first condition.*

Secondly, $\overline{\frac{2\,ar}{r^2+1}}\Big|^2$ *and* $\overline{\frac{ar^2-a}{r^2+1}}\Big|^2$ *are evidently both squares, which is the second condition. Q. E. D.*

EXAMPLES.—Let the square number 100 be proposed to be divided into two parts, which will be squares.

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[*] Mr. Bonnycastle, in his solution of the problem, (Algebra, third Edit. p. 143.) has omitted this restriction, which is evidently necessary; for if r be supposed $=1$, then will the numerator of the fraction $\frac{ar^2-a}{r^2+1}$ vanish, and the solution become nugatory.

Here $a^2=100$, *and* $a=10$. *First, assume* $r=2$, *then will* $x=\frac{2ar}{r^2+1}=(\frac{40}{5}=)\,8=$ *the side of the first square, and* $rx-a=6=$ *the side of the second square; for* $\overline{8}|^2+\overline{6}|^2=(64+36=)100$, *as was required.*

Secondly, assume $r=3$, *then will* $x=(\frac{60}{10}=)6$, *and* $rx-a=8$, *as before.*

Thirdly, assume $r=4$, *then* $x=\frac{80}{17}$, *and* $rx-a=(\frac{320}{17}-10=)$ $\frac{150}{17}$. *For* $\overline{\frac{80}{17}}\Big|^2+\overline{\frac{150}{17}}\Big|^2=(\frac{6400+22500}{289}=\frac{28900}{289}=)100$.

Divide 36 into two square numbers.

Here $a^2=36$, $a=6$; *assume* $r=2$, *then* $x=\frac{24}{5}$, *and* $rx-a=$ $(\frac{48}{5}-6=)\frac{18}{5}$.

To divide 25 into two square numbers. *Ans.* 16 *and* 9.

To divide 81 into two square numbers.

2. To find two square numbers having a given difference.

Let $d=$ *the given difference,* $a\times b=d$, *whereof* $a>b$, *and let* $x=$ *the side of the less square, and* $x+b=$ *the side of the greater; then will* $\overline{x+b}|^2-x^2=(x^2+2bx+b^2-x^2=)\ 2bx+b^2=ab$; *divide this by* b, *and* $2x+b=a$, $\therefore x=\frac{a-b}{2}=$ *the side of the less square; and* $x+b=(\frac{a-b}{2}+b=)\frac{a+b}{2}=$ *the side of the greater square: wherefore* $\overline{\frac{a+b}{2}}\Big|^2=\frac{a^2+2ab+b^2}{4}=$ *the greater square required, and* $\overline{\frac{a-b}{2}}\Big|^2=\frac{a^2-2ab+b^2}{4}=$ *the less square.* Q. E. I.

SYNTHESIS. *First,* $\overline{\frac{a+b}{2}}\Big|^2$ *and* $\overline{\frac{a-b}{2}}\Big|^2$ *are evidently both squares.*

Secondly, $\frac{a^2+2ab+b^2}{4}-\frac{a^2-2ab+b^2}{4}=(\frac{4ab}{4}=ab=)d$; *since by hypothesis* $ab=d$. *Q. E. D.*

EXAMPLES.—To find two square numbers, whereof the greater exceeds the less by 11.

Here $d=11(=11\times1)$, *let* $a=11$, $b=1$.

Then $\frac{a+b}{2}=(\frac{11+1}{2}=)6=$ *side of the greater square.*

And $\frac{a-b}{2}=(\frac{11-1}{2}=)5=$ *side of the less square.*

Whence $\overline{6}|^2=36$, *and* $\overline{5}|^2=25$, *are the squares required.*

To find two square numbers differing by 6.

Here $d=6\ (=3\times2)$, $a=3$, $b=2$.

Then $\frac{a+b}{2}=\frac{5}{2}=$ *side of the greater.*

And $\frac{a-b}{2}=\frac{1}{2}=$ *side of the less;* $\therefore \frac{25}{4}$ *and* $\frac{1}{4}$ *are the squares required.*

To find two squares, whose difference is 15. *Ans.* 64 *and* 49.

To find two squares differing by 24.

3. To find two numbers, whose sum and difference will be both squares.

Let $x=$ *one of the numbers,* $x^2-x=$ *the other; then will their sum* $(x+x^2-x=)\ x^2$, *evidently be a square number.*

And since $(x^2-x-x=)\ x^2-2x=$ *their difference, must likewise be a square; let its side be assumed* $=x-r$, *then will* $(\overline{x-r}|^2=)x^2-2xr+r^2=x^2-2x$, *or* $2xr-2x=r^2$, $\therefore x=\frac{r^2}{2r-2}$, *and*

$$x^2-x=\overline{\frac{r^2}{2r-2}}\Big|^2-\frac{r^2}{2r-2}=(\frac{r^4}{4r^2-8r+4}-\frac{r^2}{2r-2}=\frac{r^5-3r^4+4r^3-2r^2}{4r^3-12r^2+12r-4}=)\frac{r^2}{4}+\frac{r^3-r^2}{\overline{r-1})^3.4}$$

the numbers required, where r may be any number greater than 2 [h].

EXAMPLES.—Let $r=3$, *then will* $x=\frac{36}{16}$, *and* $x^2-x=\frac{45}{16}$ *the numbers sought; for* $\frac{45+36}{16}=\frac{81}{16}$ *and* $\frac{9}{16}$, *both squares.*

[h] If 2 be substituted in this example for r, both numbers will come out $=2$; that is, their sum will be 4, and difference 0; wherefore r must not only be greater than 1, (as is asserted in Bonnycastle's Algebra, p. 146.) but greater than 2.

Let $r=5$, to find the numbers.

4. To divide a given number, which is the sum of two known squares, into two other squares.

Let $a^2+b^2=$ *the number given,* $rx-a=$ *the side of the first required square,* $sx-b=$ *the side of the second, where* $r > s$.

Then will $\overline{rx-a}|^2+\overline{sx-b}|^2=(r^2x^2-2\,arx+a^2+s^2x^2-2\,bsx+b^2=)\ \overline{r^2+s^2}.x^2-\overline{2\,ar+2\,bs}.x+a^2+b^2=a^2+b^2\ \therefore\ \overline{r^2+s^2}.x^2-\overline{2\,ar+2\,bs}.x=o$, *or* $\overline{r^2+s^2}.x^2=\overline{2\,ar+2\,bs}.x$; $\therefore$ *dividing by* x, *we have* $\overline{r^2+s^2}.x=2\,ar+2\,bs$, $\therefore\ x=\frac{2.\overline{ar+bs}}{r^2+s^2}$; *consequently* $rx-a=\frac{2\,r.\overline{ar+bs}}{r^2+s^2}-a=$ *side of the first square, and* $sx-b=\frac{2\,s.\overline{ar+bs}}{r^2+s^2}-b=$ *side of the second.*

Examples.—Let $a=6$, $b=4$, $r=5$, $s=3$; *then will* $x=\frac{42}{17}$, $rx-a=\frac{108}{17}$, *and* $sx-b=\frac{58}{17}$.

Let $a=4$, $b=3$, $r=2$, and $s=1$, be given.

5. To find two numbers, of which the sum is equal to the square of the least. *Ans.* 6 *and* 3.

6. To divide the number 30 into two parts, such that their product will be a square number. *Ans.* 27 *and* 3.

7. To divide the number 129 into two parts, the difference of which will be a square number. *Ans.* 105 *and* 24.

8. What two numbers are those, whose product added to the sum of their squares, will make a square? *Ans.* 5 *and* 3.

9. To find two squares, such that their sum added to their product may likewise make a square. *Ans.* $\frac{16}{9}$ *and* $\frac{1}{9}$.

10. To find two numbers, one of which being taken from their product, the remainder will be a cube. *Ans.* 3 *and* 108.

11. To find two numbers, such that either of them being added to the square of the other, the sum will be a square. *Answer* $\frac{16}{11}$ *and* $\frac{43}{11}$.

12. To find three numbers, such that their sum, and likewise the sum of every two of them, will each be a square number. *Ans.* 42, $68\frac{1}{4}$, *and* 22.

PART VII.

ALGEBRA.

INFINITE SERIES [a].

1. A SERIES is a rank of quantities, which usually proceed according to some given law, increasing or decreasing successively; the simple quantities which constitute the series are called its terms.

2. An increasing or diverging series is that in which the terms successively increase, *as* 1, 2, 3, 4, *&c.* $a+3a+7a$, *&c.*

3. A decreasing or converging series is that in which the terms successively decrease, *as* 5, 3, 1, *&c.* $10a-7a-2a$, *&c.*

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[a] The doctrine and application of infinite series, justly considered as the greatest improvements in analysis which modern times can boast, were introduced about the year 1668, by Nicholas Mercator, who is supposed to have taken the first hint of such a method from Dr. Wallis's Arithmetic of Infinites; but it was the genius of Newton that first gave it a body and form.

The principal use of infinite series, is to approximate to the values and sums of such fractional and radical quantities, as cannot be determined by any finite expressions; to find the fluents of fluxions, and thence the length and quadrature of curves, &c. Its application to astronomy and physics is very extensive, and has supplied the means whereby the modern improvements in those sciences have been made. The intricacy of this branch of science has exercised the abilities of some of the most learned mathematicians of Europe, and its usefulness has induced many to direct their chief attention to its improvement: among those authors who have written on the subject, the following are the principal; D'Alembert, Barrow, Briggs, the Bernoullis, Lord Brouncker, Bonnycastle, Des Cartes, Clairaut, Colson, Cotes, Cramer, Condorcet, Dodson, Euler, Emerson, Fermat, Fagnanus, Goldbach, Gravesande, Gregory, Halley, De l'Hôpital, Harriot, Huddens, Huygens, Horsley, Hutton, Jones, Kepler, Keill, Kirkby, Landen, De Lagni, Leibnitz, Lorgna, Manfredi, Monmort, De Moivre, Maclaurin, Montano, Nichole, Newton, Oughtred, Riccati, Regnald, Saunderson, Slusius, Sterling, Stuart, Simpson, Taylor, Varignon, Vieta, Wallis, Waring, &c. &c.

4. A neutral series is that in which the terms neither increase nor decrease, *as* 1, 1, 1, 1, *&c.* $a+a+a+a$, *&c.*

5. An arithmetical series is that in which the terms increase or decrease by an equal difference, *as* 1, 3, 5, 7, *&c.* 9, 6, 3, 0, *&c.* $a+2a+3a$, *&c.*

6. A geometrical series is that in which the terms increase by constant multiplication, or decrease by constant division, *as* 1, 3, 9, 27, *&c.* 12, 6, 3, $\frac{3}{2}$, *&c.* $a+2a+4a+8a$, *&c.*

7. An infinite series is that in which the terms are supposed to be continued without end; or such a series, as from the nature of the law of increase or decrease of its terms requires an infinite number of terms to express it.

8. On the contrary, a series which can be completely expressed by a finite number of terms, is called a finite or terminate series.

9. Infinite series usually arise from the division of the numerator by the denominator of such fractions as do not give a terminate quotient, or by extracting the root of a surd quantity.

10. *To reduce fractions to infinite series.*

RULE I. Divide the numerator by the denominator, until a sufficient number of terms in the quotient be obtained to shew the law of the series.

II. Having discovered the law of continuation, the series may be carried on to any length, without the necessity of farther division.

1. Reduce $\frac{1}{1+x}$ to an infinite series [b].

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[b] If n be an integer, then will

1. $\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+$, &c. to $+b^{n-1}$, which series evidently terminates.

2. $\frac{a^n-b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-$, &c. which terminates in $-b^{n-1}$, when n is an *even* number, but goes on indefinitely when n is *odd*.

3. $\frac{a^n+b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-$, &c. which series terminates in $+b^{n-1}$, when n is an *odd* number, but goes on indefinitely when n is *even*.

OPERATION.

$$1+x)\,1\quad *\,(1-x+x^2-x^3+,\ \&c.\ \textit{the series required.}$$

$$\begin{array}{l} 1+x \\ \hline -x-x^2 \\ -x-x^2 \\ \hline \qquad x^2 \\ \qquad x^2+x^3 \\ \hline \qquad\quad -x^3 \\ \qquad\quad -x^3-x^4 \\ \hline \qquad\qquad\quad x^4\ \&c. \end{array}$$

Explanation.

This operation is similar to those in Art. 50. Part 3. Vol. I. It is unnecessary to proceed farther in the work, since we can readily discover the law by which the terms of the quotient proceed, viz. by constantly multiplying by x, and making the terms alternately + and —; knowing this, we may continue the quotient to any length we please, without troubling ourselves with the work.

2. Reduce $\dfrac{a}{x-z}$ to an infinite series.

OPERATION.

$$x-z)\,a\quad *\,\left(\frac{a}{x}+\frac{az}{x^2}+\frac{az^2}{x^3}+\frac{az^3}{x^4}+,\ \&c.\ \textit{the series required.}\right.$$

$$\begin{array}{l} a-\frac{az}{x} \\ \hline \quad \frac{az}{x} \\ \quad \frac{az}{x}-\frac{az^2}{x^2} \\ \hline \qquad \frac{az^2}{x^2} \\ \qquad \frac{az^2}{x^2}-\frac{az^3}{x^3} \\ \hline \qquad\quad \frac{az^3}{x^3} \\ \qquad\quad \frac{az^3}{x^3}-\frac{az^4}{x^4} \\ \hline \qquad\qquad \frac{az^4}{x^4}\ \&c. \end{array}$$

Explanation.

Here the law of continuation is manifest, the signs being all +, and each term arises by multiplying the numerator of the term immediately preceding it by z, and its denominator by x.

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4. The difference a^n-b^n is not measured by the sum $a+b$.

Hence, first, the *difference* of the nth powers of any two numbers is measured by the *difference* of the numbers, whether n be *even* or *odd*.

Secondly, it is measured by the *sum* of the numbers, when n is *even*, but not when n is *odd*.

Thirdly, the *sum* of the nth powers is measured by the *sum* of the numbers when n is *odd*, but not when n is *even*. In each of the quotients *which terminate*, the number of terms is equal to the index n. See an ingenious application of these conclusions in the Rev. Mr. Bridge's *Lectures on Algebra*, p. 248.

11. **When any quantity is common to every term, the series may be simplified by dividing every term by that quantity, putting the quotients under the vinculum, and placing that quantity before the vinculum, with the sign $\times$ between.**

Thus, in the above series $\frac{a}{x}$ *is common to all the terms, and dividing by* $\frac{a}{x}$, *the quotient is* $1+\frac{z}{x}+\frac{z^2}{x^2}+\frac{z^3}{x^3}+$, &c. *which quotient put under the vinculum and connected with the divisor* $\frac{x}{a}$ *by the sign* $\times$, *the series becomes* $\frac{a}{x}\times\overline{1+\frac{z}{x}+\frac{z^2}{x^2}+\frac{z^3}{x^3}+}$, &c. *which is a simpler form than that in the example.*

3. Reduce $\frac{1}{1-x}$ to an infinite series. *Ans.* $1+x+x^2+x^3+$, &c.

4. Reduce $\frac{az}{a-z}$ to an infinite series. *Ans.* $z+\frac{z^2}{a}+\frac{z^3}{a^2}+\frac{z^4}{a^3}+$, &c.

5. Let $\frac{a}{x+z}$ be converted into an infinite series. *Ans.* $\frac{a}{x}-\frac{az}{x^2}+\frac{az^2}{x^3}-\frac{az^3}{x^4}+$, &c. *or* $\frac{a}{x}\times\overline{1-\frac{z}{x}+\frac{z^2}{x^2}-\frac{z^3}{x^3}+}$, &c. *See ex.* 2.

6. Let $\frac{a^2}{x+b}$ be turned into an infinite series. *Ans.* $\frac{a^2}{x}\times\overline{1-\frac{b}{x}+\frac{b^2}{x^2}-\frac{b^3}{x^3}+}$, &c.

7. Reduce $\frac{1}{3}$, and likewise its equal $\frac{1}{2+1}$, to infinite series.

Ans. $\frac{1}{3}=.3333$, &c. $=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}=$

$3\times\overline{\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+}$, &c.

And $\frac{1}{2+1}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+$, &c. $=\frac{1}{2}\times\overline{1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+}$, &c. $=\frac{1}{2}\times\overline{1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+}$, &c.

12. *To reduce compound quadratic surds to infinite series.*

RULE. Extract the square root, (Art. 57. Part 3. Vol. I.) and continue the work until the law of the series be discovered; after

which the root may be carried to any length, as in the preceding rule; and it will be the series required.

EXAMPLES.—1. Convert $\overline{x^2+z^2}|^{\frac{1}{2}}$ to an infinite series.

OPERATION.

$$x^2+z^2\left(x+\frac{z^2}{2x}-\frac{z^4}{8x^3}+\frac{z^6}{16x^5}-\right., \&c. \textit{ the series required.}$$

$$x^2$$

$$x+\frac{z^2}{2x}\Big)\quad z^2$$

$$z^2+\frac{z^4}{4x^2}$$

$$x+\frac{z^2}{x}-\frac{z^4}{8x^3}\Big)-\frac{z^4}{4x^2}$$

$$-\frac{z^4}{4x^2}-\frac{z^6}{8x^4}+\frac{z^8}{64x^6}$$

$$\frac{z^6}{8x^4}-\frac{z^8}{64x^6}\ \&c.$$

Explanation.

The law of continuation is not obvious in this example, but the series may be made somewhat more simple by dividing all the terms after the first by $\frac{z^2}{2x}$, it will then become

$$x+\frac{z^2}{2x}\times 1-\frac{z^2}{4x^2}+\frac{z^6}{8x^4}-\&c.$$

2. Let $\sqrt{a^2-x^2}$ be converted into an infinite series. *Ans.* $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$, &c.

3. Change $\sqrt{a^2+b}$ into an infinite series. *Ans.* $a+\frac{b}{2a}-\frac{b^2}{8a^3}+\frac{b^3}{16a^5}-$, &c.

4. Express $\overline{1+x}|^{\frac{1}{2}}$ in an infinite series.

13. SIR ISAAC NEWTON'S BINOMIAL THEOREM [*].

For readily finding the powers and roots of binomial quantities.

RULE I. Let P= the first term of any given binomial, Q= the quotient arising from the second term being divided

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[*] This theorem was first discovered by Sir I. Newton in 1669, and sent (in the above form) in a letter dated June 13th, 1676, to Mr. Oldenburgh, at that time Secretary of the Royal Society, in order that it might be communicated to M. Leibnitz. As early as the beginning of the 16th century, Stifelius and others knew how to determine the integral powers of a binomial, not merely by continued multiplication of the root, but also by means of a table, which Stifelius had formed by addition, wherein were arranged the coefficients of the terms of any power within the limits of the table. Vieta seems also to have

by the first; then will PQ = the second term. Let $\frac{m}{n}$ = the index of the power or root required to be found, viz. m = its nume-

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understood the law of the coefficients, but the method of generating them successively one from another, was first taught by Mr. Henry Briggs, Savilian Professor of Geometry at Oxford, about the year 1600: thus the theorem, as far as it relates to *powers*, appears to have been complete, wanting only the algebraic form; this Newton gave it, and likewise extended its application and use to the extraction of *roots* of every description, by infinite series, which probably never was thought of before his time. The theorem was obtained at first by induction, and for some time no demonstration of it appears to have been attempted; several mathematicians have however since given demonstrations, of which the following is perhaps the most simple.

$$\left.\begin{array}{l}\text{Let } \overline{1+x}|^n = 1+px+qx^2+rx^3+sx^4+, \text{ \&c.}\\ \overline{1+y}|^n = 1+py+qy^2+ry^3+sy^4+, \text{ \&c.}\end{array}\right\} \text{each to } n+1 \text{ terms.}$$

Then by subtraction $\overline{1+x}|^n - \overline{1+y}|^n = p.\overline{x-y}+q.\overline{x^2-y^2}+r.\overline{x^3-y^3}+$, &c. to n terms; wherefore $\frac{\overline{1+x}|^n-\overline{1+y}|^n}{\overline{1+x}-\overline{1+y}} = \frac{p.\overline{x-y}+q.\overline{x^2-y^2}+r.\overline{x^3-y^3}}{x-y}$,

that is, (by actual division; see the preceding note,) $\overline{1+x}|^{n-1}+\overline{1+y}.\overline{1+x}|^{n-2}+$, &c. (to n terms) $=p+q.\overline{x+y}+r.\overline{x^2+xy+y^2}$ $+s.\overline{x^3+x^2y+xy^2+y^3}+$, &c. to n terms.

Let $x=y$, then $n.\overline{1+x}|^{n-1}=p+2qx+3rx^2+4sx^3+$, &c. to n terms, whence $n.\overline{1+x}|^n = \dots\dots\dots\dots\dots p+2qx+3rx^2+4sx^3+$, &c. $\times\,\overline{1+x}$

$$\left.\begin{array}{l}=p+2qx+3rx^2+4sx^3+, \text{ \&c.}\\ \quad+\; px+2qx^2+3rx^3+, \text{ \&c.}\end{array}\right\} =$$

$p+\overline{2q+p}.x+\overline{3r+2q}.x^2+\overline{4s+3r}.x^3+$, &c. ($A$). But because $\overline{1+x}|^n=1+px+qx^2+rx^3+$, &c. by the above assumption, therefore $n.\overline{1+x}|^n=n+npx+nqx^2+nrx^3+$, &c. ($B$) wherefore the two series A and B (being each equal to $n.\overline{1+x}|^n$) are equal to one another, and consequently the coefficients of the same powers of x will be equal; that is,

1. $p=n$,

2. $2q+p=np$, or $2q+n=n^2$, ∴ $2q=n^2-n=n.\overline{n-1}$, and $q=\frac{n.\overline{n-1}}{2}$,

3. $3r+2q=nq$, or $3r=\overline{n-2}.q$, ∴ $r=\frac{\overline{n-2}.q}{3}=\frac{n.\overline{n-1}.\overline{n-2}}{2.3}$; &c. &c. &c.

Hence $\overline{1+x}|^n=1+nx+\frac{n.\overline{n-1}}{2}.x^2+\frac{n.\overline{n-1}.\overline{n-2}}{2.3}.x^3+$, &c. ($C$)

Now since $a+b=a\times\overline{1+\frac{b}{a}}$, ∴ $\overline{a+b}|^n=a^n\times\overline{1+\frac{b}{a}}|^n$ = (by substituting $\frac{b}{a}$ for x in the series C) $a^n\times\overline{1+n.\frac{b}{a}+\frac{n.\overline{n-1}}{2}.\frac{b^2}{a^2}+}$, &c. $=a^n+n.a^{n-1}b+$

rator, n=its denominator; then $\overline{P+PQ}|^{\frac{m}{n}}$ will express the given binomial with the index of the required power or root placed over it.

II. Let each of the letters A, B, C, D, &c. represent the value of the term in a series, which immediately precedes the term in which that letter stands.

III. Then will the root or power of the binomial $\overline{P+PQ}|^{\frac{m}{n}}$ be expressed by the following series, viz. $\overline{P}|^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ$ $+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+$, &c.

IV. If the terms and index of any binomial, with their proper signs, be substituted respectively for those in the above general form, then will the series which arises express the power or root required.

EXAMPLES.—1. To extract the square root of a^2-z^2 in an infinite series.

Here $P=a^2$, $Q=-\frac{z^2}{a^2}$, *and (since* $\frac{1}{2}$ *is the index of the square root)* $m=1$, $n=2$; *then* $\overline{P+PQ}|^{\frac{m}{n}}=\overline{a^2-z^2}|^{\frac{1}{2}}$, *and*

$\overline{P}|^{\frac{m}{n}}=(\overline{a^2}|^{\frac{1}{2}}=)$ a=*the first term* . . . . A.

$+\frac{m}{n}AQ=(\frac{1}{2}\times A\times -\frac{z^2}{a^2}=\frac{1}{2}\times a\times -\frac{z^2}{a^2}=-\frac{az^2}{2a^2}=)-\frac{z^2}{2a}=$ *the second term* . . . . B.

$+\frac{m-n}{2n}BQ=(\frac{1-2}{4}\times B\times -\frac{z^2}{a^2}=-\frac{1}{4}\times -\frac{z^2}{2a}\times -\frac{z^2}{a^2}=)-$ $\frac{z^4}{8a^3}$=*the third term* . . . . C.

$\frac{n.n-1}{2}.a^{n-2}b^2+$, &c. in which, if $\overline{P}|^{\frac{m}{n}}$ be substituted for a^n, Q for $\frac{b}{a}$, and A, B, C, &c. for the preceding terms, the series will become $\overline{P}|^{\frac{m}{n}}+\frac{m}{n}.AQ+$ $\frac{m-n}{2n}.BQ+\frac{m-2n}{3n}.CQ+$, &c. as above.

If the index $\frac{m}{n}$ be a positive whole number, the series will terminate at the $\overline{\frac{m}{n}+1}$*th* term; but if it be negative, or fractional, the series will not terminate: all which is manifest from the above examples.

$$+\frac{m-2n}{3n}CQ=\left(\frac{1-4}{6}\times C\times-\frac{z^2}{a^2}=-\frac{1}{2}\times-\frac{z^4}{8a^3}\times-\frac{z^2}{a^2}=\right)-$$

$\frac{z^6}{16a^5}$ = *the fourth term* .... D.

$$+\frac{m-3n}{4n}DQ=\left(\frac{1-6}{8}\times D\times-\frac{z^2}{a^2}=-\frac{5}{8}\times-\frac{z^6}{16a^5}\times-\frac{z^2}{a^2}=\right)$$

$-\frac{5z^8}{128a^7}$ = *the fifth term* .... E.

$$+\frac{m-4n}{5n}EQ=\left(\frac{1-8}{10}\times E\times-\frac{z^2}{a^2}=-\frac{7}{10}\times-\frac{5z^8}{128a^7}\times-\frac{z^2}{a^2}=\right)$$

$-\frac{7z^{10}}{256a^9}$ = *the sixth term* .... F.

&c. &c. *Wherefore the square root of the given binomial, or* $\overline{a^2-z^2}|^{\frac{1}{2}}=a-\frac{z^2}{2a}-\frac{z^4}{8a^3}-\frac{z^6}{16a^5}-\frac{5z^8}{128a^7}-\frac{7z^{10}}{256a^9}$, &c. *as required.*

2. Find $\overline{a+b}|^{\frac{3}{5}}$ in an infinite series.

Here $P=a$, $Q=\frac{b}{a}$, $m=3$, $n=5$, *and* $\overline{P+PQ}|^{\frac{m}{n}}=\overline{a+b}|^{\frac{3}{5}}$, *wherefore*

$\overline{P}|^{\frac{m}{n}}=a^{\frac{3}{5}}$ *the first term of the series* .... A.

$+\frac{m}{n}AQ=\left(\frac{3}{5}\times A\times\frac{b}{a}=\frac{3}{5}\times a^{\frac{3}{5}}\times\frac{b}{a}=\right)\frac{3b}{5a^{\frac{2}{5}}}$ *the second term* B.

$+\frac{m-n}{2n}BQ=\left(\frac{3-5}{10}\times B\times\frac{b}{a}=-\frac{1}{5}\times\frac{3b}{5a^{\frac{2}{5}}}\times\frac{b}{a}=\right)-\frac{3b^2}{25a^{\frac{7}{5}}}$ *the third term* .... C.

$$+\frac{m-2n}{3n}CQ=\left(\frac{3-10}{15}\times C\times\frac{b}{a}=-\frac{7}{15}\times-\frac{3b^2}{25a^{\frac{7}{5}}}\times\frac{b}{a}=\right)$$

$\frac{7b^3}{125a^{\frac{12}{5}}}$ *the fourth term* .... D.

$$+\frac{m-3n}{4n}DQ=\left(\frac{3-15}{20}\times D\times\frac{b}{a}=-\frac{3}{5}\times\frac{7b^3}{125a^{\frac{12}{5}}}\times\frac{b}{a}=\right)-$$

$\frac{21b^4}{625a^{\frac{17}{5}}}$ *the fifth term* .... E.

&c. &c. *Wherefore* $\overline{a+b}|^{\frac{3}{5}}=a^{\frac{3}{5}}+\frac{3b}{5a^{\frac{2}{5}}}-\frac{3b^2}{25a^{\frac{7}{5}}}+\frac{7b^3}{125a^{\frac{12}{5}}}-\frac{21b^4}{625a^{\frac{17}{5}}}$ +, &c. (*which by Art.* 9.) $=a^{\frac{3}{5}}\times\overline{1+\frac{3b}{5a}-\frac{3b^2}{25a^2}+\frac{7b^3}{125a^3}-\frac{21b^4}{625a^4}}$ +, &c.

3. To find the value of $\frac{y^2}{\sqrt{y^2+z^2}}$ in an infinite series.

Here $\frac{y^2}{\sqrt{y^2+z^2}} = y^2 \times \overline{y^2+z^2}|^{-\frac{1}{2}}$; *we first find* $\overline{y^2+z^2}|^{-\frac{1}{2}}$, *and then multiply the resulting series by* y^2; *wherefore in the present case* $P=y^2$, $Q=\frac{z^2}{y^2}$, $m=-1$, $n=2$, *and* $\overline{P+PQ}|^{\frac{m}{n}} = \overline{y^2+z^2}|^{-\frac{1}{2}}$, *then* $\overline{P}|^{\frac{m}{n}} = \overline{y^2}|^{-\frac{1}{2}} = (y^{-1}=)\ \frac{1}{y}$ *the first term* .. *A.*

$+\frac{m}{n}AQ = (-\frac{1}{2} \times A \times \frac{z^2}{y^2} = -\frac{1}{2} \times \frac{1}{y} \times \frac{z^2}{y^2} =) - \frac{z^2}{2y^3}$ *the 2nd term* .... *B.*

$+\frac{m-n}{2n}BQ = (\frac{-1-2}{4} \times B \times \frac{z^2}{y^2} = -\frac{3}{4} \times -\frac{z^2}{2y^3} \times \frac{z^2}{y^2} =)\ \frac{3z^4}{8y^5}$ *the 3rd term* .... *C.*

$+\frac{m-2n}{3n}CQ = (\frac{-1-4}{6} \times C \times \frac{z^2}{y^2} = -\frac{5}{6} \times \frac{3z^4}{8y^5} \times \frac{z^2}{y^2} =) - \frac{5z^6}{16y^7}$ *the 4th term* .... *D.*

$+\frac{m-3n}{4n}DQ = (\frac{-1-6}{8} \times D \times \frac{z^2}{y^2} = -\frac{7}{8} \times -\frac{5z^6}{16y^7} \times \frac{z^2}{y^2} =)$ $\frac{35z^8}{128y^9}$ *the 5th term* .... *E.*

&c. &c. *This series multiplied by* y^2, *according to what was premised, we have* $\frac{y^2}{\sqrt{y^2+z^2}} = (y^2 \times \overline{\frac{1}{y} - \frac{z^2}{2y^3} + \frac{3z^4}{8y^5} - \frac{5z^6}{16y^7} + \frac{35z^8}{128y^9} -, \&c.} =)\ y - \frac{z^2}{2y} + \frac{3z^4}{8y^3} - \frac{5z^6}{16y^5} + \frac{35z^8}{128y^7} -$, &c. *the series required.*

4. To involve 12, or its equal 11+1, to the cube.

Here $P=11$, $Q=\frac{1}{11}$, $m=3$, $n=1$; *then, as before,* $\overline{P+PQ}|^{\frac{m}{n}} = \overline{P}|^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ +$, &c. $=(\overline{11}|^3=)1331$ $(+\frac{3}{1} \times 1331 \times \frac{1}{11} =) + 363$ $(+\frac{2}{2} \times 363 \times \frac{1}{11} =) + 33$ $(+\frac{1}{3} \times 33 \times \frac{1}{11} =) + 1$, *where (since the coefficient of the next term will be* 0) *the series must evidently terminate. Wherefore collecting the above terms,* $(1331+363+33+1=)$ 1728 *is the cube of* 12, *as was required.*

5. Find the value of $\overline{x+y}|^{\frac{1}{3}}$ in an infinite series. *Ans.* $x^{\frac{1}{3}}+\frac{y}{3x^{\frac{2}{3}}}-\frac{y^2}{9x^{\frac{5}{3}}}+\frac{5y^3}{81x^{\frac{8}{3}}}-$, &c.

6. To find $\frac{a}{b+c}$ in an infinite series. *Ans.* $\frac{a}{b}\times\overline{1-\frac{c}{b}+\frac{c^2}{b^2}-\frac{c^3}{b^3}+}$, &c.

7. Find $\sqrt{a^2+b}$ in an infinite series. *Ans.* $a+\frac{b}{2a}-\frac{b^2}{8a^3}+\frac{b^3}{16a^5}-$, &c.

8. Extract the 5th root of 248832 by infinite series. *Ans.* 12.

9. Find $\frac{1}{\overline{x+y}|^2}$ by infinite series. *Ans.* $\frac{1}{x^2}-\frac{2y}{x^3}+\frac{3y^2}{x^4}-\frac{4y^3}{x^5}+$, &c.

10. Find $\frac{r^2}{r+x}$ in an infinite series.

11. To find $\sqrt[5]{x^2-z^2}$ in an infinite series.

12. Find $y\times\overline{y-z}|^{\frac{7}{5}}$ in an infinite series.

14. *A series being given, to find the several orders of differences.*

RULE I. Subtract the first term from the second, the second from the third, the third from the fourth, and so on; the several remainders will constitute a new series, called *the first order of differences.*

II. In this new series, take the first term from the second, the second from the third, &c. as before, and the remainders will form another new series, called *the second order of differences.*

III. Proceed in the same manner for the *third, fourth, fifth, &c. orders*, until either the differences become 0, or the work be carried as far as is thought necessary [d].

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[d] Let $a, b, c, d, e,$ &c. be the terms of a given series, then if $D=$ the first term of the nth order of differences, the following theorem will exhibit the value of D: viz. $\pm a\mp nb\pm n.\frac{n-1}{2}.c\mp n.\frac{n-1}{2}.\frac{n-2}{3}.d\pm n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e\mp$, &c. (to $n+1$ terms) $=D$, where the upper signs must be taken when n is an even number, and the lower signs when n is odd.

EXAMPLES.—1. Given the series 1, 4, 8, 13, 19, 26, &c. to find the several orders of differences.

Thus 1, 4, 8, 13, 19, 26, *&c. the given series.*
Then ... 3, 4, 5, 6, 7, *&c. the first differences.*
And ..... 1, 1, 1, 1, *&c. the second differences.*
Also .... 0, 0, 0, *&c. the third differences.*

where the work evidently must terminate.

2. Given the series 1, 4, 8, 16, 32, 64, 128, &c. to find the several orders of differences.

Here 1, 4, 8, 16, 32, 64, 128, *&c. given series.*
And ... 3, 4, 8, 16, 32, 64, *&c. 1st diff.*
1, 4, 8, 16, 32, *&c. 2nd diff.*
3, 4, 8, 16, *&c. 3rd diff.*
1, 4, 8, *&c. 4th diff.*
3, 4, *&c. 5th diff.*
1, *&c. 6th diff. &c.*

3. Find the several orders of differences in the series 1, 2, 3, 4, &c. *Ans. First differences* 1, 1, 1, 1, *&c. Second diff.* 0, 0, 0, *&c.*

4. To find the several orders of differences in the series 1, 4, 9, 16, 25, &c. *Ans. First differences* 3, 5, 7, 9, *&c. Second* 2, 2, 2, *&c. Third* 0, 0, *&c.*

5. Required the orders of differences in the series 1, 8, 27, 64, 125, &c.

6. Given 1, 6, 20, 50, 105, &c. to find the several orders of differences.

7. Given the series 1, 3, 7, 13, 21, &c. to find the third and fourth orders of differences.

15. *To find any term of a given series.*

RULE I. Let a, b, c, d, e, &c. be the given series; $d^{\text{I}}, d^{\text{II}}, d^{\text{III}}, d^{\text{IV}}$, &c. respectively, the first term of the first, second, third, fourth, &c. order of differences, as found by the preceding article; n = the number denoting the place of the term required.

If the differences be very great, the logarithms of the quantities may be used, the differences of which will be much smaller than those of the quantities themselves; and at the close of the operation the natural number answering to the logarithmical result will be the answer. See *Emerson's Differential Method, prop.* 1.

II. Then will $a+\frac{n-1}{1}.d^{\text{I}}+\frac{n-1}{1}.\frac{n-2}{2}.d^{\text{II}}+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}$ $.d^{\text{III}}+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}.\frac{n-4}{4}.d^{\text{IV}}+$&c.=to the n^{th} term required *.

EXAMPLES.—1. To find the 10th term of the series 2, 5, 9, 14, 20, &c.

Here (*Art.* 12.) 2, 5, 9, 14, 20, &c. *series.*
3, 4, 5, 6, &c. 1*st diff.*
1, 1, 1, &c. 2*nd diff.*
0, 0, &c. 3*rd diff.*

Where $d^{\text{I}}=3$, $d^{\text{II}}=1$, $d^{\text{III}}=0$, *also* $a=2$, $n=10$; *wherefore* $a+\frac{n-1}{1}.d^{\text{I}}+\frac{n-1}{1}.\frac{n-2}{2}.d^{\text{II}}=(2+\frac{10-1}{1}\times3+\frac{10-1}{1}\times\frac{10-2}{2}\times 1=)\ 2+27+36=65=$ *the* 10*th term required.*

2. To find the 20th term of the series 2, 6, 12, 20, 30, &c.

Here $a=2$, $n=20$; *and Art.* 12.
2, 6, 12, 20, 30, &c. *series.*
4, 6, 8, 10, &c. 1*st diff.*
2, 2, 2, &c, 2*nd diff. or* $d^{\text{I}}=4$, $d^{\text{II}}=2$; *whence* $a+\frac{n-1}{1}.d^{\text{I}}+\frac{n-1}{1}.\frac{n-2}{2}.d^{\text{II}}=(2+\frac{19}{1}\times4+\frac{19}{1}\times\frac{18}{2}\times2=)\ 2+76+342=420=$ *the* 20*th term required.*

3. Required the 5th term of the series 1, 3, 6, 10, &c. *Ans.* 15.

4. To find the 10th term of the series 1, 4, 8, 13, 19, &c. *Ans.* 64.

5. To find the 14th term of the series 3, 7, 12, 18, 25, &c. *Ans.* 133.

6. Required the 20th term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

7. To find the 50th term of 1, 4, 8, 13, 19, &c.

8. To find the 10th term of 3, 7, 12, 18, 25, &c.

16. *If the succeeding terms of a given series be at an unit's distance from each other, any intermediate term may be found by interpolation, as follows.*

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* For the investigation of this rule, see *Emerson's Differential Method, prop.* 2.

RULE I. Let y be the term to be interpolated, x its distance from the beginning of the series, d^{I}, d^{II}, d^{III}, d^{IV}, &c. the first terms of the several orders of differences.

II. Then will $a+xd^{I}+x.\frac{x-1}{2}.d^{II}+x.\frac{x-1}{2}.\frac{x-2}{3}.d^{III}+x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}.d^{IV}+$ &c. $=y$, the term required [f].

EXAMPLES.—1. Given the logarithms of 105, 106, 107, 108, and 109, to find the logarithm of 107.5.

Series.	*Logarithms.*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
105	.0211893	41166			
106	0253059	40779	−387	−8	
107	0293838	40400	−379	−6	−2.
108	0334238	40027	−373		
109	0374265				

Here $x=(107.5-105=2.5)\ \frac{5}{2}=$ *the distance of the term* y, $a=.0211893$, $d^{I}=41166$, $d^{II}=-387$, $d^{III}=-8$, $d^{IV}=-2$.

Then $y=a+xd^{I}+x.\frac{x-1}{2}.d^{II}+x.\frac{x-1}{2}.\frac{x-2}{3}.d^{III}+x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}d^{IV}=(a+\frac{5}{2}d^{I}+\frac{5}{2}\times\frac{3}{4}\times d^{II}+\frac{5}{2}\times\frac{3}{4}\times\frac{1}{6}\times d^{III}+\frac{5}{2}\times\frac{3}{4}\times\frac{1}{6}\times-\frac{1}{8}\times d^{IV}=a+\frac{5}{2}d^{I}+\frac{15}{8}d^{II}+\frac{5}{16}d^{III}-\frac{5}{128}d^{IV}=.0211893+\frac{5}{2}\times41166+\frac{15}{8}\times-387+\frac{5}{16}\times-8-\frac{5}{128}\times.-2=)$ $0211893+102915-725-2.5+.078=.031407128$, *the logarithm required.*

2. Given the logarithmic sines of $3^{0}\ 4^{1}$, $3^{0}\ 5^{1}$, $3^{0}\ 6^{1}$, $3^{0}\ 7^{1}$, and $3^{0}\ 8^{1}$, to find the sine of $3^{0}\ 6^{1}\ 15^{11}$.

Series.	*Logarithms.*	*1st diff.*	*2nd diff.*	*3rd diff.*
$3^{0}\ 4^{1}$	8.7283366	23516		
3 5	8.7306882	23390	−126	1
3 6	8.7330272	23263	−127	−4
3 7	8.7353535	23140	−123	
3 8	8.7376675			

Here $x=(3^{0}\ 6^{1}\ 15^{11}-3^{0}\ 4^{1}=2^{0}\ 15^{1}=)\frac{9}{4}=$ *the distance of the term* y, *to be interpolated*; $a=8.7283366$, $d^{I}=23516$, $d^{II}=-126$,

[f] This rule is investigated in Emerson's *Differential Method, prop.* 5.

$d^{111}=1$, *and* $y=a+xd^{1}+x.\frac{x-1}{2}.d^{11}+x.\frac{x-1}{2}.\frac{x-2}{3}.d^{111}=(a+\frac{9}{4}$ $d^{1}+\frac{45}{32}d^{11}+\frac{15}{128}d^{111}=)8.7283366+.0052911-.00001771875+$ $.0000000117=8.73360999296$, *the log. sine required.*

3. Given the series $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, to find the term which stands in the middle, between $\frac{1}{52}$ and $\frac{1}{53}$. *Ans.* $\frac{1}{105}$.

4. Given the logarithmic sines of $1^{\circ}\ 0^{1}$, $1^{\circ}\ 1^{1}$, $1^{\circ}\ 2^{1}$, and $1^{\circ}\ 3^{1}$, to find the logarithmic sine of $1^{\circ}\ 1^{1}\ 40^{11}$. *Ans.* 8.2537533.

5. Given the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c. to find the middle term between $\frac{1}{5}$ and $\frac{1}{6}$.

17. *If the first differences of a series of equidifferent terms be small, any intermediate term may be found by interpolation, as follows.*

RULE I. Let a, b, c, d, e, &c. represent the given series, and $n=$ the number of terms given.

II. Then will $a-nb+n.\frac{n-1}{2}.c-n.\frac{n-1}{2}.\frac{n-2}{3}.d+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e+$, &c.$=o$, from whence, by transposition, &c. any required term may be obtained [s].

EXAMPLES.—1. Given the square root of 10, 11, 12, 13, and 15, to find the square root of 14.

Here $n=5$, and e is the term required.

$a=(\sqrt{10}=)3.1622776$
$b=(\sqrt{11}=)3.3166248$
$c=(\sqrt{12}=)3.4641016$
$d=(\sqrt{13}=)3.6055512$
$f=(\sqrt{15}=)3.8729833$

And since $n=5$, the series must be continued to 6 terms.

Therefore $a-nb+n.\frac{n-1}{2}.c-n.\frac{n-1}{2}.\frac{n-2}{3}.d+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e-n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.f=o.$

[s] For the investigation of this rule, see *Emerson's Differential Method, prop.* 6.

Whence, by transposition, in order to find e, we shall have $n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e = -a + nb - n.\frac{n-1}{2}.c + n.\frac{n-1}{2}.\frac{n-2}{3}.d + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.f$; *this in numbers becomes* $5e = -3.1622776 + 5 \times 3.3166248 - 10 \times 3.4641016 + 10 \times 3.6055512 + 3.8729833 = 56.5116193 - 37.8032936 = 18.7083257$, *and* $e = \frac{18.7083257}{5} = 3.74166514$ = *the root, nearly.*

2. Given the square roots of 37, 38, 39, 41, and 42, to find the square root of 40. *Ans.* 6.32455532.

3. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. *Ans.* 3.684033.

4. Given the logarithms of 108, 109, 110, 111, 112, and 114, to find the logarithm of 113. *Ans.* 2.0530784.

18. *To revert a given series.*

When the powers of an unknown quantity are contained in the terms of a series, the finding the value of the unknown quantity in another series, which involves the powers of the quantity to which the given series is equal, and known quantities only, is called reverting the series[h].

RULE I. Assume a series for the value of the unknown quantity, of the same form with the series which is required to be reverted.

II. Substitute this series and its powers, for the unknown quantity and its powers, in the given series.

III. Make the resulting terms equal to the corresponding terms of the given series, whence the values of the assumed coefficients will be obtained.

EXAMPLES.—1. Let $ax + bx^2 + cx^3 + dx^4 +$, &c. $= z$ be given, to find the value of x in terms of z and known quantities.

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[h] Various methods of reversion may be seen, as given by Demoivre, in the Philosophical Transactions, No. 240. in Maclaurin's Algebra, p. 263, &c. Colson's Comment on Newton's Fluxions, p. 219; Horsley's Ed. of Newton's Works, vol. I. p. 291, &c. Stuart's Explanation of Newton's Analysis, p. 455. Simpson's Fluxions, &c. &c.

Let $z^n=x$, then it is plain that if z^n and its powers be substituted in the given series for x and its powers, the indices of z will be n, $2n$, $3n$, $4n$, &c. and 1; whence $n=1$, and the differences of these indices are 0, 1, 2, 3, 4, &c. Wherefore the indices of the series to be assumed, must have the same differences; let therefore this series be $Az+Bz^2+Cz^3+Dz^4+$, &c.$=x$. And if this series be involved, and substituted for the several powers of x, in the given series, it will become

$$\left.\begin{array}{llll} aAz+aBz^2+aCz^3+aDz^4+, \&c. \\ * \quad +bA^2z^2+2bABz^3+2bACz^4+, \&c. \\ * \quad\quad * \quad\quad\quad * \quad\quad + \quad bB^2z^4+, \&c. \\ * \quad\quad * \quad +cA^3z^3+3cA^2Bz^4+, \&c. \\ * \quad\quad * \quad\quad\quad * \quad + \quad dA^4z^4+, \&c. \end{array}\right\}=z.$$

Whence, by equating the terms which contain like powers of z, we obtain $(aAz=z$, *or*$)\,A=\frac{1}{a}$; $(aBz^2+bA^2z^2=o$, *whence*$)\,B=(-\frac{bA^2}{a}=)-\frac{b}{a^3}$, $(aCz^3+2bABz^3+cA^3z^3=o$; *whence*$)\,C=(-\frac{2bAB+cA^3}{a}=)\frac{2b^2-ac}{a^5}$; $D=(-\frac{2bAC+bB^2+3cA^2B+dA^4}{a}=)\frac{5abc-5b^3-a^2d}{a^7}$, &c. *and consequently* $x=(Az+Bz^2+Cz^3+$, &c. $=)\frac{z}{a}-\frac{bz^2}{a^3}+\frac{2b^2-ac}{a^5}z^3-\frac{5b^3-5abc+a^2d}{a^7}z^4+$, &c. *the series required.*

This conclusion forms a general theorem for every similar series, involving the like powers of the unknown quantity.

2. Let the series $x-x^2+x^3-x^4+$, &c.$=z$, be proposed for reversion.

Here $a=1$, $b=-1$, $c=1$, $d=-1$, &c. these values being substituted in the theorem derived from the preceding example, we thence obtain $x=z+z^2+z^3+z^4+$, &c. the answer required.

3. Let $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$, &c.$=y$, be given for reversion.

Substituting as before, we have $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{3}$, and $d=-\frac{1}{4}$, &c. These values being substituted, we shall have $x=y+\frac{y^2}{2}+\frac{y^3}{6}+\frac{y^4}{24}+$, &c. from which if y be given, and sufficiently small for the series to approximate, the value of x will be known.

4. Let $x^m+bx^{m+p}+cx^{m+2p}+dx^{m+3p}+$, &c.$=z$.

Let $z^n=x$, *then, if* z *be transposed, the indices will be* 1, nm, $nm+np$, $nm+2np$, $nm+3np$, *&c. where, if the two least,* 1 *and* nm, *be made equal to each other, we shall have* $n=\frac{1}{m}$; *and the differences are* $\frac{p}{m}$, $\frac{2p}{m}$, $\frac{3p}{m}$, $\frac{4p}{m}$, *&c. The series therefore to be assumed for* x *is* $Az^{\frac{1}{m}}+Bz^{\frac{1+p}{m}}+Cz^{\frac{1+2p}{m}}+Dz^{\frac{1+3p}{m}}+$, *&c.*$=z$: *this series being involved, and the like terms of both compared as before, we have* $A=1$, $B=-\frac{b}{m}$, $C=\frac{\overline{1+m+2p}.bb-2mc}{2m^2}$, $D=-\frac{\overline{2m^2+9mp+9p^2+3m+6p+1}.b^3}{6m^3}+\frac{\overline{1+m+3p}.bc}{m^2}-\frac{d}{m}$, *&c. from whence the value of* x *being found, theorems for innumerable cases may thence be deduced.*

5. Revert the series $z+\frac{z^3}{6}+\frac{3z^5}{40}+\frac{5z^7}{112}+$, &c.$=x$. *Ans* $z=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+$, &c.

6. Revert the series $ax+bx^2+cx^3+dx^4+$, &c.$=gz+hz^2+kz^3+lz^4+$, &c.

19. *To find the sum of n terms of an infinite series.*

RULE I. Let a, b, c, d, e, &c. be the given series, $s=$the sum of n terms, and d', d'', d''', d^{iv}, &c. respectively the first terms of the several orders of differences, found by Art. 12.

II. Then will $na+n.\frac{n-1}{2}.d'+n.\frac{n-1}{2}.\frac{n-2}{3}.d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.d'''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.d^{iv}+$, &c.$=s$, the sum of n terms of the series, as was required[1].

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[1] This rule is investigated by Mr. Emerson, in his *Differential Method, prop.* 3. The investigations of this and some of the foregoing rules, although not difficult, are rather prolix, and require too much room to be admitted within the compass of notes; for this reason they are omitted. The following problems on the summation of series, which afford but a very imperfect specimen of that noble branch, were taken mostly from *Dodson's Mathematical Repository*, vol. I. where a great number of problems on the subject, with in-

PROB. 1. To find the sum of n terms of the series 1, 2, 3, 4, 5, &c.

First, by Art. 12. 1, 2, 3, 4, 5, &c. *the given series.*
1, 1, 1, 1, &c. *first differences.*
0, 0, 0, &c. *second differences.*

Here $a=1$, $d^1=1$, $d^{11}=o$; *then will* $na+n.\frac{n-1}{2}.d^1=$ $(\frac{2na+\overline{n^2-n}.d^1}{2}$, *which, (since* a *and* d^1 *each* $=1)=\frac{2n+n^2-n}{2}=)$ $\frac{n.\overline{n+1}}{2}=s$, *the sum required.*

The sum of n terms of this series may likewise be found as follows.

Let $1+2+3+4+5+$, &c. . . . $+n=s$

Invert this series, and $n+\overline{n-1}+\overline{n-2}+\overline{n-3}+\overline{n+4}+$, &c. . . . . $+1=s$.

Add both series together, and $\overline{n+1}+\overline{n+1}+\overline{n+1}+\overline{n+1}+\overline{n+1}$ $+$, &c. . . . $+\overline{n+1}=2s$; *that is,* $n.\overline{n+1}=2s$, *whence* $s=\frac{n.\overline{n+1}}{2}$, *as before.*

EXAMPLES.—1. Let the sum of 20 terms of the above series be required.

Here $n=20$, *and* $s=\frac{n.\overline{n+1}}{2}=\frac{20\times21}{2}=110$, *the answer.*

2. Let the sum of 1000 terms be required. *Ans.* 500500.

3. Let the sum of 12345 terms be required.

PROB. 2. To find the sum of n terms of the series 1, 3, 5, 7, 9, &c.

Here 1, 3, 5, 7, 9, &c. *the given series.*
2, 2, 2, 2, &c. . . *first difference.*
0, 0, 0, &c. . . *second difference.*

Wherefore $a=1$, $d'=2$, $d''=o$, *and* $na+n.\frac{n-1}{2}.d'=(na+\frac{n^2-n}{2}.d'=$ *(since* $a=1$ *and* $d=2)$ $n+n^2-n=)$ $n^2=s$, *the sum required.*

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genious solutions, may be seen. The doctrine of Infinite Series will probably never be complete; but it would require a very large treatise to do ample justice to the subject, even in its present state.

Or thus,

Let $1+3+5+7+9+$, &c. . . . . $+\overline{2n-1}=s$.

This inverted, is $\overline{2n-1}+\overline{2n-3}+\overline{2n-5}+\overline{2n-7}+\overline{2n-9}+$, &c. . . . . . . $+1=s$.

The sum of both is $\overline{2n+2n+2n+2n+2n+}$, &c. . . $+2n=2s$.

Whence n terms of this sum is $2\,n.n=2\,s$, *or* $s=n^2$, *as before.*

EXAMPLES.—1. To find the sum of 10 terms of the above series.

Here $n=10$, *and* $s=(n^2=)$ 100, *the answer.*

2. To find the sum of 50 terms. *Ans.* 2500.

3. To find the sum of 1928 terms.

PROB. 3. To find the sum of n terms of the series of squares 1, 4, 9, 16, 25, &c.

Here 1, 4, 9, 16, 25, &c. *the series.*

3, 5, 7, 9, &c. . . . . 1*st diff.*

2, 2, 2, &c. . . . . 2*nd diff.*

0, 0, &c. . . . . 3*rd diff.*

Whence $a=1$, $d'=3$, $d''=2$, $d'''=o$, *and* $na+n.\frac{n-1}{2}.d'+n.\frac{n-1}{2}.\frac{n-2}{3}.d''=(n+3\,n.\frac{n-1}{2}+2\,n.\frac{n-1}{2}.\frac{n-2}{3}=\frac{3\,n^2-n}{2}+\frac{n^3-3\,n^2+2\,n}{3}=)\ \frac{n.\overline{n+1}.\overline{2\,n+1}}{6}=s$, *the sum required.*

EXAMPLES.—1. Let the sum of 30 terms of the above series be required.

Here $n=30$; *wherefore* $\frac{n.\overline{n+1}.\overline{2\,n+1}}{6}=\frac{30\times 31\times 61}{6}=9455$, *the answer.*

2. Let the sum of 70 terms be required.

PROB. 4. To find the sum of n terms of the series $a+\overline{a+d}+\overline{a+2\,d}+\overline{a+3\,d}+$, &c.

Here, by the rule, $na+n.\frac{\overline{n-1}}{2}.d=na+\frac{n.\overline{n-1}.d}{2}=s$, *the sum required.*

Or thus,

Since the series $a+\overline{a+d}+\overline{a+2\,d}+\overline{a+3\,d}+$, &c.

$$=\left\{\begin{array}{l}\overline{+1+1+1+1+1+,\ \&c.}\times a\\ \overline{+0+1+2+3+4+,\ \&c.}\times d\end{array}\right\}=s,$$

We have the sum of the first of these, $+1+1+1+1+$, &c. (*to*

n terms) $=n$: *and the sum of the latter,* $+0+1+2+3+$, *&c.* *(to n terms)* $=\frac{n.\overline{n-1}}{2}$, *(theor. 22. Arithmetical Progression,)* *wherefore* $na+\frac{n.\overline{n-1}}{2}.d=s$, *as before.*

Or thus,

Because $\overline{a}+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+$, *&c.......* $+a+\overline{n-1}.d=s$,

And $a+\overline{nd-d}+a+\overline{nd-2d}+a+\overline{nd-3d}+a+\overline{nd-4d}+$, *&c.......* $+a=s$,

The sum of both, $\}\ 2a+\overline{nd-d}+2a+\overline{nd-d}+2a+\overline{nd-d}+2a+\overline{nd-d}+$, *&c.......* $+2a+\overline{nd-d}=2s$.

That is, $\overline{2a+nd-d}.n=2s$, *or* $s=(\frac{\overline{2a+\overline{n-1}.d}.n}{2}=)\ na+\frac{n.\overline{n-1}}{2}.d$, *as before.*

Prob. 5. To find the sum of n terms of the series $1, x, x^2, x^3$, &c.

Let $1+x+x^2+x^3+$, *&c. (to* x^{n-1}*)* $=s$; *multiply this series by* x, *and* $x+x^2+x^3+x^4+$, *&c. (to* x^n*)* $=sx$; *subtracting the upper from the lower, we have* $-1+x^n=sx-s$; *whence* $s=\frac{x^n-1}{x-1}$, *the sum required.*

When x is a proper fraction, the sum of the series in infinitum may be found in the same manner.

Thus $1+x+x^2+x^3+$, *&c.* $=s$.

And $x+x^2+x^3+x^4+$, *&c.* $=sx$; *whence, subtracting as before,* $-1=sx-s$; *and* $s=\frac{1}{1-x}$, *the sum of the series in infinitum.*

Prob. 6. To find the sum of an infinite number of terms of the circulating decimal .99999, &c.

First, .99999, *&c.* $=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+$, *&c.* $=s$, *that is,* $9\times\overline{\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}}+$, *&c.* $=s$; *divide by* 9, *and* $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+$, *&c.* $=\frac{s}{9}$; *multiply this by* 10, *and* $1+\frac{1}{10}$

$+\frac{1}{100}+\frac{1}{1000}+$, &c.$=\frac{10s}{9}$; *subtract the last but one from the last, and* $1=(\frac{10s}{9}-\frac{s}{9}=)\frac{9s}{9}$, *or* $s=1$, *the sum required.*

Hence,

$$\text{The sum of}\left\{\begin{array}{l} .1111,\ \&c.\ or\ \frac{1}{9} \\ .2222,\ \&c.\ or\ \frac{2}{9} \\ .3333,\ \&c.\ or\ \frac{1}{3} \\ .4444,\ \&c.\ or\ \frac{4}{9} \\ .5555,\ \&c.\ or\ \frac{5}{9} \\ .6666,\ \&c.\ or\ \frac{2}{3} \\ .7777,\ \&c.\ or\ \frac{7}{9} \\ .8888,\ \&c.\ or\ \frac{8}{9} \end{array}\right\}\text{of } .9999,\ \&c.=\left\{\begin{array}{l} \frac{1}{9} \\ \frac{2}{9} \\ \frac{1}{3} \\ \frac{4}{9} \\ \frac{5}{9} \\ \frac{2}{3} \\ \frac{7}{9} \\ \frac{8}{9} \end{array}\right.$$

PROB. 7. To find the sum of n terms of the series $a^2+\overline{a+d}|^2+\overline{a+2d}|^2+\overline{a+3d}|^2+$, &c.

First, by actually squaring the terms, we have

$$\begin{array}{l} a^2 \quad =a^2 \\ \overline{a+d}|^2 \ =a^2+2\times 1\,ad+\ 1\ d^2 \\ \overline{a+2d}|^2=a^2+2\times 2\,ad+\ 4\ d^2 \\ \overline{a+3d}|^2=a^2+2\times 3\,ad+\ 9\ d^2 \\ \overline{a+4d}|^2=a^2+2\times 4\,ad+16\,d^2 \\ \quad\&c. \qquad\qquad \&c. \end{array}$$

$$\left.\begin{array}{l} \textit{Whence}\ \overline{1+1+1+1+\&c.\ (to\ n\ terms)}\times a^2 \\ +\overline{0+1+2+3+\&c.\ (to\ n\ terms)}\times 2ad \\ +\overline{0+1+4+9+\&c.\ (to\ n\ terms)}\times d^2 \end{array}\right\}=s.$$

$$\begin{array}{l}\textit{But} \\ \textit{the sum of}\end{array}\left\{\begin{array}{l} 1+1+1+1+\&c. \\ 0+1+2+3+\&c. \\ 0+1+4+9+\&c. \end{array}\right\}\textit{to } n \textit{ terms}=\left\{\begin{array}{l} n \\ \frac{n.\overline{n-1}}{1\times 2} \\ \frac{n.\overline{n-1}.\overline{2n-1}}{1\times 2\times 3} \end{array}\right.$$

$$\text{Whence } \frac{(n.a^2+n.\overline{n-1}.ad+\dfrac{n.\overline{n-1}.\overline{2n-1}}{1\times2\times3}.d^2=)}{n}$$

$$n.a^2+\overline{n-1}.ad+\frac{\overline{n-1}.\overline{2n-1}}{6}.d^2=s, \text{ the sum required.}$$

PROB. 8. To find the sum of the infinite series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, &c.

First, let $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, &c. *ad infinitum*$=s$,

or, which is the same,

$\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}+$, &c. *ad infinitum* $=s$,

which, divided by 2, becomes

$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$, &c. *ad infinitum* $=\frac{s}{2}$,

or, which is the same,

$(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+$, &c.$=\frac{s}{2}$,

that is,

$$\left.\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{\&c.}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-, \text{\&c.}\end{array}\right\}=\frac{s}{2}.$$

Whence $1=\frac{s}{2}$, *and therefore* $s=2$, *the sum required.*

PROB. 9. To find the sum of n terms of the above series.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$, &c. *to* $\frac{1}{n}$.

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$, &c. *to* $\frac{1}{n}$.

And $z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$, &c. *to* $\frac{1}{n+1}$;

Whence, subtracting the third from the first,

$\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+$, &c. *to* $\frac{1}{n.\overline{n+1}}$;

That is, $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+$, &c. *to* $\frac{1}{n.\overline{n+1}}$;

This, multiplied by 2, becomes

$$\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+, \text{\&c. to } \frac{2}{n.\overline{n+1}};$$

That is, the sum of $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, *&c. to n terms* $=\frac{2n}{n+1}$.

PROB. 10. To find the sum s of the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ +, &c.

Let $x=\frac{1}{2}$, *then will* $x+x^2+x^3+x^4+x^5+$, *&c.* $=s$;

Substitute $\frac{z}{1-x}=(s=)\ x+x^2+x^3+x^4+x^5+$, *&c.*

Then will $z=\overline{1-x}.\overline{x+x^2+x^3+x^4+x^5}+$, *&c. which quantity, by actual multiplication, comes out* $=x$, *that is,* $x=z$; *and therefore, substituting* x *for* z *in the second step, it becomes* $x+x^2+x^3+x^4+x^5=\frac{x}{1-x}=s$; *in which, by restoring the value of* x, *we have* $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+$, *&c.* $(=\frac{\frac{1}{2}}{1-\frac{1}{2}})=1=s$, *the sum required.*

PROB. 11. To find the sum of 1000 terms of the series $1+5+9+13+17+$, &c. *Ans.* 1999000.

PROB. 12. To find the sum of 20 terms of the series $1+3+9+27+81+$, &c. *Ans.* 1743392200.

PROB. 13. To find the sum of 12 terms of the series $4+9+16+25+$, &c. *Ans.* 1562.

PROB. 14. To find the sum of n terms of the series $a^3+\overline{a+d}|^3+\overline{a+2d}|^3+\overline{a+3d}|^3+$, &c. *Ans.* $na^3+\frac{n.\overline{n-1}.3a^2d}{2}+\frac{n.\overline{n-1}.\overline{2n-1}.3ad^2}{6}+\frac{\overline{n^4-2n^3+n^2}.d^3}{4}$.

PROB. 15. To find the sum of n terms of the series $1+3+7+15+31+$, &c. *Ans.* $2^{n+1}-\overline{2+n}$.

PROB. 16. Required the sum of the infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+$, &c. *Ans.* $\frac{1}{3}$.

PROB. 17. To find the sum of the infinite series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+$, &c. *Ans.* 2.

PROB. 18. To find the sum of $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+$, &c. ad infinitum. *Ans.* $1\frac{1}{2}$.

PROB. 19. To find the sum of the infinite series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$, &c. *Ans.* $\frac{1}{4}$.

PROB. 20. To find the sum of n terms of the above series.
Ans. $\frac{\overline{n+1}.\overline{n+2}-2}{4.\overline{n+1}.\overline{n+2}}$.

PROB. 21. To find the sum of the infinite series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+$, &c. *Ans.* $\frac{1}{18}$.

PROB. 22. To find the sum of n terms of the above series.
Ans. $\frac{1}{18}-\frac{1}{3.\overline{n+1}.\overline{n+2}.\overline{n+3}}$.

20. THE INVESTIGATION OF LOGARITHMS.

Let there be given $a^x=N$, in which expression x is the logarithm of a^x; it is required to find the value of x, that is, the logarithm of ($a^x=$) the number N.

Let $a=1+b$, *and* $N=1+n$; *then will* $\overline{1+b}|^{x}=1+n$, *from which, extracting the* y^{th} *root, we obtain* $\overline{1+b}|^{\frac{x}{y}}=\overline{1+n}|^{\frac{1}{y}}$, ∵ (*Art.* 11.) $\overline{1+b}|^{\frac{x}{y}}=1+\frac{x}{y}.b+\frac{x}{y}.\overline{\frac{x}{y}-1}.\frac{b^2}{2}+\frac{x}{y}.\overline{\frac{x}{y}-1}.\overline{\frac{x}{y}-2}.\frac{b^3}{2.3}$ $+$, &c.

$$\overline{1+n}|^{\frac{1}{y}}=1+\frac{1}{y}.n+\frac{1}{y}.\overline{\frac{1}{y}-1}.\frac{n^2}{2}+\frac{1}{y}.\overline{\frac{1}{y}-1}.\overline{\frac{1}{y}-2}.\frac{n^3}{2.3}+, \&c.$$

Here, if y *be assumed indefinitely great, the quantities* $\frac{x}{y}$,

$\frac{1}{y}$, *may be considered as* $=o$, *since they will in that case be indefinitely small with respect to the numbers* 1, 2, 3, 4, &c.

Wherefore $\frac{x}{y}-1=-1$, $\frac{1}{y}-1=-1$, $\frac{x}{y}-2=-2$, $\frac{1}{y}-2=-2$, &c.

These values being substituted in the above series, we shall have $(\overline{1+b}|^{\frac{x}{y}}=\overline{1+n}|^{\frac{1}{y}}=)\ 1+\frac{x}{y}.b-\frac{x}{y}.\frac{b^2}{2}+\frac{x}{y}.\frac{b^3}{3}-$, &c. $=1+\frac{1}{y}.n-\frac{1}{y}.\frac{n^2}{2}+\frac{1}{y}.\frac{n^3}{3}-$, &c. $\therefore \frac{x}{y}.\overline{b-\frac{1}{2}b^2+\frac{1}{3}b^3-, \&c.}=\frac{1}{y}.\overline{n-\frac{1}{2}n^2+\frac{1}{3}n^3-, \&c.}$ *or,* $x=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-, \&c.}{b-\frac{1}{2}b^2+\frac{1}{3}b^3-, \&c.}=$ *(by substituting for n and b, their equals* $N-1$ *and* $a-1$*)*

$\frac{\overline{N-1}-\frac{1}{2}\overline{N-1}|^2+\frac{1}{3}\overline{N-1}|^3-, \&c.}{\overline{a-1}-\frac{1}{2}\overline{a-1}|^2+\frac{1}{3}\overline{a-1}|^3-, \&c.}$. *Let* $M=$ *the denominator of either of the two latter fractions, then the last but one will become* x *(or the log. of* $1+n$*)* $=\frac{1}{M}.\overline{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+, \&c.}$ *which series, when n is a whole number, does not converge, and therefore is of no use; but we may obtain by means of it a series which will converge sufficiently fast for our purpose, as follows:*

21. *Since log.* $\overline{1+n}=\frac{1}{M}.\overline{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\frac{1}{5}n^5-, \&c.}$ *for n let* $-n$ *be substituted, and the above expression becomes log.* $\overline{1-n}=\frac{1}{M}.\overline{-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\frac{1}{5}n^5-, \&c.}$

And if the lower equation be subtracted from the upper, the remainder is (log. $\overline{1+n}$ *− log.* $\overline{1-n}=$*) log.* $\frac{1+n}{1-n}=\frac{1}{M}.\overline{2n+\frac{2}{3}n^3+\frac{2}{5}n^5+, \&c.}=\frac{2}{M}.\overline{n+\frac{1}{3}n^3+\frac{1}{5}n^5+, \&c.}$ *Let* $\frac{1}{N-1}$ *be substituted for n in this equation, and it will become* $\frac{N}{N-2}=\frac{2}{M}.\overline{\frac{1}{N-1}+\frac{1}{3.\overline{N-1}|^3}+\frac{1}{5.\overline{N-1}|^5}+, \&c.}$ *that is, log.* $N-$ *log.* $\overline{N-2}=\frac{2}{M}.\overline{\frac{1}{N-1}+\frac{1}{3.\overline{N-1}|^3}+\frac{1}{5.\overline{N-1}|^5}+, \&c.}$

Whence, by transposition,

$$Log.\ N=\frac{2}{M}.\overline{\frac{1}{N-1}+\frac{1}{3.\overline{N-1}|^3}+\frac{1}{5.\overline{N-1}|^5}+,\ \&c.}+log.\ \overline{N-2},$$

which latter is a very convenient series for finding the logarithm of any whole number N, provided N be greater than 2, and the logarithm of $\overline{N-2}$ previously known.

22. *Since $a^x=N$, it follows from the nature of logarithms, (see Vol. I. P. 2. Art. 18, 37.) that $x\times log.\ a=log.\ N$; but (Art. 20.) $x=log.\ N$: wherefore $log.\ a=1$; and $log.\ \frac{a}{a}=log.\ a-log.\ a=o$. Wherefore, (since $\frac{a}{a}=1$,) $log.\ 1=o$.* Having therefore the logarithm of 1 given, we can thence find the logarithm of 3; *for let $N=3$, then $N-2=1$, the logarithm of which is o, as we have shewn; wherefore, by substituting 3 for N in the above expression, we shall have* $log.\ 3=\frac{2}{M}.\overline{\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+,\ \&c.}+(log.\ 1=)\ o.$

23. Having found the logarithm of 3, we may thence find those of all the odd numbers in succession; thus,

Let $N=5$; then, $log.\ 5=\frac{2}{M}.\overline{\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+,\ \&c.}+log.\ 3.$

Let $N=7$; then, $log.\ 7=\frac{2}{M}.\overline{\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+,\ \&c.}+log.\ 5.$

Let $N=9$; then, $log.\ 9=\frac{2}{M}.\overline{\frac{1}{8}+\frac{1}{3.8^3}+\frac{1}{5.8^5}+,\ \&c.}+log.\ 7.$

Let $N=11$; then, $log.\ 11=\frac{2}{M}.\overline{\frac{1}{10}+\frac{1}{3.10^3}+\frac{1}{5.10^5}+,\ \&c.}+ log.\ 9.$

24. The logarithm of the number 2 is thus found.

Log. of 4 (by what has been shewn above)$=\frac{2}{M}.\overline{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+,\ \&c.}+log.\ 2.$

But $log.\ 4=log.\ 2^2=2\times log.\ 2$; therefore $2\times log.\ 2=\frac{2}{M}.\overline{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+,\ \&c.}+log.\ 2$; *whence, by transposition,* $(2\times log.\ 2-log.\ 2=)\ log.\ 2=\frac{2}{M}.\overline{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+,\ \&c.}$

25. Having shewn the method of finding the logarithms of all the prime numbers, those of the composite numbers will be readily obtained by addition only; thus,

$$\textit{Because}\left\{\begin{array}{c} 4=2\times 2\\ 6=3\times 2\\ 8=4\times 2\\ 9=3\times 3\\ 10=5\times 2\\ 12=6\times 2\\ \&c.\end{array}\right\}\ \textit{therefore}\left\{\begin{array}{l} log.\ \ 4=log.\ 2+log.\ 2.\\ log.\ \ 6=log.\ 3+log.\ 2.\\ log.\ \ 8=log.\ 4+log.\ 2.\\ log.\ \ 9=log.\ 3+log.\ 3.\\ log.\ 10=log.\ 5+log.\ 2.\\ log.\ 12=log.\ 6+log.\ 2.\\ \quad\quad \&c.\end{array}\right.$$

26. But before we can apply the above expressions to the actual construction of logarithms, the value of the quantity M must be determined; it is called the *modulus*[k] of the system, and may be assumed equal to any number whatever: whence it is plain that (by varying the value of M) innumerable systems of logarithms may be formed for the same scale of numbers, in each of which the magnitude of the logarithm of any number will depend on the value of M; moreover M depends on the value of a, (since $M=\overline{a-1}-\frac{1}{2}.\overline{a-1}|^2+\frac{1}{3}.\overline{a-1}|^3-$, &c.) which therefore is called the *base* of the system, and may be varied at pleasure.

If $M=1$, then will log. $N=\overline{N-1}-\frac{1}{2}.\overline{N-1}|^2+\frac{1}{3}.\overline{N-1}|^3-$, &c. the logarithms of this system are denominated *Napier's* or *hyperbolic* logarithms.

Let $\overline{N-1}-\frac{1}{2}.\overline{N-1}|^2+\frac{1}{3}.\overline{N-1}|^3-$, *&c.*$=p$; *then if* M *be the modulus, we shall have log.* $N=\frac{p}{M}$, *if* $M=1$, *then will hyp. log.* $N=p$; *and if this value of* p *be substituted in the preceding equation, it becomes log.* $N=\frac{hyp.\ log.\ N}{M}$, *whence also hyp. log.* $N=M\times log.\ N$.

27. Hence hyperbolic logarithms are changed into others,

[k] The name *modulus* was first given to this factor by Mr. Cotes, in a learned paper on the nature and construction of logarithms, printed in the *Philosophical Transactions*, No. 338, and afterwards in a tract entitled Logometria. The modulus is a fourth proportional to the fluxion of the number, the fluxion of the logarithm, and the number itself; or it is the number which expresses the subtangent of the *logarithmic* or *logistic* curve.

whose modulus is M, by dividing the former by M: and logarithms whose modulus is M, are changed into hyperbolic logarithms, by multiplying the former of these by M.

Let $N=a$, then since log. $N=\frac{\text{hyp. log. } N}{M}$, we shall have by substitution, log. $a=\frac{\text{hyp. log. } N}{M}$; but it has been shewn that log. $a=1$, wherefore by multiplication $(aM=1\times M=)M=$ hyp. log. a.

But since the value of a may be assumed at pleasure, let $a=10$; substitute this value for a in the above equation, and $M=$ hyp. log. 10.

Logarithms derived from this assumption are usually called Briggs's, or the Common Logarithms; and to construct a table of them, it is plain we must first find the hyperbolic logarithm of 10, which has been shewn to be the modulus of that system.

Now log. $10=\text{log. }\overline{2\times5}=\text{log. }2+\text{log. }5$, and the modulus of the system of hyperbolic logarithms is unity, or $M=1$.

Therefore, (Art. 24.) hyp. log. $2=2\times\overline{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+}$, &c.$=$ .69314718.

Hyp. log. $3=2\times\overline{\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+}$, &c. $=1.09861228$.

Hyp. log. $5=2\times\overline{\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+}$, &c.$+$ log. $3=1.60943791$.

28. Having found the hyperbolic logarithms of 2 and 5, we have from the nature of logarithms, *hyp. log.* $10=$ *hyp. log.* 2 $+$ *hyp. log.* $5=(.69314718+1.60943791=)2.30258509=M$, *the modulus of the system of common logarithms; and since $\frac{2}{M}=\frac{2}{2.30258509}=.868588964$, this quotient being substituted for its equal $\frac{2}{M}$, will become a constant multiplier of the general series, that is, com. log. $N=.868588964\times\overline{\frac{1}{N-1}+\frac{1}{3.\overline{N-1}|^3}+\frac{1}{5.\overline{N-1}|^5}+}$, &c.$+$ log. $\overline{N-2}$;* which is a general theorem for finding the common logarithms of all the prime numbers above 2; the theorem for finding the logarithm of the number 2 being

$.868588964\times\overline{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+}$, &c. (*Art.* 24.) and since the logarithms of the composite numbers are derived from those of the prime numbers *by addition only*, we are now in possession of the means of constructing a complete table of these useful numbers.

29. To construct a table of common logarithms.

Let $A=.868588964$, *then the above theorem for finding the logarithm of* 2 *will become* $\frac{A}{3}+\frac{A}{3.3^3}+\frac{A}{5.3^5}+$, &c. *whence is derived the following practical rule for finding the logarithm of the number* 2.

RULE I. Divide the factor .868588964 by 3, and reserve the quotient.

II. Divide the reserved quotient by 9, and in like manner reserve the quotient; divide this last quotient by 9, and reserve the quotient; and so on, continually dividing by 9, as long as division can be made.

III. Set the reserved quotients in order, under one another, and divide them respectively by the odd numbers 1, 3, 5, 7, 9, &c. placing the quotients one under another as before.

IV. Add the last mentioned quotients together, and the sum will be the logarithm of 2, as was required.

EXAMPLES.—1. To find the logarithm of the number 2.

OPERATION.

3)	.868588964	1)	.289529654(	.289529654
9)	.289529654	3)	32169962(	10723321
9)	32169962	5)	3574440(	714888
9)	3574440	7)	397160(	56737
9)	397160	9)	44129(	4903
9)	44129	11)	4903(	446
9)	4903	13)	545(	42
9)	545	15)	60(	4
9)	60		*Ans. log. of* 2=	.301029995
	6			

Explanation.

The first (or left hand) column contains the divisors 3, 9, 9, &c. the second contains the dividend, and successive quotients, which arise by dividing each number in it by the opposite divisor; the third contains the divisors, 1, 3, 5, 7, &c. In the fourth column the reserved quotients above mentioned are arranged under one another in order, each opposite its respective divisor. The fifth consists of the quotients arising from the division of each of the reserved quotients by its proper divisor; the sum of these latter, subjoined at the bottom, is the logarithm required.

Note. In some of the above *divisions*, where the remainder is very large, the

last quotient figure is assumed greater by one than it ought strictly to be; this, as it serves only to make up for other small remainders lost, will be productive of no error of consequence in the result.

2. To find the common logarithm of the number 3.

Here, by assuming A as before, the general theorem for finding the common logarithms of all numbers greater than 2, will become $\frac{A}{N-1}+\frac{A}{3.\overline{N-1}|^3}+\frac{A}{5.\overline{N-1}|^5}+$, &c. $+\log.\ \overline{N-2}$. *In this case* $N=3$, $\because N-1=2$, $\overline{N-1}|^3=2\times4$, $\overline{N-1}|^5=2\times4\times4$, $\overline{N-1}|^7=2\times4\times4\times4$, $\overline{N-1}|^9=2\times4\times4\times4\times4$, &c. &c. whence it is plain, that the first column of divisors must be 2, 4, 4, 4, 4, &c. and the other column of divisors, in this and every other case, will be the odd numbers, 1, 3, 5, 7, &c. and proceeding as before, the work will stand thus:

2).868588964	1).434294482(	.434294482
4).434294482	3).108573620(	36191207
4).108573620	5) 27143405(	5428681
4) 27143405	7) 6785851(	969407
4) 6785851	9) 1696463(	188496
4) 1696463	11) 424116(	38556
4) 424116	13) 106029(	8156
4) 106029	15) 26507(	1767
4) 26507	17) 6627(	389
4) 6627	19) 1657(	87
4) 1657	21) 414(	19
4) 414	23) 103(	4
4) 103	25) 25(	1
25	*Sum*	.477121252

To which add (*log.* $\overline{N-2}$=)*log.* 1=.000000000

The sum is the log. of 3=.477121252

In a similar manner the logarithms of the other prime numbers are obtained, and by means of them those of the composite numbers, as has been already shewn.

3. To find the logarithm of 5. *Ans.* .698970004.

4. To find the logarithm of 7. *Ans.* .845098040.

5. To find the logarithm of 4. *Ans.* .602059991.

6. To find the logarithms of 8, 9, 10, 11, 12.

PART VIII.

GEOMETRY.

HISTORICAL INTRODUCTION.

GEOMETRY[a] is the science of magnitude, or local extension; it teaches and demonstrates the properties of lines, surfaces, solids, ratios, and proportions, in a general manner, and with the most unexceptionable strictness and precision.

Geometry, or measuring, must have been practised as an art at the commencement of society, or shortly after, when men began to build, and to mark out the limits of their respective territories. That this art had reached a considerable degree of perfection at the time of the general deluge, can hardly be doubted from that stupendous monument of human folly, the Tower of Babel, which was begun about 115 years after that period: Herodotus informs us, that this vast building had a square base, each side of which was a furlong in length; Strabo affirms that its height was likewise a furlong; and Glycas says, that the constant labour of forty years was consumed in erecting this unfinished and useless fabric. The Pyramids, Obelisks, Temples, and other public edifices with which Egypt abounded, existed prior to any authentic date of profane history: many of these had been in ruins probably

[a] The name *Geometry* is derived from γη the earth, and μετρειν to measure. The invention of measuring is ascribed to the Egyptians by Herodotus, Diodorus, Strabo, and Proclus; to Mercury by others among the ancients; and to the Hebrews by Josephus.

for ages before the earliest historians lived, who speak of their magnificence as surpassing that of the most splendid structures in Greece [b]. Can it be supposed possible, that buildings, whose magnificent remains alone were sufficient to excite the wonder and admiration of a learned and polished nation like the Greeks, could have been raised without the assistance of Geometry?

The priests of Memphis informed Herodotus, that their king Sesostris divided the lands bordering on the Nile among his subjects, requiring that the possessor should pay an annual tribute proportionate to the dimensions of the land he occupied; and if the overflowing of that river occasioned any diminution, the king, on being applied to, caused the land to be measured, and claimed tribute in proportion only to what remained. "I believe," adds Herodotus, "that here Geometry took its birth, and hence it was transmitted to the Greeks." On the strength of this conjecture we frequently hear it affirmed, that "Geometry derived its origin from the annual inundation of the Nile;" but it is plain that this as-

[b] Several instances of this may be given. The tomb of Osymandyas, one of their kings, is said to have been uncommonly magnificent; it was surrounded by a circle of gold, 365 cubits in circumference, divided into as many equal parts, which shewed the rising and setting of the sun for every day in the year: this circle was carried away by Cambyses, king of Persia, when he conquered Egypt, A. C. 525. *Goguet Orig. des Lois, &c. T. 2. liv. 3. Rollin's Anc. Hist. vol. I. p.* 3. The famous Labyrinth contained 12 palaces surrounded by 1500 rooms, adorned with innumerable ornaments and statues of the finest marble, and most exquisite workmanship; there were besides, 1500 subterraneous apartments, which Herodotus (who surveyed this noble and beautiful structure) was not permitted to see, because the sepulchres of their kings were there, and likewise the sacred crocodiles and other animals, which a nation so wise in other respects worshipped as gods: "Who" (says the learned and pious Rollin) "can speak this without confusion, and without deploring the blindness of man!" The magnificent city of Thebes, with its numerous and splendid palaces and other public edifices, which was ruined by Cambyses, is the last instance to be mentioned, although many more might be added. It extended above 23 miles, had an hundred gates, and could send out at every gate 20,000 fighting men, and 200 chariots.

sertion deserves little credit; for as a science, Geometry never existed in Egypt before the time of Alexander, and as an art it must have been known there (as we have shewn above) long before the age of Sesostris; for according to the very probable conclusions of our most accurate and best informed chronologers, Sesostris was the Egyptian king, who invaded Jerusalem, A. C. 971; on which occasion he is mentioned in 1 Kings, ch. xiv. v. 25, under the name of Shishak: now we have direct proofs, on the most unquestionable authority, that measuring was understood by the Jews who came from Egypt, many centuries earlier than that date; see Genesis, ch. vi. v. 15, 16. Exodus, ch. xxv. xxvi. xxvii. and various other parts of the Mosaic history.

Not to take up the reader's time with conjectures about the origin of Geometry, which at best must be vague and uncertain, we hasten to inform him, that the Greeks, to whose taste and industry almost every science stands indebted, were the first people who collected the scattered principles and practices of Geometry, which they found in Egypt and other eastern countries, and moulded them into a form and consistence. Until it passed through their masterly hands, Geometry could not by any accommodation of language be properly termed a science; but by their consummate skill and indefatigable labours, a few scanty and detached principles and rules, heretofore chiefly applied to the measuring of land, (as the name Geometry imports,) at length grew into and became the most complete and elegant science in the world. We adore that benign Providence, who has repeatedly condescended to make even wicked and idolatrous nations useful instruments for promoting the execution of his merciful designs to man.

Thales [c] ranks among the earliest of the Grecian philoso-

[c] Thales, the father of the Greek philosophy, and the first of the seven wise men of Greece, was born at Miletum, A. C. 640; after acquiring the best learn-

phers, who travelled into foreign countries in quest of that knowledge which their own could not supply, A. C. 640. He became not only an able geometer, but was likewise very skilful in every branch of Mathematics and Physics, as these sciences then stood. We are unacquainted with the particulars of his acquirements and discoveries in Geometry, but he is mentioned as being the first who measured the height of the pyramids at Memphis, by means of their shadows, and who applied the circumference of a circle to the measuring of angles.

Pythagoras[d] was another eminent Grecian philosopher, who

ing his own country afforded, he travelled in the East, and returned with a mind enriched with the knowledge of Geometry, Astronomy, Natural Philosophy, &c. which he improved by his own skill and application. He divided the celestial sphere into five zones; he observed the apparent diameter of the sun, making it half a degree; he understood the cause and course of eclipses, calculated them with accuracy, and divided the year into 365 days. He disliked monarchy, because he considered it as little better than tyranny, to every species of which he was an avowed enemy. One evening as he walked out to contemplate the stars, he had the misfortune to fall into a ditch, on which an old woman, who saw him, exclaimed, "How can you possibly know what is doing in the heavens, when you cannot see what is even at your feet!" He died at the Olympic Games, at the age of upwards of 90 years. Thales was the founder of the *Ionian sect*, and had for his scholars some of the most eminent philosophers of antiquity, among whom are mentioned Anaximander, Anaximenes, and Pythagoras. It is uncertain whether he left any writings; Augustine mentions some books on Natural Philosophy ascribed to him; Simplicius, some on Nautic Astrology; Laërtius, two treatises on the Tropics and Equinoxes; and Suidas, a work on Meteors, written in verse.

[d] Pythagoras, a celebrated philosopher of Samos. He was early instructed in music, poetry, astronomy, and gymnastic exercise, with whatever else might tend to enlighten his mind, and invigorate his body. At the age of eighteen he resolved to travel for that instruction, which the ablest philosophers of Samos were incompetent to supply: he spent 25 years in Egypt, where having ingratiated himself with the priests, he became acquainted with all the learning of that country; having travelled through Chaldea, and visited Babylon, he returned, passing through Crete, Sparta, and Peloponnesus, from whence he crossed over into Italy, and finally fixed his residence at Crotona. Here he opened a school, which, by the fame of his mental and personal accomplishments, was soon crowded with pupils, many of whom came from distant parts of Greece and Italy. His scholars, who were called the *Italian sect*, were formed by

was endowed with an equal thirst for useful knowledge, and employed the same means to gratify it, A. C. 590. The 32nd and 47th propositions of the first book of Euclid's Elements are ascribed to him; from the latter of which he was led to determine, that the diagonal of a square is incommensurable to its side: every person moderately acquainted with Geometry will acknowledge, that the useful purposes to which these important propositions may be applied are innumerable.

About this time, or shortly after, the following celebrated

the rules obtained from the Egyptian priests; among other austerities, he enjoined them a five years silence, during which they were only to hear; after this they were allowed to propose doubts, ask questions, &c. in which they were permitted to say, not *a little* in *many words*, but *much* in *as few words as possible*. Query. Might not the prattling, self-sufficient young gentlemen in some of our academies, be admirably benefited by an institution of this kind?

Besides the propositions mentioned above, Pythagoras was the author of the following, viz. only three rectilineal figures can fill up the space about a point; namely, the equilateral triangle, the square, and the hexagon. He invented the multiplication table; the obliquity of the ecliptic was first discovered by him; he called the world κοσμος, and asserted that it was made in musical proportion; the sun he called *the fiery globe of unity*, and maintained that the seven planets move round him in an harmonious motion at distances corresponding to the musical divisions or intervals of the monochord: he taught the true solar system, which had been asserted by Philolaus of Crotona, but being forgotten and lost during many ages after, was at length revived by Copernicus, and demonstrated by the illustrious Newton.

The modesty of Pythagoras was not less conspicuous than his attainments; on being addressed at a public assembly with the splendid appellation of σοφος, *wise man*, he disclaimed the title, and requested that they would rather call him φιλοσοφος, *a lover of wisdom;* a circumstance which first gave rise to the terms *philosophy* and *philosopher*.

Some authors affirm, that Pythagoras offered 100 oxen as a sacrifice to Apollo, in gratitude for the discovery which that god enabled him to make of the 47th proposition of the first book of Euclid; this is extremely improbable, as he was a firm believer in the doctrine of the transmigration of souls, which forbade taking away the life of any animal: nor is it much more credible that he substituted little oxen made of flour, clay, or wax; no, this would doubtless have been considered as an intolerable affront, which the meanest heathen god in the catalogue would disdain to put up with. The whole story is perhaps nothing better than a fiction, an ingenious sample of ancient priest-craft.

problems began to be agitated among the learned; namely, the rectification and quadrature of the circle, the trisection of an angle, the finding two mean proportionals, and the duplication of the cube *. Some of the ancients solved these problems, but their solutions were either *mechanical*, by *approximation*, or depended on the properties of certain curves not considered as geometrical; consequently their methods did not fulfil the necessary condition, requiring that these problems, which without dispute are elementary, should be solved by pure elementary Geometry. Some of the most eminent geometers of both ancient and modern times have engaged in this arduous undertaking, and not one among them all has succeeded: no solution of either of these famous problems, strictly and purely geometrical, has ever yet appeared. What a useful lesson does this address to the noisy advocates for the omnipotency of reason! they may hence learn, that the reasoning powers of the human mind, although unquestionably great and excellent, have their limits, narrower perhaps than these philosophers have been accustomed or are willing to allow; and consequently that reason, although the most noble and distinguishing boon that Heaven has ever conferred on man, was not given him to be deified. Let them contemplate with becoming attention the

* The *rectification* of a circle is the finding a right line equal to its circumference, and its *quadrature* is the finding a square equal to its area. The finding two mean proportionals consists in this; having two right lines given, thence to find two others, such, that the four lines will be *continued* proportionals. The duplication of the cube consists in finding the side of another cube, which cube shall be in magnitude just double the former: the two latter problems depend on each other, and form but one, known by the name of the *Delian problem*, which it obtained from the following circumstance: a plague threatening to depopulate Athens, the oracle of Apollo at Delphos was consulted, and returned for answer, "Double the altar and the plague shall cease." The geometers immediately set to work to find the side of a cube double of this altar, which was likewise cubical; but after much labour they found to their great mortification, that the solution could not be effected by any of the methods then in use.

numerous insurmountable obstacles which oppose themselves at the very threshold of almost every department of knowledge, and candour will oblige them to confess that the mental powers are still very imperfect, and consequently that superior attainments in any science ought always to be accompanied with modesty, diffidence, and humility.

Of those who engaged with ardour in the above-mentioned difficult researches, Anaxagoras of Clazomene was one of the earliest, A. C. 500; he was an excellent geometer, and composed a treatise expressly on the quadrature of the circle, which, according to Plutarch, was written during his imprisonment at Athens. Œnopidus of Chios and Zenodorus flourished about A. C. 480; to the former are ascribed the 9th, 11th, 12th, and 23d propositions of Euclid's first book of Elements. Zenodorus proved, that figures of equal areas are not necessarily contained by equal boundaries, as some had asserted: one only of his treatises has escaped the ravages of time; it has been preserved by Theon in his Commentaries, and is the earliest piece on Geometry at present extant.

The school of Pythagoras produced a great number of learned geometricians: with the names of some of them we are acquainted, but scarcely any thing is known of their discoveries and improvements; as most of their writings, through the constant mutability of human affairs, during a long lapse of ages, have been destroyed or lost. One famous discovery in Geometry, however, remains to be noticed as originating among the disciples of Pythagoras, namely, the ingenious theory of the five regular bodies [f].

[f] They are likewise denominated the *Platonic bodies*, and are as follow. 1. The *Tetraëdron*, or regular triangular pyramid, contained by four equilateral and equal triangular faces. 2. The *Hexaëdron*, or cube, contained by six equal square faces. 3. The *Octaëdron*, contained by eight equal equilateral triangular faces. 4. The *Dodecaëdron*, contained by twelve equal and regular

Hippocrates [g] of Chios, A. C. 450. distinguished himself as the first who squared a curvilineal space [h]; in his attempts to solve the celebrated problem of doubling the cube, he discovered, that if two mean proportionals between the side of a given cube and double that side be found, the least of these means will be the side of the required cube: the same is demonstrated in Euclid 33. 11. but it was soon discovered that the difficulty, instead of being removed, was only a little disguised; for the two mean proportionals themselves could not be found by any pure geometrical process, and the problem continues, to the present hour, to bid defiance to the united skill and labours of the ablest geometricians.

Geometry was cultivated with the greatest attention by Plato [i]; his school was a school of geometers, as appears from

pentagonal faces; and 5. The *Icosaëdron*, contained by twenty equal and equilateral triangular faces. These five, together with the *Sphere*, which may be considered as a sixth, are all the regular solids that can possibly be made. The following are called *mixed* solids, each being compounded of two of the former: viz. 1. The *Exoctoëdron*, contained by fourteen planes, viz. six equal squares, and eight equal and equilateral triangles. 2. The *Icosidodecaëdron*, contained by thirty-two planes, viz. twelve equal and regular pentagons, and twenty equal and equilateral triangles. See a treatise on the Regular and Mixed Solids, by Flussas, subjoined to *Barrow's Euclid. London*, 1751. The five regular solids may be constructed with pasteboard, the method of doing which was first shewn by Albert Durer, an ingenious magistrate of Nuremberg, in his *Institutiones Geometricæ. Paris*, 1532. See also Hawney's *Complete Measurer*, 9th Ed. p. 268. Bonnycastle's *Introduction to Mensuration, &c.* 4th Ed. p. 181. &c. Hutton's *Math. Dictionary*, vol. I. p. 215, and vol. II. p. 355. &c.

[g] I am equally uncertain whether there be any further particulars of this geometrician in existence, and whether the above date be correct: he must not be confounded with a learned physician of the same name, in the Island of Cos, who was much esteemed for skill and fidelity in his profession.

[h] This curve is the *lunula*: if three semicircles be described on the three sides of a right angled triangle, their intersections will form two lunar spaces, the sum of which is equal to the area of the triangle; the proof of which depends on Euclid 47. 1, 31. 6, and 2. 12. Proclus ascribes the lunula to Œnopidus.

[i] The original name of this eminent philosopher was Aristocles, and he received that of Plato from the broadness of his shoulders; he was born at

the following inscription which he caused to be placed over the door; LET NO ONE PRESUME TO ENTER HERE, WHO IS UNSKILLED IN GEOMETRY. Like his predecessors, Plato attempted the duplication of the cube; for this purpose he contrived an instrument, consisting of straight rules, moving in grooves perpendicularly to each other, by means of which he was enabled to find two mean proportionals: but the pro-

Athens about 430 years before Christ, and educated with the greatest attention both to his mental and corporeal improvements; having in his early years acquired considerable skill in music, painting, poetry, philosophy, gymnastic exercises, &c. he at 20 years old became a disciple of Socrates, who stiled him *the Swan of the Academy*. Plato, on the death of his beloved master, retired to Megara, where he was kindly entertained by Euclid the philosopher: from thence he passed over into Italy, where he perfected himself in natural philosophy under Archytas and Philolaus; from Italy he went to Cyrene, where he received instructions in geometry from Theodorus: he afterwards travelled into Egypt, where he acquired arithmetic, astronomy, and, as it is supposed, an acquaintance with the writings of Moses; after visiting Persia, he returned to Athens, where he opened a school, and taught philosophy in the Academia, whence his disciples were called *Academics*. Plato afterwards made several excursions abroad, in one of which being at Syracuse, he had the misfortune to displease Dionysius, and narrowly escaped with his life. The tyrant, however, delivered him into the hands of an envoy from Lacedemon, which then was at war with Athens, and he was sold for a slave to a Cyrenian merchant, who immediately liberated and sent him to Athens. The ancients thought more highly of Plato than of all their philosophers, calling him *the divine Plato; the most wise; the most sacred; the Homer of philosophers, &c.* The orator Cicero was so enthusiastic in his praise, that he one day exclaimed, "*errare mehercule malo cum Platone, quam cum istis vera sentire.*" The Platonic philosophy appears to be founded chiefly on the Mosaic account of the creation, &c. hence, in the early ages of the church, Platonism and Christianity were incorporated and blended together by some of the fathers of the Eastern church; but this union is severely and justly censured by Gisborne, Milner, and others, as extremely detrimental to the genuine spirit of Christianity. After the death of Plato, which happened A. C. 348, two of his disciples, Xenocrates and Aristotle, succeeded him: the former taught in the Academy, and his disciples were called *Academics;* the latter taught in the Lycæum, and his scholars obtained the name of *Peripatetics*, from the circumstance of their receiving their instructions, not sitting, as is usual, but walking. The works of Plato are numerous: they are all, except twelve letters, written in the form of dialogue; the best editions are those of Lyons, 1588. Frankfort, *fol.* 1602. and Deuxponts, 12 vol. 8vo. 1718.

cess was *mechanical*, and consequently could not be admitted as a *geometrical* solution of the problem.

The circle was the only curve hitherto admitted into Geometry, but Plato introduced into that science the theory of the conic sections, or those curves which are formed by a plane cutting a cone in various directions. The numerous properties of these celebrated curves, and their usefulness in Geometry, soon became apparent, and excited the attention of mathematicians, who considered this branch of Geometry of a distinct and more exalted nature than that which treated of the circle and rectilineal figures only; and hence it obtained the name of the *higher* or *sublime* Geometry. By means of the properties of these curves, Archytas of Tarentum [k], the master of Plato, taught the method of finding two mean proportionals, and thence the duplication of the cube, A. C. 400. Menechmus accomplished the same thing about that period, or shortly after: they both effected the solution by means of the intersection of two conic sections; a circumstance which merits particular notice, as being the origin of the celebrated theory of *geometrical loci*, of which so many important applications have been made by both ancient and modern geometricians. Were it possible to describe the conic sections by one simple continued motion, like the circle, the above solutions would possess all the advantages of geometrical construction, according to the sense applied to the term by the ancients; but failing in that particular, they do not fulfil the necessary condition.

The great problems we have so frequently mentioned,

[k] Archytas is said to be the inventor of the crane and screw; he contrived also a wooden pigeon, which could fly: the ten categories of Aristotle are ascribed to him; as are also several works, but none of them have descended to us. He was a wise legislator, and a skilful and valiant general, having commanded the army seven times without having been once defeated. He was at last shipwrecked and drowned in the Adriatic Sea.

although now given up as impossible to be solved by the proposed method, were studied by the ancients with incessant ardour; and the researches to which speculations of this kind gave birth, proved a fruitful source of discoveries in Geometry.

The numerous and extensive applications of Geometry to other branches of knowledge, especially to Astronomy, made a systematic arrangement of its principles and conclusions, according to their logical connexion and dependance, indispensable. Of those who undertook to compose Elements of Geometry, Hippocrates, Eudoxus, Leon, Thaetetus, Theudius, and Hermotimus, were the chief, and the usefulness of their labours in this respect was apparent; but their treatises, of which scarcely any thing is known, were all superseded by the Elements of Euclid [1], which have maintained their superiority over other systems of the kind through every succeeding age to the present, and still hold their rank as the only classical standard of elementary Geometry. Euclid's Elements, as we now have them, are comprised in fifteen books, and the subjects they treat of may be arranged in three divisions; of which the first includes the theory of superficies, the second that of numbers, and the third that of solids: the first four books explain and demonstrate the properties of lines, angles, and planes; the fifth treats in a general manner of the ratios and proportions of magnitudes;

[1] Euclid was one of the most celebrated mathematicians of the Alexandrian school; he was born at Alexandria, and taught with great applause, A. C. 280. He wrote several works, as mentioned in the text, of which the Elements is the chief. In this work he availed himself of the labours of those who had gone before him, collecting and properly arranging the principles and propositions which had already been given by others, supplying the deficiencies, and strengthening and confirming the demonstrations. The particulars of his life, and time of his death, are unknown: it is said that King Ptolemy Lagus, on examining the Elements, asked him if it was not possible to arrive at the same conclusions by a shorter method; to which Euclid replied, "There is no *royal* road to Geometry."

the sixth of the proportions, &c. of plane figures; the seventh, eighth, and ninth, explain and prove the fundamental properties of numbers; the tenth contains the theory of commensurable and incommensurable lines and spaces; and the remaining five books unfold the doctrine of solids.

The first six books, with the eleventh and twelfth, are all that are now usually studied; the modern improvements in analysis having furnished much shorter and more convenient methods of attaining to an adequate knowledge of the subjects contained in the remaining books, than those given in the Elements.

The Elements of Euclid furnish all that is necessary for determining the perimeters and areas of rectilineal figures, the superficies and solid contents of bodies contained by rectilineal planes, and for describing them on paper: in them it is proved, that a cone is equal to one-third of its circumscribing cylinder; that the solid content of a cylinder is found by multiplying the area of its base into its altitude: we are likewise taught, what ratio similar plane figures, and also similar solids, have to one another; that the peripheries of circles are as their diameters, and the areas as the squares of their diameters; that angles are measured and compared by means of the intercepted circumferences, &c. These and several other properties of the circle are given in the Elements, but it is no where directly shewn how the circumference (that is, its ratio to the *given* diameter) or how the area of a circle may be found: it is true, that a method of approximation both to the circumference and area seems to be implied in the second proposition of the twelfth book, but no further notice is taken of it in any of the subsequent propositions.

In his demonstrations, Euclid has observed for the most part all that strictness, for which the ancients were so distinguished: from a small number of definitions and self-evident

principles, he has deduced with incontestable evidence the truth of all the propositions which he proposed for proof. This rigorous strictness has, however, sometimes led him necessarily into an indirect and complicated chain of reasoning, which makes his demonstrations in a few instances tedious and difficult. To remedy this defect, several of the moderns have undertaken with success to simplify and render more direct and appropriate, such of the demonstrations as seemed to require improvement; but others, who have lessened the number of propositions by retrenching those which they deemed superfluous, have in general been less happy: by removing those links, which appeared to them unnecessary, the chain of demonstration has in many cases been broken and spoiled.

The Elements have been translated into the language of every country where learning has been encouraged, and enriched with numerous and valuable commentaries. The Arabs were the first people who engaged in this way: on the revival of learning among them, their grand care was to obtain the mathematical works of the best Greek authors, and translate them into the Arabic language. There were probably several translations of Euclid; one in particular is mentioned as made by Honain Ebn Ishak al Ebadi, a learned physician, who flourished in the reign of the Khalif Al Motawakkel, A. D. 847. Adelard, a monk of Bath, in the twelfth century, appears to have been the first who made a Latin translation of the Elements, which he did from the Arabic, as no Greek copy of Euclid had then been discovered. Campanus of Novara translated and commented on the Elements in 1250, which work was revised and further commented on by Lucas De Burgo, about 1470. Orontius Finæus published the first six books with notes in 1530, which is said to have been the first edition that appeared in print. Peletarius published the first six books in 1557, and about the same time Tartalea gave a commentary on the whole of the fifteen books.

In 1570 Billingsley's Euclid appeared, with a very plain and useful preface and notes by the learned and eccentric Dr. John Dee. Candalla published the Elements, with additions and improvements, in 1578, which work was afterwards reprinted with a prolix commentary by Clavius the Jesuit. Many editions of the Elements have since appeared, the chief of which are those of De Chales, Tacquet, Herigon, Barrow, Ozanam, Keill, Whiston, and Stone; but Dr. Robert Simson's translation of the first six and the eleventh and twelfth books, with the Data, first published in the year 1756, is that now most generally used in the British Empire. Playfair's Euclid is an improvement on Simson's; and Ingram's edition contains some particulars chiefly relating to practical Geometry, which are not to be found in either. Before we conclude this enumeration, it will be necessary to observe, that Dr. David Gregory [m], the Savilian Professor of Astronomy, published at Oxford, in 1703, the whole of the works of Euclid in Greek and Latin; this he is said to have done in prosecution of a design of Dr. Bernard [n], his prede-

[m] David Gregory was born at Aberdeen in 1661; here and at Edinburgh he received his mathematical and classical education: in 1684 he was elected Professor of Mathematics in the University of Edinburgh; and it deserves to be noticed, that he, in conjunction with his brother James, first introduced the Newtonian philosophy into Scotland. Through the friendly interference of Newton and Flamstead, our author obtained the Savilian Professorship of Astronomy at Oxford, where he was honoured with the degree of M.D. His works are *Exercitatio Geometrica*, &c. 4to. Edinb. 1684. *Catoptricæ et Dioptricæ Sphæricæ Elementa*, Oxon. 1695. *Astronomiæ, Physicæ, et Geometricæ Elementa*, and some others: he died in 1710, at Maidenhead in Berkshire.

[n] Dr. Edward Bernard rendered himself famous by being the first who undertook to collect the works of the ancient mathematicians for publication; he likewise brought to England the 5th, 6th, and 7th books of the Conics of Apollonius, being a copy of the Arabic Version which the celebrated Golius had obtained in the East. He succeeded Dr. Wren in the Professorship in 1673, and resigned it in 1691, on being presented to the Rectory of Brightwell in Berkshire. He died in 1696, in the 58th year of his age. His works on mathematical subjects are mostly inserted in the Philosophical Transactions: they consist of *Observations on the Obliquity of the Ecliptic*, various *Astronomical* and *Chronological Tables*, &c.

cessor, and in obedience to a precept of Sir Henry Saville *, the founder of the Professorship, requiring that those who fill the chairs of Geometry and Astronomy should publish the mathematical works of the ancients. Dr. Gregory's is the completest edition of Euclid extant.

According to Pappus and Proclus, several mathematical treatises, besides the Elements, were written by Euclid: his Data, a work still extant, is calculated to facilitate the method of resolution, or analysis, shewing from certain things given by hypothesis, what other things may thence be found. His three books of Porisms are said to have been a curious collection of important particulars relating to the analysis of the more difficult and general problems; but no part of this work, or of any other on the same subject written by the ancients, has been preserved, except a small specimen by Pappus: from whence several modern geometricians, particularly Fermat, Bulliald, Albert Girard, Halley, Simson, and Playfair, have attempted to restore either completely, or in part, what the ancients are supposed to have delivered on the subject. Euclid wrote, besides these, a work on the Division of

* Henry Saville was born at Bradley in Yorkshire, A. D. 1549, and entered at Merton College, Oxford, in 1561, of which college he was chosen a fellow, and took his degree of M. A. in 1570. In 1578 he travelled through different parts of Europe for improvement, and on his return was appointed Greek Tutor to Queen Elizabeth. In 1585 he was made Warden of Merton College, over which he presided 36 years, with equal credit to himself and advantage to that learned body. He was chosen Provost of Eton College in 1596, and received the honour of knighthood from King James I. in 1604, after declining the most flattering offers of preferment in either church or state. Sir Henry Saville was an accomplished gentleman, a profound scholar, and a munificent patron of learning, to which (on the death of his only son) he devoted his whole fortune. In 1619 he founded two professorships at Oxford, one for Geometry, and one for Astronomy, each of which he endowed with estates. In addition to the several legacies he left to the University, he bestowed on it a great quantity of mathematical books, rare and curious manuscripts, Greek types, &c. &c. He died at Eton College in 1722, leaving behind him several works, of which the only one pertaining to our present subject is his *Collection of Mathematical Lectures on Euclid's Elements*, 4to. 1621.

Superficies; Loci ad Superficiem; four books on Conic Sections; and treatises on other branches of the Mathematics.

Archimedes [p], one of the greatest geometricians of antiquity, was the first who approximated to the ratio of the cir-

[p] Archimedes was born at Syracuse, and related to Hiero, King of Sicily: he was remarkable for his extraordinary application to mathematical studies, but more so for his skill and surprising inventions in Mechanics. He excelled likewise in Hydrostatics, Astronomy, Optics, and almost every other science; he exhibited the motions of the heavenly bodies in a pleasing and instructive manner, within a sphere of glass of his own contrivance and workmanship; he likewise contrived curious and powerful machines and engines for raising weights, hurling stones, darts, &c. launching ships, and for exhausting the water out of them, draining marshes, &c. When the Roman Consul, Marcellus, besieged Syracuse, the machines of Archimedes were employed: these showered upon the enemy a cloud of destructive darts, and stones of vast weight and in great quantities; their ships were lifted into the air by his cranes, levers, hooks, &c. and dashed against the rocks, or precipitated to the bottom of the sea; nor could they find safety in retreat: his powerful burning glasses reflected the condensed rays of the sun upon them with such effect, that many of them were burned. Syracuse was however at last taken by storm, and Archimedes, too deeply engaged in some geometrical speculations to be conscious of what had happened, was slain by a Roman soldier. Marcellus was grieved at his death, which happened A. C. 210, and took care of his funeral. Cicero, when he was Questor of Sicily, discovered the tomb of Archimedes overgrown with bushes and weeds, having the sphere and cylinder engraved on it, with an inscription which time had rendered illegible.

His reply to Hiero, who was one day admiring and praising his machines, can be regarded only as an empty boast. "Give me," said the exulting philosopher, "a place to stand on, and I will lift the earth." (Δος μοι που στω, και την γην κινησω.) This however may be easily proved to be impossible; for, granting him a place, with the simplest machine, it would require a man to move swifter than a cannon shot during the space of 100 years, to lift the earth only *one inch* in all that time.—Hiero ordered a golden crown to be made, but suspecting that the artists had purloined some of the gold and substituted base metal in its stead, he employed our philosopher to detect the cheat; Archimedes tried for some time in vain, but one day as he went into the bath, he observed that his body excluded just as much water as was equal to its bulk; the thought immediately struck him that this discovery had furnished ample data for solving his difficulty; upon which he leaped out of the bath, and ran through the streets homewards, crying out, εὕρηκα! εὕρηκα! *I have found it! I have found it!*—The best edition of his works is that of Torelli, edited at the Clarendon Press, Oxford, fol. 1792, by Dr. Robertson, Savilian Professor of Astronomy.

cumference of a circle to its diameter, A. C. 250: this he effected by circumscribing about, and inscribing in the circle regular polygons of 96 sides, and making a numerical calculation of their perimeters; by means of this process he made the ratio as 22 to 7, which is a determination near enough the truth for common practical operations, where great exactness is not required, and has the advantage of being expressed by small numbers. He was the next after Hippocrates, who squared a curvilineal space; he applied himself with ardour to the investigation of the measures, proportions, and properties of the conic sections, spirals, cylinders, cones, spheres, conoids, spheroids, &c. On these subjects the following works of his are still extant, viz. two books on the Sphere and Cylinder; and treatises on the Dimensions of the Circle; on Spirals; on Conoids and Spheroids; and on the Centres of Gravity.

The next geometer of note after Archimedes, was Apollonius Pergæus, A. C. 230: this great man studied for a long time in the schools of Alexandria under the disciples of Euclid, and was the author of several valuable works on Geometry, which were so much esteemed, that they procured him the honourable title of *the great Geometrician.* His principal work, and the most perfect of the kind among the ancients, is his treatise on the Conic Sections, in eight books; seven only of these have been preserved, the four first in the original Greek, and the 5th, 6th, and 7th in an Arabic version [q].

[q] According to Pappus and Eutocius, the following works were likewise written by Apollonius, viz. 1. The Section of a Space. 2. The Section of a Ratio. 3. The Determinate Section. 4. The Inclinations. 5. The Tangencies, and 6. The Plane Loci; each of these treatises consisting of two books. Pappus has left us some particulars of the above works, which are all concerning them that now remain; but from these scanty materials, many restorations have been made, viz. by Vieta, Snellius, Ghetaldus, Fermat, Schooten, Alex. Anderson, Halley, Simson, Horsley, Lawson, Wales, and Burrow. The best edition of the Conics of Apollonius is that by Dr. Halley, *fol. Oxon.* 1710.

The age of Archimedes and Apollonius has with justice been stiled, THE GOLDEN AGE OF ANCIENT GEOMETRY, as the science never acquired so great a degree of brilliancy at any other period of the Grecian history.

The duplication of the cube, quadrature of the circle, trisection of an angle, &c. were problems of which the ancients never lost sight; many of the propositions in the Elements, particularly prop. 27, 28, and 29 of the sixth book, are intimately connected with the solution, and probably originated in the attempts to obtain it. The application of the conic sections to this purpose by Menechmus, has been already noticed: about the same time Dinostratus invented the quadratrix, a mechanical curve, possessing the triple advantage of trisecting and multiplying an angle, and squaring the circle. The conchoid of Nicomedes, who flourished A. C. 250, has been applied by both ancient and modern geometers equally to the trisection, finding two mean proportionals, and the construction of other solid problems; for which purposes this curve has been preferred by Archimedes, Pappus, and Newton, to any other. (See Newton's *Arithmetica Universalis*, p. 288, 289.) The cissoid, another curve, being an improvement on the conchoid, was invented by Diocles about 150 years before Christ.

Hero, Dositheus, Eratosthenes, and Hypsicles, who flourished in the second century before Christ, and Geminius who flourished in the first, were all eminent for their skill in Geometry: indeed the science continued to be cultivated with ardour by a numerous list of geometricians produced by the Alexandrian school, until that famous seat of learning fell a prey to the blind and merciless bigotry of the Arabs. The first who wrote on the sphere and its circles to any consi-

* Here the eighth book is supplied, and likewise is added, Serenus' treatise on the Section of the Cylinder and Cone, printed from the original Greek, with a Latin translation.

derable extent, at least whose works have been preserved, was Theodosius, A. C. 60: this work, in which the propositions are demonstrated with equal strictness and elegance, forms the basis of spherical Trigonometry, as practised by the moderns. About the same time, or shortly after, Menelaus wrote his treatise on Chords, which is lost; but his work on Spherical Triangles, containing the construction and trigonometrical method of resolving them, according to the antient practice, is still extant. We are particularly indebted to Pappus, A. D. 380, and Proclus, A. D. 430, for their laborious researches; many particulars relating to the sciences of the Greeks would have been lost to posterity, but for their writings: the former was an eminent mathematician of Alexandria, and author of several learned and useful works, particularly eight books of Mathematical Collections, of which the first and part of the second are wanting. These books contain a great variety of useful information relating to Geometry, Arithmetic, Mechanics, &c. with the solution of problems of different sorts. Proclus likewise studied at Alexandria, and afterwards presided over the Platonic school at Athens; he wrote, besides many other works, Commentaries on the first book of Euclid, on the Mathematics, on Philosophy; also a treatise De Sphaera, which was published by Dr. Bainbridge, Savilian Professor of Geometry at Oxford, in 1620.

The writings of the Greek geometricians were translated and commented on by several learned Arabians, but the improvements they introduced were chiefly of the practical kind; among these may be mentioned the fundamental propositions of Trigonometry, in which, by the substitution of sines instead of the chords, and other convenient abridgements, they greatly simplified the theory and solutions of plane and spherical triangles. These improvements are ascribed to Mahomet Ebn Musa, a geometer of whom there still exists a work on Plane and Spherical Figures. We like-

wise possess a work on Surveying, written by Mahomet of Baghdad, which some modern authors have ascribed to Euclid.

A few learned men, famous for their skill in Geometry, flourished in the West during the fifteenth century. Of these the chief were the Cardinals Bessarion and Cusa [r], Purbach, Nicholas Oresme, Bianchini, George of Trabezonde, Lucas de Burgo, Schonerus, Walther, and Regiomontanus; the latter wrote a treatise on Plane and Spherical Trigonometry, A. D. 1464; in which, among other improvements, he introduced the use of the tangents, and applied Algebra to the solution of geometrical problems: this is the more surprising, as it occurred several years before the publication of any of the works of De Burgo, who is generally supposed to have been the introducer of Algebra into Europe.

On the revival of learning in Europe about the beginning of the sixteenth century, the study of Geometry began to be cultivated with great attention; the works of the Greek geometricians were eagerly sought after and translated into Latin or Italian, and served as guides to those who had a taste for that correct reasoning, for which the ancient Geometry is so justly famed, or were desirous of availing themselves of the knowledge of its application and use, as connected with the necessary business of life. As early as 1522, John Werner, a celebrated astronomer of Nuremberg, published some tracts on the Conic Sections, and on other geometrical subjects. Tartalea composed a treatise on Arithmetic, Algebra, Geometry, Mensuration, &c. entitled, *Trattato di Numeri et Misure*, 1556, being the first modern work

[r] Nicolas De Cusa was born of poor parents, A. D. 1401; his application to learning and his personal merit, however, raised him to the rank of bishop and cardinal: his claim to the honour of having squared the circle was ably refuted by Regiomontanus; nevertheless he was a man of very extraordinary parts, and excelled in the knowledge of law, divinity, natural philosophy, and geometry, on which subjects he is said to have written some excellent treatises. He died in 1464.

which teaches how to find the area of a triangle by means of its three sides, without the aid of a perpendicular. Maurolicus was a respectable geometer, and wrote on various subjects; his treatise on the Conic Sections is remarkable for its perspicuity and elegance. Aurispa, Batecombe, Butes, Ramus, Xylander, Fortius, Cardan, Fregius, Bombelli, Ficinus, Durer, Zeigler, Fernel, Ubaldi, Clavius, Barbaro, Byrgius, Commandine, Pelletier, Dryander, Nonius, Linacre, Sturmius, Saville, Ghetaldus, R. Snellius, and many others who flourished at this period, were cultivators of Geometry; and if they made few discoveries, still their labours as translators, commentators, or teachers, were beneficial in diffusing knowledge, and merit our grateful acknowledgments.

Various approximations to the ratio of the circumference of a circle to its diameter, were given about the beginning of the 17th century, approaching much nearer the truth than any that had hitherto appeared; viz. by Adrian Romanus, Willebrord Snellius, Peter Metius, and Ludolph Van Ceulen; according to the conclusion of Metius, if the diameter be 113, the circumference will be 355, which is very near the truth, and has the advantage of being expressed by small numbers. By continual bisection of the circumference, Van Ceulen found, that if the diameter be 1, the circumference will be 3,14159, &c. to 36 places of decimals; which discovery was thought so curious, that the numbers were engraved on his tomb in St. Peter's Church-yard, at Leyden *.

* The simplest (and consequently least accurate) ratio of the diameter to the circumference is as 1 to 3; a ratio somewhat nearer than this, is as 6 to 19. We have noticed before that Archimedes determined the ratio to be as 7 to 22 nearly, which is nearer than the above.

A nearer approximation is as	106 to 333
That of Metius is still nearer, viz. as	113 to 355
A nearer approximation than the last is	as 1702 to 5347
A still nearer is	as 1815 to 5702, &c.

Geometrical problems had long before this period been solved algebraically, by Cardan, Tartalea, Regiomontanus, and Bombelli; but a regular and general method of applying Algebra to Geometry, was first given by Vieta, about the year 1580; as also the elements of angular sections. Des Cartes improved the discovery of Vieta, by introducing a general method of representing the nature and circumstances of curve lines by algebraic equations, distributing curves into classes, corresponding to the different orders of equations by which they are expressed; A.D. 1637. A method of tangents, and a method *de maximis et minimis*, much resembling that of fluxions or increments, owe their origin to Fermat, a learned countryman and competitor of Des Cartes, with whom he disputed the honour of first applying Algebra to curve lines, and to the geometrical construction of equations, secrets of which he was in possession before Des Cartes' Geometry appeared. About this time, or a little earlier, Galileo invented the cycloid; its properties were afterwards demonstrated by Torricellius.

The improvement of Des Cartes, now called the *new Geometry*, was cultivated with ardour and success by mathematicians in almost every part of Europe; his work was translated out of French into Latin, and published by Francis Schooten, with a commentary by Schooten, and notes by M. de Beaune, 1649. The Indivisibles of Cavalerius, published in 1635, was a new and useful invention, applied to

Van Ceulen's numbers, as mentioned above, were extended to 72 places of figures by Mr. Abraham Sharp, about 1706; Mr. Machin afterwards extended the same to 100 places, and M. De Lagni has carried them to the amazing length of 128 places: thus, if the diameter be 1000, &c. (to 128 places) the circumference will be 31415, 92653, 58979, 32384, 62643, 38327, 95028, 84197, 16939, 93751, 05820, 97494, 45923, 07816, 40628, 62089, 98628, 03482, 53421, 17067, 98214, 80865, 13272, 30664, 70938, 446+, or 7—. This number (which includes those of Van Ceulen, Sharp, and Machin) is sufficiently near the truth for any purpose, so that except the ratio could be completely found, we need not wish for a greater degree of accuracy.

determine the areas of curves, the solidities of bodies generated by their revolution about a fixed line, &c. Roberval, as early as 1634, had employed a similar method, which he applied to the cycloid, a curve at that time much celebrated for its numerous and singular properties; he likewise invented a general method for tangents, applicable alike to geometrical and mechanical curves. The inverse method of tangents derived its origin from a problem, which De Beaune proposed to his friend Des Cartes, in 1647. In 1655 the learned Dr. Wallis published his Arithmetica Infinitorum; being either a new method of reasoning on quantities, or else a great improvement on the Indivisibles of Cavalerius above mentioned; speculations which led the way to infinite series, the binomial theorem, and the method of fluxions: this work treats of the quadrature of curves and many other problems, and gives the first expression known for the area of a circle by an infinite series.

One of the greatest discoveries in modern Geometry was the theory of evolutes, the author of which was Christian Huygens, an ingenious Dutch mathematician, who published it at the Hague in 1658, in a work entitled, Horologium Oscillatorium, sive de Motu Pendulorum, &c.

In 1669 were published Dr. Barrow's Optical and Geometrical Lectures, containing many very ingenious and profound researches on the dimensions and properties of curves, and especially a method of tangents, by a mode of calculation differing from that of fluxions or increments in scarcely any particular, except the notation. About this time the use of geometrical loci for the solution of equations, was carried to a great degree of perfection by Slusius, a canon of Liege, in his *Mesolabium et Problemata Solida*: he likewise inserted in the Philosophical Transactions, a short and easy method of drawing tangents to all geometrical curves, with a demonstration of the same; and likewise a tract on the

Optic Angle of Alhazen. Besides those we have mentioned, many others of this period devoted their attention to the rectification and quadrature of curves, &c. of whom Van Heuraet, Rolle, Pascal, Briggs, Halley, Lallouère, Torricellius, Herigon, Niell, Sir Christopher Wren, Faber, Lord Brouncker, Nicholas Baker, G. St. Vincent, Mercator, Gregory, and Leibnitz, were the principal.

The seventeenth century is famed for giving birth to two noble discoveries; namely, that of logarithms in 1614 by Lord Napier, whereby the practical applications of Geometry are greatly facilitated; and that of fluxions, to which problems relating to infinite series, the quadrature and properties of curves, and other geometrical subjects connected with Astronomy, Physics, &c. and which were formerly considered as beyond the reach of human sagacity, readily submit. For this sublime discovery, the learned are indebted either to the profound and penetrating genius of Sir Isaac Newton [t], or

[t] Sir Isaac Newton, one of the greatest mathematicians and philosophers that ever lived, was born in Lincolnshire, in 1642. Having made some proficiency in the classics, &c. at the grammar school at Grantham, he (being an only child) was taken home by his mother (who was a widow) to be her companion, and to learn the management of his paternal estate: but the love of books and study occasioned his farming concerns to be neglected. In 1660 he was sent to Trinity College, Cambridge: here he began with the study of Euclid, but the propositions of that book being too easy to arrest his attention long, he passed rapidly on to the Analysis of Des Cartes, Kepler's Optics, &c. making occasional improvements on his author, and entering his observations, &c. on the margin. His genius and attention soon attracted the favourable notice of Dr. Barrow, at that time one of the most eminent mathematicians in England, who soon became his steady patron and friend. In 1664 he took his degree of B. A. and employed himself in speculations and experiments on the nature of light and colours, grinding and polishing optic glasses, and opening the way for his new method of fluxions and infinite series. The next year, the plague which raged at Cambridge obliged him to retire into the country; here he laid the foundation of his universal system of gravitation, the first hint of which he received from seeing an apple fall from a tree; and subsequent reasoning induced him to conclude, that the same force which brought down the apple might possibly extend to the moon, and retain her in her orbit: he afterwards extended the doctrine to all the bodies which compose the solar system, and

to that of Leibnitz, or to both, for both laid claim to the invention. No sooner was the method made public, than a

demonstrated the same in the most evident manner, confirming the laws which Kepler had discovered, by a laborious train of observation and reasoning; namely, that "the planets move in elliptical orbits;" that "they describe equal areas in equal times;" and that "the squares of their periodic times are as the cubes of their distances." Every part of natural philosophy not only received improvement by his inimitable touch, but became a new science under his hands: his system of gravitation, as we have observed, confirmed the discoveries of Kepler, explained the immutable laws of nature, changed the system of Copernicus from a probable hypothesis to a plain and demonstrated truth, and effectually overturned the vortices and other imaginary machinery of Des Cartes, with all the improbable epicycles, deferents, and clumsy apparatus, with which the ancients and some of the moderns had encumbered the universe. In fact, his *Philosophiæ Naturalis Principia Mathematica* contains an entirely new system of philosophy, built on the solid basis of experiment and observation, and demonstrated by the most sublime Geometry; and his treatises and papers on optics supply a new theory of light and colours. The invention of the reflecting telescope, which is due to Mr. James Gregory, would in all probability have been lost, had not Newton interposed, and by his great improvements brought it forward into public notice.

In 1667 Newton was chosen fellow of his College, and took his degree of M. A. Two years after, his friend Dr. Barrow resigned to him the mathematical chair; he became a Member of Parliament in 1688, and through the interest of Mr. Montagu, Chancellor of the Exchequer, who had been educated with him at Trinity College, our author obtained in 1696 the appointment of Warden, and three years after that of Master, of the Mint: he was elected in 1699 member of the Royal Academy of Sciences at Paris; and in 1703 President of the Royal Society, a situation which he filled during the remainder of his life, with no less honour to himself than benefit to the interests of science.

In 1705, in consideration of his superior merit, Queen Anne conferred on him the honour of knighthood: he died on March 20th, 1727, in the 85th year of his age. Virtue is the brightest ornament of science: Newton is indebted to this for the *best* part of his fame; he was a *great* man, and *good* as he was great: to the most exemplary candour, moderation, and affability, he added every virtue necessary to constitute a truly moral character; above all, he felt a firm conviction of the truth of Revelation, and studied the Bible with the greatest application and diligence. But such is the folly of man, that the tribute, which is due to the GREAT FIRST CAUSE ALONE, we transfer to the instrument; Newton, Marlborough, Nelson, Wellington, &c. have *all* our praise, while the GREAT SOURCE of knowledge, strength, victory, and every benefit we enjoy, is forgotten. How would the modest Newton have reddened with shame and indignation, could he have heard all the extravagant encomiums, little short of adoration, which have with foolish and

sharp and virulent contest ensued: at length the Royal Society was appealed to, and a Committee appointed to examine letters, papers, and other documents, and thence to form a decision on the claim of each. The result of the inquiry was, "That Sir I. Newton had invented his method before the year 1669, and consequently fifteen years before M. Leibnitz had given any thing on the subject in the Leipsic Acts:" the same Report in another part says, "that it did not appear that M. Leibnitz knew any thing of the differential calculus, before his letter of the 21st of June, 1677." It appears however that this decision, which confirmed the claim of our illustrious countryman, did not give entire satisfaction to the continental mathematicians of that period, nor are their successors better disposed to yield the palm to Newton; they still contend that Leibnitz, admitting that he was not the *first* inventor, (and some refuse to concede this point,) borrowed nothing of his method from his rival; a fact which some well informed Englishmen have much questioned.

Other tracts containing improvements in Geometry were given by Newton; as, 1. *Enumeratio Linearum Tertii Ordinis.* 2. *Tractatus Duo de Speciebus et Magnitudine Figurarum Curvilinearum.* 3. *Genesis Curvarum per Umbras:* in these, as well as in his *Principia* and other works, he has for the most part employed his own new analysis, by which the doctrine of curves has been amazingly extended and improved.

Geometry had hitherto consisted of two kinds, *Elementary*, or that which treats of right lines, rectilineal figures, the circle, and solids terminated by these; and *Higher*, or *Transcendent Geometry*, which treats of all sorts of curves,

impious profusion been lavished on his memory! His works, collected in 5 volumes 4to. with a valuable Commentary by Dr. Horsley, were published in 1784.

except the circle, and the solids generated by their revolution: to these, as has been observed, the discoveries of Sir Isaac Newton have added a third, viz. the *Sublime Geometry*, or the doctrine and application of fluxions [a].

Of those authors, who have since applied themselves to the cultivation and improvement of the new calculus, (as the doctrine of fluxions was called,) and to the extension of its applications, the following are the names of some of the chief; viz. Agnesi, D'Alembert, Bossut, the Bernoulli's, Cheyne, Cotes, Craig, Clairaut, Colson, Cagnoli, Condorcet, Emerson, Euler, Fontaine, Fagnanus, Guisnee, Le Grange, L'Hôpital, Hayes, Hodson, Harris, Hutton, Jones, Jack, Landen, Lorgna, De Lagni, Manfredi, Maseres, Maclaurin, Nicole, Nieuwentyt, Reyneau, Riccati, Raphson, Rowe, Smith, Sterling, Saunderson, Simpson, Taylor, Vince, Walmsley, Waring, &c.

The following inventions, which are either nearly allied to the method of fluxions or capable of similar application, have been already noticed in the Introduction to Part III. viz. Dr. Brook Taylor's *Methodus Incrementorum*, 1715; Kirkby's *Doctrine of Ultimators*, 1748; Landen's *Residual Analysis*, 1764; and Major Glenie's *Doctrine of Universal Comparison*, 1789, and his *Antecedental Calculus*, 1793.

It has been the error and misfortune of some eminent

[a] "On peut diviser la Géométrie de différentes manières. En élémentaire, et en transcendante. La Géométrie élémentaire ne considère que les propriétés des lignes droites, des lignes circulaires, et des solides terminés par ces figures. Le cercle est la seule figure curviligne dont on parle dans les élémens de Géométrie.

"La Géométrie transcendante est proprement celle qui a pour objet toutes les courbes différentes du cercle, comme les sections coniques, et les courbes d'un genre plus élevé.

"Par là, on auroit trois divisions de la Géométrie: Géométrie élémentaire, ou des lignes droites, et du cercle; Géométrie transcendante, ou des courbes; et Géométrie sublime, ou des nouveaux calculs." *D'Alembert, Encyclopedie*, mot *Géométrie*.

and otherwise deserving characters, to direct their attention almost *exclusively* to *mathematical demonstration*, whereby they have been induced to deny or undervalue the force and evidence of *moral certainty*; the celebrated Dr. Edmund Halley [x] was one of these. Revelation is a subject, which among very many others does not admit of *mathematical* proof; and therefore he affirmed with equal rashness and impiety, that "the doctrines of Christianity are incomprehensible, and the religion itself is a cheat." This hardy declaration roused the indignation of Dr. Berkeley [y], the

[x] Edmund Halley was born in London, A. D. 1656. After making considerable progress in the classics at St. Paul's School, and obtaining some knowledge of the mathematics, he was sent in 1673 to Oxford, where he applied himself closely to mathematics and astronomy. Having conceived the design of completing the catalogue of stars, by increasing it from his own observation by those in the southern hemisphere, he embarked for St. Helena in November, 1676; he returned in 1678, having completed his catalogue, on which occasion the University of Oxford honoured him with the degree of M. A. and the Royal Society elected him one of their Fellows. In 1691 he applied for the appointment of Savilian Professor, but being charged with infidelity and scepticism, and his pride scorning to disavow the charge, he did not succeed; however in 1703 he succeeded Dr. Wallis as Professor of Geometry at Oxford, and had the degree of LL. D. conferred on him. In 1713 he became Secretary to the Royal Society, an office which six years after he resigned, on being appointed Astronomer Royal: in prosecuting the duties of this office, he is said to have missed scarcely a single observation during eighteen years which he held it; he died in 1742. Dr. Halley's numerous observations on the heavenly bodies, the winds and tides, the variation of the magnetic needle, and other valuable tracts on mathematical subjects, published separately or in the Philosophical Transactions, have rendered his name famous all over Europe.

[y] George Berkeley was born at Kilerin in Ireland, in the year 1684: after receiving the first part of his education at Kilkenny school, he became a Pensioner of Trinity College, Dublin, in 1699, and a Fellow in 1707: in 1721 he took the degrees of B. D. and D. D. and three years after was promoted to the Deanery of Derry, and to the Bishopric of Cloyne in 1733; in 1752 he removed with his family to Oxford, where he died the following year. Besides the replies and rejoinders to which the above dispute gave birth, Dr. Berkeley wrote *Arithmetica absque Algebra, aut Euclide Demonstrata*, 1707; a *Mathematical Miscellany*, inscribed to Mr. Molineux; *Theory of Vision*, 1709; *The Principles of Human Knowledge*, 1710; *Dialogues between Hylas and Philonus*, 1713. In the two latter it is attempted to be proved, that the common notion of the existence of matter is false; that we cannot be certain that

learned and virtuous bishop of Cloyne, who, to assert the truth and honour of injured religion, published in 1734 *The Analyst*. In this work, which is addressed to Halley as an *infidel mathematician*, he shews that the mysteries in faith, &c. are unjustly objected to, especially by the mathematicians, who, he affirms, admit much greater mysteries, and even falsehoods, into science; of which, he says, the doctrine of fluxions furnishes an example. This avowed attack on a new branch of science, the principles of which had not then in every particular been established with sufficient firmness, called forth the zeal and abilities of its admirers; and produced, besides a direct answer, as it is supposed by Dr. Jurin, Robins's *Discourse concerning the Method of Fluxions*, &c. 1735; Walton's *Vindication*, &c. 1735; and Smith's *New Treatise of Fluxions, with answers to the principal objections in the Analyst*, 1737: but the most complete vindication of the method of fluxions to which this contest gave rise, together with a firm establishment of its principles, &c. are to be found in Maclaurin's *Complete System of Fluxions, with their application to the most considerable Problems in Geometry and Natural Philosophy*, in 2 vol. 4to. published at Edinburgh, in 1742: this is indeed the most complete and comprehensive work on the science that has ever yet appeared.

Of the modern elementary writers on Geometry, who have given systems of their own, and not strictly followed Euclid, the following are the principal; viz. Borelli, Pardies, Wolfius,

there are any such things as external sensible objects; and that they are, as far as we can know, nothing more than mere impressions made upon the mind by the immediate act of God, according to certain rules called laws of nature. He was a truly excellent man, and the line by which Pope has characterised him, by ascribing to him "every virtue under heaven," is said not to have far exceeded the truth. In addition to the above works, he wrote *The Minute Philosopher*; some tracts on religious and political subjects; *Siris*, or the virtues of Tar Water; and another piece on the same subject.

Sturmius, Robertson, Marchetti, Hamilton, Emerson, Simpson, Bonnycastle, and Hutton; those of the three last are valuable and useful performances. Those who have written on the subject of practical Geometry, are Bayer, Bonnycastle, Clavius, Cantzlerus, Gregory, Herigon, Hawney, Hulsius, Kepler, Lightbody, Le Clerc, Mallet, Ozanam, Ramus, Reinhold, Scliwinterus, Scheffelt, Tacquet, Voigtel, Wolfius, and many others.

ON THE USEFULNESS OF GEOMETRY.

NO question is more frequently asked by beginners in Geometry, than the following: *Of what use is the study of Euclid's Elements?* The industrious, the idle, the sensible, and the dull, from different motives, are equally concerned in the inquiry: they almost daily agitate it with a degree of importunity, which sometimes proves troublesome to the Tutor, because he finds himself incapable of answering the question *completely* to his own or their satisfaction. The difficulty however lies not in the ignorance of the Tutor, or the want of usefulness in the science, but in the nature of things: for no art or science whatever can teach its own use; how then can one, who is learning merely the principles of Geometry, expect to understand fully its usefulness, or that his Tutor, however learned he may be, can by any explanation do justice to a science, of which the various and useful applications will perhaps never be completely determined? To try to satisfy all the absurd and vexatious scruples, which the idle, the querulous, or the captious, please to start against any branch of learning, would perhaps be a vain attempt; but it will be proper to advise the diligent and well-disposed student, (and to such the advice can hardly be needful,) that it is his duty, and will be to his advantage, to study attentively and without scruple, any branch of learning which his friends may think proper to recommend to him as useful, and which the experience of wise and good men in every age has proved to be so.

But in the present instance, an implicit reliance on authority is not at all necessary; the *obvious* uses of Geometry are sufficient to recommend it to the candid and impartial inquirer; some of these we shall briefly enumerate. Geometry is useful, as it applies to the businesses and concerns of society, and as fundamental to other sciences and arts connected with them. Whatever relates to the comparison, estimation, &c. of distances, spaces, and bodies, belongs to Geometry; and consequently on its principles and conclusions immediately depend Mensuration, Surveying, Perspective, Architecture, Navigation, Fortification, with many other branches equally conducive to public benefit: in short, it is difficult to acquire a tolerable degree of know-

ledge in philosophy, or any art or science, without some acquaintance with Geometry.

In addition to the direct and practical uses of the science, there is another, which Lord Bacon calls "collateral and intervenient." Geometry strengthens, corroborates, and otherwise improves the reasoning faculties, inuring the mind to patient labour, teaching it method, and supplying it with the means of contriving and adopting proper expedients for the prosecution of its researches. Geometry may then be justly considered as a highly valuable science, both with respect to its practical application, and as a complete model of strict demonstration: and in the latter view it recommends itself to the diligent attention of every lover of truth.

In what follows, we shall treat of Geometry in the two-fold view above explained, by briefly shewing the practical application of Euclid's doctrine, and likewise by considering it purely as a system of demonstration.

The demonstration of a proposition does not depend on the correctness of the diagram, which therefore may be drawn by hand; but in the practical uses of the propositions which we mean to exemplify, accurate figures should be made, and for this purpose instruments must be employed: we will therefore give a brief description of such instruments as are necessary for the construction of figures, and explain their farther uses hereafter, repeating, that *the instruments are by no means necessary to the demonstration.*

DESCRIPTION OF A CASE OF MATHEMATICAL INSTRUMENTS.

A common pocket case of Mathematical Instruments contains, 1. a pair of Plain Compasses; 2. a pair of Drawing Compasses; to the latter belong 3. a Port Crayon, 4. a Dotting Pen, and 5. a Steel Pen: 6. a Drawing Pen, with 7. a Pointer; 8. a Protractor; 9. a Plain Scale; 10. a Sector; 11. a Parallel Ruler; and 12. a Black-lead Pencil [a].

[a] Cases of Mathematical Instruments may be had at all prices, from five shillings to six guineas; a case that costs twenty-five or thirty shillings will be suf-

The PLAIN COMPASSES are used for the following purposes:

1. To draw a blank or obscure line by the edge of a ruler, through any given point or points.

2. To take the distance between two points, and apply it to any line or scale; or to take the length of one line, and apply it to another.

3. To measure any line by taking its length between the points of the compasses, and apply them to the divisions of a proper scale.

4. To set off any proposed distances on a given line.

5. To describe obscure circles, intersecting arcs, &c.

6. To lay off any proposed angle, and to measure a given angle, by means of a scale of chords, &c.

The DRAWING COMPASSES [b]; one of the legs is filled

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ficiently good to answer the learner's purpose, and he should not go much under that price. *A Magazine*, or complete collection of every kind of useful drawing instrument, will cost from five to forty guineas.

In using the instruments, lines and figures should be drawn as fine, neat, and exact as possible; the paper on which the drawing is made should, if possible, not be pricked through or deeply scratched with the compasses; it should be laid on a quire of blotting, or other paper, during the operation; and the drawer should sit so that the light may be on his left, and not by any means in front. The drawing pen should not be dipped in the ink, but ink should be taken from the stand with a common pen, and put into it. The points of the instruments should be cleaned and wiped quite dry after they have been used, and every means employed to guard against rust, which will otherwise spoil the instruments.

[b] In the best sort of compasses, the pin or axle is made of steel, as also half the joint itself, as the opposite metals rubbing on each other are found to wear more equally; the points should be of hard well-polished steel, and the joint work with a smooth, easy, and uniform motion. In the drawing compasses, the shifting point is sometimes made with a joint, and furnished with a fine spring and screw; so that, having opened the compasses *nearly* to the required extent, by turning the screw the point will be moved to the *true* extent within a *hair's breadth*, for which reason they are named *Hair Compasses*.

There are various other kinds of compasses not appertaining to a common case of instruments, which are not less useful to the practical geometrician than those we have described; viz.

1. *Bow Compasses*, a small sort which shut up in a hoop; their use is to describe the circumferences and arcs of very small circles.

with a triangular socket and screw, to receive and fasten for use the following supplementary parts; viz. 1. a STEEL POINT; which being fixed in the socket, makes the compasses a plain pair, having all the uses above described. 2. A PORT CRAYON, with a short piece of black-lead or slate pencil, finely pointed and fitted on it for drawing circles and arcs on paper, or on a slate. 3. A STEEL PEN, for drawing lines or circles with ink; the small adjusting screw passing through the sides of the pen, serves to open or close them, for the purpose of drawing lines as thick or fine as may be thought necessary. 4. A DOTTER [c], which is a small indented wheel, fixed at the end of a common steel drawing pen; from which it receives ink for the purpose of drawing dotted lines or circles.

In the Port Crayon, Dotter, and Steel Pen, there is a joint for setting the lower part of the instrument perpendicular to the paper, which must be done in order to draw a line well.

The DRAWING PEN is fixed in a brass handle, and its use is to draw straight ink lines by the edge of a ruler. The handle or shaft unscrews near the middle, and in the end of the

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2. *Spring Compasses*, or *Dividers*, made of hardened steel, having an arched head, which by its spring opens the legs; the opening being directed by a circular screw, and worked with a nut.

3. *Proportional Compasses*, both *simple* and *compound*; their uses are to divide a given line into any number of equal parts; to find the sides of similar planes or solids in any given ratio; to divide a circle into any number of equal parts, &c.

4. *Trisecting Compasses*, invented by M. Tarragon, for trisecting arcs and angles.

5. *Triangular Compasses* with three legs, for taking three points at once.

6. *Turn-up Compasses* are the plain compasses, with two additional points fixed near the bottom of the legs, the one carrying a port crayon, and the other a drawing pen; these are made with a joint to turn up, so as to be used or not, as occasion may require.

7. *Beam Compasses* for describing very large circles.

8. *Elliptical Compasses* for describing ellipses.

9. *Spiral Compasses*, for describing spirals.

10. *Cylindrical* and *Spherical Compasses*, or *Calipers*, for measuring the diameters of cylindrical and spherical bodies, &c. &c.

[c] The Dotting Pen, not being easily cleaned, soon becomes rusty and useless; the best way to draw a *dotted* line is first to draw the line in pencil, and then to dot it with the writing or drawing pen.

upper part is fixed a fine *Steel Pin* or POINTER, for making dots, small, neat, and with the greatest exactness.

The PROTRACTOR[d] is a brass semicircle divided into 180 degrees, and numbered each way from end to end; the external edge of the Protractor's diameter is called the *fiducial edge*, and is the diameter of the circle, the small notch in the middle of the fiducial edge being the centre. The use of the Protractor is to measure any angle, to make an angle of any proposed number of degrees, to erect perpendiculars, &c.

The PLAIN SCALE is used for measuring and laying down distances: it contains on one side, a line of 6 inches, a line of 50 equal parts, and a diagonal scale. On the other side it has a line of chords marked C, and seven decimal scales of various sizes.

The line or scale of inches has each inch divided into 10 equal parts, and is used for taking dimensions in inches and tenth parts of an inch.

The line of 50 equal parts being 6 inches in length, is properly a decimal scale of a foot, for by it the foot is divided into 10 and likewise 100 equal parts. By this line, and the line of inches above described, any given decimal of a foot may be reduced into inches; and likewise any given number of inches to the decimal of a foot.

EXAMPLES.—1. Reduce .2 of a foot into inches.

Here, opposite 20 *in the second line* (*for* $20=\frac{20}{100}=\frac{2}{20}=.2$) *stands* $2\frac{4}{10}$ *inches, in the first: therefore* .2 *foot*$=2\frac{4}{10}$ *inches.*

2. Reduce $5\frac{4}{10}$ inches to the decimal of a foot.

Opposite $5\frac{4}{10}$ *in the first line, stands* 45 *in the second; therefore* $5\frac{4}{10}$ *inches*$=.45$ *foot.*

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[d] A Protractor in the form of a right angled parallelogram, is not only more convenient for the case than the semicircular one, but likewise measures some angles with greater exactness, and is therefore to be preferred. The Protractor, Scales, and Sector, should be made of either ivory, steel, or silver, rather than brass, for brass injures the sight when used long together, especially by candle-light.

The improved Protractor has an index moving about the centre, cutting the circumference, and will set off an angle true to a single minute.

3. To find the value of 3 inches. *Ans.* .25 *foot.*

4. To find the value of .15 foot. *Ans.* 1 *inch* $\frac{8}{10}$.

The Diagonal Scale is likewise a centesimal scale, for by it an unit is divided into 100 equal parts; and any number of those parts may be taken in the compasses, and laid down on paper with sufficient exactness for most practical purposes.

To explain the construction and use of the Diagonal Scale, let *ABCD* be a section of the scale, which is equally divided (suppose into inches) from *B* towards *A* in the points *E*, 1, 2, 3, &c. *Let BC=BE:* and let each of these be divided into 10 equal parts in the points marked by the small figures, 1, 2, 3, 4, &c. *I, II, III, IV*, &c. also divide *CF* in the same manner in the points *a, b, c, d*, &c. and let the lines passing through *B, E*, 1, 2, 3, be perpendicular to *AB*, and the lines *kI, nII, mIII, oIV*, &c. parallel to it, join 9 *C*, 8 *a*, 7 *b*, 6 *c*, 5 *d*, &c.

Since $9B=BI=aC$, and $9C$ by its inclination to BC meets it in C, if the distance of $9C$ and BC at B, that is $9B$, be called 1, then will their distance on the next parallel marked *I* be $\frac{9}{10}$, and at the next parallel marked *II*, it will be $\frac{8}{10}$; at the next marked *III*, it will be $\frac{7}{10}$; at the next marked *IV*, it will be $\frac{6}{10}$; and so on, decreasing successively by $\frac{1}{10}$, down to the point *C*, where the lines meet, and consequently the distance is nothing.

If $8B$ be called 2, then will the distance from $8a$ to BC on the parallel marked *I*, be $1\frac{9}{10}$; on the parallel marked *II*, $1\frac{8}{10}$; on the parallel marked *III*, $1\frac{7}{10}$; on the parallel marked *IV*, $1\frac{6}{10}$; and the like for other divisions.

Examples.—1. Let it be required to find 3.4 on the scale.

Here it will be convenient to begin at E; wherefore if the distance of the lines EF and 3 f be taken in the compasses on the parallel marked IV, it will be 3.4, the number required.

2. To find 7.8 on the scale.

Extend the compasses from EF to 7 b on the parallel marked VIII, and it will be the distance.

3. To find 3.45 by the scale.

In this case we must take each of the primary divisions marked with the large figures, 1, 2, 3, &c. for unity, and then the smaller divisions, E 1, &c. will each represent one tenth, and the parallel differences each one hundredth; wherefore we must extend the compasses from 3 D to 4 e on the parallel marked V, and it will be the distance required.

Whence if each primary division, as *EB*, be called	Then will each subdivision, as *B*9, be	And each successive parallel difference
1000	100	10
100	10	1
10	1	.1
1	.1	.01
.1	.01	.001
.01	.001	.0001
&c.	&c.	&c.

The Diagonal Scale has the decimal and centesimal division at each end, the unit of one being double that of the other, for the convenience of drawing figures of different sizes[*].

The other side of the Plain Scale contains seven lines decimally divided and subdivided; these are called *Plotting Scales*, and serve to construct the same figure of seven different sizes: by the help of these we can accommodate the figure to the dimensions of the page or sheet on which it is required to be drawn.

The number at the beginning of each of these lines shews how many of its subdivisions make an inch.

The line of chords marked *C* on this side of the Plain Scale, is used for the same purposes as the Protractor, viz. to measure and lay down angles, &c. The method of using both will be explained hereafter.

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[*] The method of diagonals was invented by Richard Chanseler, an Englishman, and first published by Thomas Digges, Esq. in his *Alæ, seu Scalæ Mathematicæ*, London, 1573.

The SECTOR [f] is an instrument consisting of two flat rulers or legs, moveable on a joint or axis, the middle point of which is the centre; it contains all the lines usually set on the Plain Scale, and several others, which the peculiar construction of this useful instrument renders universal.

The lines on the Sector are distinguished into two kinds, single and double.

The single lines on the best Sectors are as follow;

1.	A line of Inches decimally divided.	
2.	A line of a Foot centesimally divided *on the edge.*	
3.	Gunter's line of the Logarithms of Numbers, marked	*n*
4.	Logarithmic Sines	*s*
5.	Logarithmic Tangents	*t*
6.	A line of Chords	*Cho.*
7.	. . . Sines	*Sin.*
8.	. . . Tangents	*Tang.*
9.	. . . Rhumbs	*Rhum.*
10.	. . . Latitude	*Lat.*
11.	. . . Hours	*Hou.*
12.	. . . Longitude	*Lon.*
13.	. . . Inclination of Meridians	*In. Mer.*
14.	. . . Logarithmic Versed Sines	*V. Sin.*

The double lines are,

1.	A line of Lines, or equal parts . . . . marked	*Lin.*
2.	. . . Chords	*Cho.*
3.	. . . Sines	*Sin.*
4.	. . . Tangents to 45 degrees	*Tan.*
5.	. . . Secants	*Sec.*
6.	. . . Tangents above 45 degrees	*Tang.*
7.	. . . Polygons	*Pol.*

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[f] The first printed account of the Sector appeared at Antwerp in 1584, by Gasper Mordente, who says that his brother Fabricius invented the Sector in the year 1554. Some ascribe the invention to Guido Ubaldo, A. D. 1568: others again to Justus Byrgius, a French mathematical instrument maker, who also flourished in the 16th century. Daniel Speckle next treated of the Sector, viz. at Strasburg in 1589, and Dr. Thomas Hood wrote on the same subject at London in 1598, as did Samuel Foster, in a posthumous work published at London by Leybourne, in 1661. Many others have since explained the nature and uses of this instrument; but the most complete account of any will be found in Mr. Robertson's Treatise of Mathematical Instruments.

The scales of Lines, Chords, Sines, Tangents, Rhumbs, Latitudes, Longitudes, Hours, and Incl. Merid. being set on one leg only, may be used with the instrument either shut or open. The scales of Inches, Decimals, Log. Numbers, Log. Sines, Log. Versed Sines, and Log. Tangents, are on both legs, and must be used with the instrument open to its utmost extent.

The double lines proceed from the centre or joint of the Sector obliquely, and each is laid *twice* on the same face of the instrument, viz. once on each leg. To perform operations peculiar to the Sector, or, as it is called, "to resolve problems *sectorwise*," its legs must be set in an angular position, and then distances are taken with the compasses, not only "laterally," (or in the direction of its length,) but "transversely," or "parallelwise," viz. from one leg to the other.

The PARALLEL RULER[s] consists of two straight flat rules, connected by two equal brass bars, which turn freely on four pins or axes, fixed two on each rule at equal distances, so that the rules being opened, or separated to any distance within the limits of the bars, they will always be parallel, and consequently the lines drawn by them will be parallel.

The BLACK LEAD PENCIL should be made of the best black lead, and its point scraped very fine and smooth; it is used for drawing lines by the edge of a scale or ruler where ink lines are not wanted. Plans and figures which require exactness, should be first drawn with the pencil, and then if they are not right, it will be easy to take out the faulty part with a piece of India rubber, and make the necessary correction; after which the pencil lines may be drawn over with ink. The pencil is not less convenient as a substitute for the pen in writing, calculating, &c. A piece of good clean India rubber, of a moderate size and thickness, must always accompany a case of Mathematical Instruments.

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s The Parallel Ruler usually put into a case of Instruments is only six inches long, and too small for most purposes; the better sorts are from six inches to two feet in length, and sold separate.

The Double Parallel Ruler consists of three rules, so connected that the two exterior rules move not only parallel, but likewise opposite to each other; for some account of its construction and use see *Martin's Principles of Perspective*, p. 28.

The foregoing short description was deemed necessary, but the uses of the Instruments must be deferred, until the learner has acquired sufficient skill in Geometry to understand them.

OF GEOMETRY, CONSIDERED AS THE SCIENCE OF DEMONSTRATION.

As the reader is supposed to be unacquainted with logic, it will be proper in this place to introduce a few particulars taken from that art, which may serve as an introduction.

1. The mind becomes conscious of the existence of external objects by the impressions it receives from them. There are five inlets or channels, called the *organs of sense*, by which the mind receives all its original information; namely, the eye, the ear, the nose, the palate, and the touch: hence seeing, hearing, smelling, tasting, and feeling, are called the five senses. This great source of knowledge, comprehending all the notices conveyed to the mind by impulses made by external objects on the organs of sense, is called SENSATION.

2. PERCEPTION is that whereby the mind becomes conscious of an impression; thus, when I feel cold, I hear thunder, I see light, &c. and am conscious of these effects on my mind, this consciousness is called *perception*.

3. AN IDEA results from perception; it is the representation or impression of the thing perceived on the mind, and which it has the power of renewing at pleasure.

4. The power which the mind possesses of retaining its ideas, and renewing the perception of them, is called MEMORY; and the act of calling them up, examining, and reviewing them, is called REFLECTION.

5. In addition to the numerous class of ideas derived by *sensation* wholly from without, the mind acquires others by *reflection;* thus by turning our thoughts inward, and observing what passes in our own minds, we gain the ideas of hope, fear, love, thought, reason, will, &c. The ideas derived by means of sensation are called SENSIBLE IDEAS, and those obtained by reflection, INTELLECTUAL IDEAS.

6. From these two sources alone (viz. sensation and reflection) the mind is furnished with ample store of materials for its future operations; sensation supplies it with the original

stock derived from without, and reflection increases that stock, deriving other ideas by means of it from within.

7. A SIMPLE IDEA is that which cannot be divided into two or more ideas; thus the ideas of green, red, hard, soft, sweet, &c. are simple.

8. A COMPLEX IDEA is that which arises from joining two or more simple ideas together; thus the ideas of beer, wine, falsehood, a house, a square, are complex, being each made up of the ideas of the several ingredients or particulars which compose it, together with that of their manner of combination.

9. In receiving its impressions, the mind is wholly passive; it cannot create one new simple idea: those from without obtrude themselves on it by means of the senses, and those from within, which arise from the mind's contemplating the impressions it has already received, are equally spontaneous and (with respect to the mind) involuntary. But although the mind cannot create one original simple impression, yet when it is stored with a number of simple ideas, it possesses a wonderful power over them: it can combine several simple ideas together, so as to form a complex one, and vary the combinations at pleasure; it can compare its ideas, and readily determine in what particulars they agree, and in what they disagree. Having combined several simple ideas so as to form a complex one, the mind can again separate or resolve this complex idea into its component simple ones: this it can do both completely, and in part; it can retain just as many of the simple ideas in composition (out of the number which forms the entire complex one) as it chooses, and reject the rest; and if to this arbitrary combination a *name* be given, whenever we hear that name pronounced, the idea compounded of the whole of the parts prescribed, and no more, occurs immediately to the mind.

10. From the comparison of ideas arises what is called RELATION; and among other relations that which in mathematics is called RATIO, being a relation arising from the comparison of quantities in respect of their *magnitude* only.

11. In comparing several complex ideas together, we find, that although they differ with respect to some of the simple ideas of which they are compounded, yet they agree in some general character: thus, a triangle and a square differ with respect to

their form, the number of their sides, and the number and magnitude of their angles; but they agree in one general character, they are both *figures*. A lion and a sheep differ widely from each other in many particulars; but in their general character they agree, viz. they are both *animals*.

12. This most important power of the mind over its complex ideas is called ABSTRACTION, and the general idea produced by its operation is called AN ABSTRACT IDEA.

13. An abstract idea then comprehends in one general class, not only all the simple ideas, but all the complex ones from which it is abstracted: thus the idea of *beast* is a complex idea, and includes the ideas of lion, horse, bear, wolf, rabbit, &c. the idea of *animal* is likewise complex, including those of man, beast, bird, fish, insect, &c.

14. Hence an abstract or general idea is merely a creature of the mind, and can have no existing pattern or architype: we can form *in the mind* the abstract idea of a triangle, viz. one that shall include the ideas of all particular triangles; but we cannot describe *on paper* any figure capable of representing a triangle in general, viz. all the varieties of triangles that can be made.

15. Hence also whatever is true of an abstract idea is likewise true of every particular complex or simple idea included under it; thus, if it be proved generally that two sides of a triangle are together greater than the third, it follows that the same thing *is* thereby proved, and must be true of each and every individual triangle: in like manner whatever is proved of plane rectilineal figures in general, will necessarily be true (not only of every kind, but) of every particular rectilineal figure that can be made; thus, since it follows from prop. 32. book 1. of Euclid, that all the interior angles (taken together) of *every* rectilineal figure are equal to twice as many right angles, wanting four, as the figure has sides, the same thing must be true of each particular kind of such figure; as of squares, triangles, trapeziums, polygons, &c. and likewise of every particular figure included under those kinds.

16. Upon an examination of our ideas of the objects that surround us, we shall find that several of them resemble each other, except in one, two, or perhaps more circumstances;

now if we leave out from our consideration the particulars in which they disagree, and retain those only in which they agree, we shall obtain the abstract idea of a SPECIES, which, as it is supposed to arise from the lowest possible degree of abstraction, is called THE INFERIOR SPECIES; and the individuals which compose it, being supposed capable of no subordinate arrangement, are called PARTICULARS. If this idea of *species* be compared with our ideas of other species, we shall in like manner perceive that they disagree in some of their circumstances only; wherefore by leaving these out as before, we shall obtain the idea of a species superior to the former, viz. which includes the former, and one, two, or more others. In like manner by continual abstraction we pass through the successive gradations of species, until at length we arrive at a point where no further abstraction is possible: the ultimate idea thus obtained, as including the ideas of all the several species, is called a GENUS.

17. Thus by successive acts of abstraction, a guinea is gold, metal, substance, being; a herring is fish, animal, substance, being; Tray is greyhound, dog, beast, animal, substance, being; an oak is tree, vegetable, substance, being; James is scholar, man, animal, substance, being, &c. In the examples here proposed it may be observed, that *substance* is common to them all; the idea of substance includes therefore those of metal, animal, and vegetable, and consequently the subordinate ideas of guinea, herring, Tray, oak, and James. Substance then is to be considered as the PROXIMATE GENUS of these, including them all; BEING is the HIGHEST OF SUPERIOR GENUS, and implies merely existence.

18. As a general knowledge of the operations of the mind in compounding, comparing, and abstracting its ideas, is necessary to those who would fully understand the plan and scope of Euclid, so it will be equally profitable to shew, in as plain a manner as possible, how our abstract and other complex ideas are unfolded, so as to make them intelligible by words (expressed either by the voice or writing) to others.

19. And first, simple ideas are expressed by words arbitrarily assumed as their representatives; so that whenever any word is read or pronounced, the idea it stands for immediately occurs to the mind of the reader or hearer: but should it happen in any

instance otherwise, the object which produces the idea must be presented to him, and he must be informed that *such a word* is the sign of that idea; or should the idea have two or three different words to express it, these should all be pronounced, and probably the idea will occur to him from one of them: there is no other method of communicating a simple idea from one mind to another. I point a person to the object, I tell him its name, and immediately his mind associates the latter with the idea of the former, making the name the constant representative of the idea.

20. But although simple ideas cannot be conveyed to the mind by any verbal description, the case is different with respect to complex ideas; these may be communicated with great facility: for since a complex idea is composed of several simple ones, if the names of the latter be pronounced, together with their mode of connection, the complex idea will immediately occur to the hearer; provided his mind be previously furnished with its component simple ideas, together with a knowledge of the names or signs by which they are expressed.

21. It has been shewn, that if the *difference* between individuals, agreeing in their general and most remarkable properties and circumstances, (and which is called their NUMERAL DIFFERENCE,) be rejected, we obtain the abstract idea of a species; if the difference between this species and another species (called the SPECIFIC DIFFERENCE) be rejected, we get the idea of a species, which includes and is superior to the former; and if in like manner we continually drop the successive specific differences, we shall at length arrive at the genus, or summit of our research.

22. Hence an easy method presents itself of unfolding a complex idea, or of communicating our complex ideas to other persons by means of definitions, namely by following a contrary order; we name the genus or kind, to this name we join that of the specific difference, and both together will convey to the mind of the hearer the complex idea we mean to describe. Again, if we consider this species as a *genus*, and join to it the next lower *specific difference*, the result will give a precise idea of the next inferior species; proceeding in this manner through all the successive ranks of species to the lowest, to which joining the *numeral difference*; we at length obtain the idea of a particular

individual: this process is exemplified in the definitions prefixed to the Elements of Euclid.

23. It may be noticed, that in laying down a definition there is no necessity to have recourse to the *highest* genus, or even to *remote* species; the *proximate superior species* may in all cases be taken for the *genus*, and as that is always supposed to be known, we have only to add to its name that of the specific difference.

24. Thus, in defining a right angled triangle, I describe it to be a *triangle* having a *right angle*: *triangle* is the species or kind to which the figure belongs, and its having a *right angle* is the circumstance by which it differs from every other species of triangle. I do not say, "a right angled triangle is a *being*," or "a *figure*," or "a *plain figure*," these species are too remote; but I call it a "*triangle*," which is the proximate species to *right angled* triangle: now the idea of triangle being previously known, that of a right angled triangle will likewise be known by specifying that *it has a right angle*.

25. The obvious use of definitions is to fix our ideas, so that whenever a definition is repeated, the precise idea intended by it, and no other, may immediately occur to the mind; and whenever an idea is present to the mind, its definition may as readily occur.

26. Adequate and precise definitions may then be considered as the true foundation of every system of instruction; for when our ideas are fitly represented by words whose signification is fixed, there can be no danger of mistake either in communicating or receiving knowledge.

27. There are some ideas of which the mind perceives their agreement or disagreement *immediately*, without the necessity of argument or proof; this necessary determination of the mind is called a JUDGMENT, and the evidence or certainty with which it spontaneously acquiesces in this determination, is called INTUITION; also the irresistible force with which the mind is impelled to its determination, is called INTUITIVE EVIDENCE.

28. The faculty by which we perceive the validity of self-evident truth, is called COMMON SENSE [s], which signifies "that instinctive persuasion of truth which arises from *intuitive evi-*

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s See *An Essay on the nature and immutability of Truth, by James Beattie, LL. D.* p. 1. c. 1.

dence:" it is antecedent to science, and although no part of it, yet "it is the foundation of all reasoning."

29. There are some ideas of which the mind cannot perceive the agreement or disagreement, without the intervention of others, which the logicians call *middle terms;* the proper choice and management of these are the chief business of science.

30. These middle terms serve as a chain to connect two remote ideas, that is, to connect the subject of our inquiry with some self-evident truth: thus, suppose *A* and *D* to be two ideas, of which the truth of *A* is self-evident, but that of *D* not so; and let it be admitted that *A* and *D* cannot be brought together, so as to afford the requisite means of comparison for determining their relation; in this case I must seek for some intermediate ideas, the first of which is connected with *A*, the last with *D*, and the successive intervening ones with each other: let these be *B* and *C*; now if it be *intuitively* certain, that *B* agrees with *A*, that *C* agrees with *B*, and that *D* agrees with *C*, it follows with no less certainty that *D* agrees with *A*: this latter certainty is however not *intuitive*, but of the kind which is called *demonstrable* [h], and the process by which the mind becomes conscious of this demonstrable certainty is called REASONING, or DEMONSTRATION.

31. Every well ordered system of science will therefore consist of DEFINITIONS and PROPOSITIONS: *definitions* are used to explain distinctly the meaning of the terms employed, and to limit and fix our ideas respecting them with absolute precision. That which affirms or denies any thing, is called a *proposition:* I am; the sun shines; vice will inevitably be punished; two and three are five, &c. are propositions.

32. Propositions are either self-evident, or demonstrable; and since there can be no evidence superior to intuition, it follows that self-evident propositions not only *require* no proof, as some have said, they *admit* of none [i].

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[h] Every step of a demonstration must follow from truths *previously known* with *intuitive* certainty; but the conclusion or thing to be proved, depending on a connected series of intuitions, and no less certain than each of the preceding steps, is nevertheless not dignified with the name of *intuition;* it is obtained (as we have noticed above) by *demonstration.*

[i] For every proposition is proved by means of others which are more evident than itself, but nothing can be more evident than that which is *self-evident;* wherefore a self-evident proposition can admit of no proof.

33. Demonstrable propositions are such as do not admit of a determination by any single effort of the mind; to arrive at a consciousness of their truth, we are obliged frequently (as we have observed above) to have recourse to several intermediate steps, the first of which rests with intuitive certainty on some self-evident truth, the rest with the same intuitive certainty depend on each other in succession, and the proposition, or truth to be proved, depends with like intuitive certainty on the last of these; so that the thing to be proved must evidently be true, since it depends on a self-evident truth, which dependance is constituted and shewn by a series of truths following or flowing from each other with intuitive certainty.

34. Propositions are likewise divided into *practical* and *theoretical*. A practical proposition is that which proposes some operation, or is immediately directed to, and terminates in practice; thus, to draw a straight line, to describe a circle, to construct a triangle, &c. are *practical propositions*.

35. A theoretical proposition is that in which some truth is proposed for consideration, and which terminates in theory: thus, the whole is greater than its part; contentment is better than riches; two sides of a triangle are together greater than the third, &c. are *theoretical propositions*.

36. Propositions, both practical and theoretical, are either *self-evident* or *demonstrable*.

37. A *self-evident practical proposition* is named by Euclid a POSTULATE, and a *self-evident theoretical proposition*, an AXIOM.

38. A *demonstrable practical proposition* is called a PROBLEM, and a *demonstrable theoretical proposition*, a THEOREM.

39. Hence, postulates and axioms being intuitive truths or maxims of common sense, admit of no demonstration; but problems and theorems not being self-evident, therefore require to be demonstrated.

40. Definitions, postulates, and axioms, are the sole principles on which demonstration is founded: this foundation, narrow and slight as it may seem, is continually extended and strengthened by the constant accession of new materials; for every truth, as soon as it is demonstrated, becomes a principle of equal force and validity with truths which are self-evident, and reasoning may be built on it with the same degree of certainty as on them: thus reasoning, by its progress, continually in-

creases its basis, and the powers of the mind, ample as they are, must hence be inadequate to the use of all that vast accumulation and variety of means, provided for their employment.

41. When from the examination and comparison of two known truths a third follows as an evident consequence, the known truths are called PREMISES, the truth derived an INFERENCE; and the act of deriving it from the *premises* is called DRAWING, OR MAKING AN INFERENCE.

Thus, if *two and two* be equal to four, and *three and one* be equal to four, these being the *premises*, it follows as an *inference* that *two and two*, and *three and one*, are equal to the same (viz. to four): now, since *things that are equal to the same are equal to one another*, it follows as a further *inference*, that *two and two* are equal to *three and one*.

42. This example will furnish a general, although necessarily an imperfect, notion of Euclid's method of proving his propositions: his demonstrations are nothing more than a regular and well connected chain of successive intuitive inferences, the *first* of which is drawn from self-evident premises, and the *last* the thing which was proposed to be proved.

43. Hence, although demonstration is necessarily founded on self-evident truth, it is not at all necessary in every case that we should have recourse to first principles, for this would make demonstration a most unwieldy machine, requiring too much labour to be of extensive use: every inference fairly drawn from self-evident principles is of equal validity with intuitive truth, and may be employed for the same purposes; thus Euclid, in his demonstrations, makes use of the truths he has before demonstrated with a confidence as well founded as though they were self-evident, and merely refers you to the proposition where the truth in question is proved. This saves a great deal of trouble, for truths once established may with the strictest propriety be employed as principles for the proof and discovery of others.

44. It frequently happens in the course of a demonstration, that an inference presents itself, which is useful in other cases, although not immediately so with respect to the proposition under consideration; when such an inference is made, it is called a COROLLARY, and the act of making it DEDUCING A COROLLARY.

45. A LEMMA is a proposition not immediately connected with the subject in hand, but is assumed for the sake of shortening

the demonstration of one or more of the following propositions.

46. A SCHOLIUM is a note or observation, serving to confirm, explain, illustrate, or apply the subject to which it refers.

47. Euclid in his Elements employs two methods for establishing the truth of what he intends to prove; namely, *direct* and *indirect*, both proceeding by a series of inferences in the manner explained above. Art. 41, 42.

48. A DIRECT DEMONSTRATION is that which proceeds from *intuitive* or *demonstrated* truths, by a chain of successive inferences, the last of which is the thing to be proved.

49. AN INDIRECT OF APOLOGICAL DEMONSTRATION, or as it is frequently named, REDUCTIO AD ABSURDUM, consists in assuming as true a proposition which directly contradicts the one we mean to prove; and proceeding on this assumption by a train of reasoning in all respects like that employed in the *direct* method, we at length deduce an inference which contradicts some *self-evident* or *demonstrated* truth, and is therefore absurd and false; consequently the proposition assumed must be false, and therefore the proposition we intended to prove must by a necessary consequence be true, since two *contrary* propositions cannot be both true or both false at the same time [k].

NOTES AND OBSERVATIONS ON SOME PARTS OF THE FIRST BOOK OF EUCLID'S ELEMENTS.

50. The first book of Euclid's Elements contains the principles of all the following books; it demonstrates some of the most general properties of straight lines, angles, triangles, parallel lines, parallelograms, and other rectilineal figures, and likewise the possibility and method of drawing those lines, angles, and figures. It begins with *definitions*, wherein the technical terms necessarily made use of in this book are explained, and our ideas

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[k] Mathematical demonstrations "are nothing more than series of enthymemes; every thing is concluded by force of syllogism, only omitting the premises, which either occur of their own accord, or are recollected by means of quotations." This might easily be shewn by examples, but the necessary explanations, &c. would take up too much room. See on this subject *The Elements of Logic, by William Duncan, Professor of Philosophy in the Marischal College of Aberdeen*, 9th Ed. a book which ought to be recommended to the perusal of students in Geometry.

respecting them ascertained and fixed; next are laid down the *postulates* and *axioms*, or those *self-evident* truths, which constitute the basis of geometrical reasoning; and lastly, the *propositions* (whether *problems* or *theorems*) are given in the order of their connexion and dependance, the demonstrations of which depend solely on the definitions, postulates, and axioms, previously laid down; and from the demonstrations useful corollaries are occasionally derived.

On the Definitions.

51. DEFINITION I. The first definition, as given by Euclid, and likewise in Dr. Simson's translation, has been justly complained of as defective; it includes no positive property of a point; we learn from it not what a point is, but what it is not; "it has no parts, nor magnitude:" now since every adequate definition admits of conversion, let us try the experiment on this; when converted it will stand thus, "that which has no parts nor magnitude is a point;" but this is evidently untrue, for although a point be without extension, that which is without extension is not necessarily a point, it may be *nothing*.

It is therefore necessary to substitute another definition of a point, which shall include a *positive* property as well as the negative one above described; this will help the student over a difficulty, which (notwithstanding Dr. Simson's illustration in his note on this definition) might have discouraged him in his first attempt at Geometry. Instead then of Dr. Simson's definition and note, let the following be substituted:

52. *Def.* "A point is that which has position, but not magnitude [1]."

53. The idea of a point (as above defined) is evidently an abstract idea: a mathematical point cannot therefore be made on paper or exhibited to the eye; we may indeed represent it by a *dot*, but this dot, make it as small as you possibly can, will have length, breadth, and thickness too; still it may be used as a *mark* or *representation* of position or situation, shewing simply

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[1] This improvement was probably first suggested by Dr. Hooke, who says, that "a point has position, and a relation to magnitude, but has itself no magnitude;" his ideas on this subject have been adopted by both Playfair and Ingram.

to where, or from whence, lines are to be drawn, distances estimated, &c. A point then, as made on paper, is to be considered as a mark *indicating* merely *position*; this mark must necessarily have magnitude, but it is made the *representative* of that which has not.

54. *Def.* 2. "A line is length without breadth." The observations contained in the foregoing article may with equal propriety be applied to this definition. To *represent* a *mathematical* line, which is without breadth or thickness, (or rather to represent the idea of such a line,) we are obliged to have recourse to a line which has both. The line we draw on paper is not therefore the line we have defined, but merely the mark by which the idea of such a line is represented. The abstract idea of length (without breadth and thickness) is perfectly familiar to every one; thus, if it be asked, "what is the length (or distance) from hence to London?" the answer is, "thirteen miles:" this would, as we might suppose, be satisfactory; but should the inquirer farther ask, *how wide?* or *how thick?* every one would pity or despise him for his stupidity.

55. *Def.* 3. This is not properly a *definition*, but an *inference* from the two former, for "that which terminates a line can have no breadth, since the line in which it is has none; and it can have no length, for in that case it would not be a termination, but a part of that which is supposed to be terminate," and would consequently itself have terminations or extremities: whence the termination of a line can have no magnitude, and having necessarily position, it must therefore be a point, by Art. 53.

56. *Def.* 4. We have before remarked, (Art. 7, 19, 20.) that a simple idea admits of no definition; no definition can possibly be given of *straightness*; to lie "evenly between its extreme points" is a very awkward paraphrasis of the word *straight*, and will not perhaps be so well understood by a learner, as the definition would be were it to run thus, "a straight line is *that which is not crooked*."

57. Hence it follows, that "a straight line is the shortest distance between its extreme points;" this has been proposed instead of Euclid's definition by some, but it has been objected to by others. Professor Playfair has given the following, which is certainly an improvement, viz. "lines which cannot coincide

in two points, without coinciding altogether, are called straight lines;" but it may be added, that neither of the two latter definitions is sufficiently simple and perspicuous to stand at the beginning of a system of Elements.

58. All other lines besides straight lines are called *curve lines*, or simply *curves*; and hence we define curves to be "those lines which do not lie evenly between their extreme points," or "which are not the shortest distance between their extreme points."

59. *Def.* 5. We have shewn that the idea of length only (or of what the mathematicians call a line) is perfectly familiar to every one; the idea of a superficies (or of length and breadth *without thickness*) may be shewn to be equally so: in calculating the content of a field, it is well known that the *superficial* content is always understood, in which length and breadth *only* are concerned; thickness does not enter at all into the consideration.

60. Our ideas of a geometrical solid, superficies, line, and point, are obtained by abstraction. (See Art. 12—17.) Thus in contemplating any *material* body that first offers itself to our consideration, we shall find that besides being made up of matter, it has extension, or, length, breadth, and thickness; now, if from the complex idea of this body, we exclude the idea of matter, there will remain the abstract idea of extension, or of length, breadth, and thickness *only*, namely, of that which in geometrical language is called a *solid*. If from the complex idea of this solid we exclude the idea of thickness, we thence obtain the abstract idea of length and breadth *only*, or of a geometrical *superficies*. Again, if from the complex idea of a superficies we exclude the idea of breadth, the result will furnish us with the abstract idea of length *only*, or of a geometrical *line*. And lastly, if from the idea of line we exclude that of length, "we get the very abstract idea of a *point*: though I confess," says Mr. Ludlam, "the operation of the mind in this case is so very subtile, that it can hardly be distinctly and clearly traced out."

61. *Def.* 6. To this definition we may add, that if the extremities or boundaries of the superficies be straight lines, it is called a *rectilineal* superficies; if curves, it is called a *curvilineal* superficies; and if some of the boundaries are straight lines, and the rest curves, it is called a *mixtilineal* superficies.

62. The definition of a plane superficies, as originally given

by Euclid, is as follows: "A plane superficies is that which *lies evenly* between its extreme lines;" the term "lies evenly" has already been objected to as obscure. (Art. 56.) Dr. Simson, convinced of its impropriety, has substituted another definition, which has the advantage of including the essential property of a plane, and consequently of distinguishing it from every other kind of superficies: for besides a plane, there are various kinds of superficies, as the *spherical, cylindrical, conical,* and many others. According to this definition, a plane superficies "is that in which *any two* points being taken, the straight line between them lies wholly in that superficies;" the term "plane," in popular language, means that which is *perfectly flat,* or *level*; now if two points be taken in a superficies which is not perfectly flat, it is plain that the intermediate parts of the straight line, which joins those points, will fall either *above* or *below* the superficies; we see moreover not only the propriety, but the absolute necessity, of the distinction "*any* two points," for *two* points may be taken (in *one* particular direction) in the surface of a cone or cylinder, which will agree with the definition, but not *any* two points.

63. *Def.* 8. To give the learner an idea of what is here meant by the term "angle," or "inclination of two lines," it will not be amiss to have recourse to a familiar example: let a pair of compasses be opened to several different extents, these will be so many different angles; when the legs are opened to but a small distance, this opening, or (as it is here called) inclination of the legs, will be a small angle; when opened wider, the legs will form at their meeting a larger angle than before, and so on.

64. The two lines which form, or (as it is usually expressed) contain an angle, are sometimes called the *legs.* The magnitude of any angle does not at all depend on the length of the legs, or lines which contain it; in the example above proposed, the legs of the compasses may be an inch, a foot, or any other length, or one may be longer than the other, and yet the *opening, inclination,* or angle contained by them may still remain the same.

65. *Def.* 9.[m] The object of the eighth definition is to define

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[m] "The first nine definitions might have been given in the form of an introduction, for none of them are geometrical, except the seventh as amended by

in general every angle which can be described on a plane, whether such angle be contained by straight or curve lines; but since curvilineal angles are not treated of in the Elements, that definition might have been omitted. In the ninth, where "a plane rectilineal angle" is defined, the word "plane" is a redundancy; for the angular point, as well as every point in the lines which contain any rectilineal angle, must necessarily be all in one and the same plane, as is proved in the second proposition of the eleventh book. The note subjoined to Def. 9. in the Elements is merely to shew how we are to read, write, or to determine the place of an angle when it is read to us: if an angle be expressed by three letters, as is usual, the angular point is always understood to be at the middle letter; thus, if *ABC* denote an angle, this angle is *always* understood to be at the middle letter *B*, and not at either *A* or *C*.

66. *Def.* 10, 11, 12. When a straight line meets another straight line, (without crossing or *cutting* it,) two angles are formed at the point where they meet; if these angles be equal to each other, they are called *right angles;* but if one be greater than the other, the former (which is greater than a right angle) is called an *obtuse angle;* and the latter (which is less than a right angle, see *prop.* 13. *book* 1.) is called an *acute angle.*

67. *Def.* 13. In the sense of this definition, points are the boundaries of a line, lines of a superficies, and superficies of a solid.

68. *Def.* 14. Hence, according to Euclid, neither a line nor an angle can be called a figure, because they are not either of them "inclosed by one or more boundaries."

Dr. Simson;" this is Mr. Ingram's opinion, and he adds, "The terms by which a line and a superficies are defined, give some explanation of the meaning of these words, but give no geometrical criteria by which to know them; and the best way of acquiring proper ideas of them, is by considering their relation to a solid and to one another, as Dr. Simson has done." See on this subject the note on Def. 1, *Simson's Euclid*, 13th Ed. p. 289. A definition then may be said to be *geometrical*, when it furnishes some criterion to which we may refer, and by which the idea of the thing defined may be completely arrived at and obtained, at the result of any demonstration where it is concerned: other definitions are usually called *metaphysical;* they are employed in all cases where geometrical definitions cannot be given, as necessary for explaining in the best manner possible the nature of the things defined, the meaning of terms, &c.

69. *Def.* 15. We have here a complete and satisfactory instance of the method of defining a species by means of the genus and special difference. (Art. 23, 24.) "*A circle is a plane figure,*" it belongs to that class of figures, which have all their parts in the same plane, and consequently agrees in this general character with a triangle, a square, a polygon, an ellipsis, &c. it is "*contained by* one *line called the circumference;*" here we have a limitation whereby all such figures as are contained by more than one line, as the triangle, square, polygon, &c. are excluded; "*and is such that all straight lines drawn from a certain point within the figure*" (called in the next following definition "*the centre*") *to the circumference, are equal to one another:* this latter clause operates as an additional limitation, which excludes the ellipsis and all irregular curvilineal figures from the definition, because there is no point in either of those figures, from whence all the straight lines drawn to the circumference are equal. Here then we are informed, *first,* to what general class of figures a circle belongs, and *secondly,* by what it differs from every other figure of that class; whence the definition furnishes us with an adequate and precise idea of the figure called a circle.

70. Another definition, in substance the same as Euclid's, is this: "A circle is a figure generated (or formed) by a straight line revolving (or turning) in a plane about one of its extreme points, which remains fixed," the fixed point being the *centre,* and the line described by the revolving point the *circumference.*

71. The circumference of a circle is likewise called the *periphery:* it is sometimes improperly named *the circle;* a circle, in the proper acceptation of the term, means the space included within the circumference, and not the circumference exclusively.

72. To describe a circle with the compasses, you have only to fix one foot at the point where the centre is intended to be, and (the compasses being opened to a proper extent) turn the other foot quite round, and it will trace out the circumference.

73. After Def. 17. add the following, which is in continual use, viz. "a radius, or semidiameter of a circle, is a straight line drawn from the centre to the circumference."

74. *Def.* 18, 19. Any part of a circle cut off by a straight line, is called a *segment of a circle;* if the straight line pass through the centre, it is a diameter, (Def. 17.) and divides the

circle into two *equal* segments, called *semi-circles*: but if the straight line which cuts the circle does not pass through the centre, it will divide the circle into two *unequal* segments, the greater of which is said to be "*a segment greater than a semi-circle*," and the less "*a segment less than a semi-circle*." By the terms "segment of a circle," and "semi-circle," we are always to understand the *space* included between a part of the circumference and the straight line by which that part is cut off, unless the contrary be expressed.

75. Any part of the circumference is called *an arc*, and the straight line which joins the extremities of an arc, (or which divides the circle into two segments,) is called *a chord*, viz. it is the common chord of both the arcs into which it divides the whole circumference.

76. *Def.* 23. We have nothing to do professedly with polygons in the first book, yet since the definition is introduced, it may not be improper to observe, that a polygon, having all its sides and angles respectively equal, is called an *equilateral*, *equiangular*, or *regular* polygon. These figures are named according to their number of sides; thus,

A polygon having $\left\{\begin{array}{l}\text{five,}\\\text{six,}\\\text{seven,}\\\text{eight,}\\\text{\&c.}\end{array}\right\}$ sides, is called $\left\{\begin{array}{l}\text{a Pentagon,}\\\text{a Hexagon,}\\\text{a Heptagon,}\\\text{an Octagon,}\\\text{\&c.}\end{array}\right.$

77. *Def.* 24, 25, 26, 27, 28, *and* 29. Triangles are distinguished into three varieties with respect to their sides, and three with respect to their angles: the three varieties denominated from their sides, (as laid down in Def. 24, 25, and 26.) are *equilateral*, *isosceles*, and *scalene*; the latter, although defined here, does not occur under that name in any other part of the Elements. The three varieties which respect their angles, are *right-angled*, *obtuse-angled*, and *acute-angled*. Def. 27, 28, and 29.

78. *Def.* 30. A square, which according to this definition "has all its sides equal, and all its angles right angles," must evidently be just as wide as it is long; hence there can be no such thing as a *long* square, although we read of such a figure in some books [a].

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[a] Euclid's definition of a square may be considered as faulty, for with the essential properties of a square he has incorporated an *inference*, which is the

79. *Def.* 31. Since the word *oblong* does not once occur in any subsequent part of the Elements, it should not have found a place here. The figure defined is a species of that which is called in the second book, and elsewhere, a *rectangle.*

80. *Def.* 35. In the definition of parallel lines as here laid down, Dr. Simson has improved on Euclid, and his definition is better adapted to the learner's comprehension than either of those approved by Wolfius, Boscovich, Thomas Simpson, D'Alembert, or Newton; the truth is, that no inference can be drawn from any definition hitherto given, sufficient to fix the doctrine of parallel lines on the firm basis of unobjectionable evidence *.

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subject of the cor. to prop. 46. b. 1. It would be more strictly scientific to define a square to be "a four-sided figure having all its sides equal, and *one* of its angles a right angle;" for that "an equilateral four-sided figure is a parallelogram," and that "every parallelogram having one right angle has all its angles right angles," are plainly inferences from the definition given in this note, and that of a parallelogram, prop. 34. b. 1. the like observations extend to Def. 32. In both instances Euclid has abandoned his own plan, and transgressed a rule which ought never to be violated without absolute necessity; the departure is however justifiable in the present instance, as Euclid's definition will be more easily understood by a beginner than that which we have proposed.

* Having explained the definitions as they stand in Euclid, we may be allowed to remark, that a more methodical arrangement of them would be a desirable improvement; should any future Editor think this hint worth his attention and adopt it, it will be conducive to elegance, correctness, clearness, and simplicity, which are undoubtedly points of importance, especially at the beginning of the Elements. The alterations I would propose are as follow:

Def. 18. A segment of a circle is the figure contained by a straight line, and the circumference it cuts off.

19. If the straight line be a diameter, the segment is called a semi-circle.

From the 20th to the 29th inclusive, may stand as at present.

30. Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

31. A parallelogram is a four-sided figure, of which the opposite sides are parallel.

32. The diameter or *diagonal* of a parallelogram is a straight line which joins any two of its opposite angles.

33. A rhombus is a parallelogram which has all its sides equal, but its angles are not right angles.

On the Postulates.

81. A postulate, as we have before observed, is a self-evident practical proposition: on this subject Mr. Ludlam very justly remarks, that "Euclid does not here require a practical dexterity in the management of a ruler and pencil, but that the postulates are here set down that his readers may admit the *possibility* of what he may hereafter require to be done." On this we remark, that our conviction of the possibility of any operation depends on our having actually performed it in some particular instance ourselves, or known that it has been performed by others; having thus satisfied itself of the possibility in particular instances, the mind immediately perceives that the possibility extends to every instance, or that the operation is true in general. On these considerations it has been affirmed, that "the mathematical sciences are sciences of experiment and observation, founded solely upon the induction of particular facts, as much so as mechanics, astronomy, optics, or chemistry [p]." This doctrine, to its fullest extent, it would perhaps be unsafe to adopt.

82. In applying the postulates, we proceed in an order the converse of that laid down in the preceding article: we admit what is affirmed in the postulate to be *true in general*, i. e. in all cases; and since it is true in all cases, it follows as a necessary inference, that it is true in the particular case under consideration. We will now begin to exemplify the use of the mathe-

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34. A rhomboid is a parallelogram of which all its sides are not equal, nor any of its angles right angles.

35. A rectangle is a parallelogram which has all its angles right angles (or which has *one* of its angles a right angle; see the foregoing note.)

36. A square is a rectangle which has all its sides equal.

37. All other four-sided figures besides these are called trapeziums.

Note. A trapezium which has two of its sides parallel, is sometimes called a trapezoid, and a straight line joining the opposite angles of a trapezium is called its diagonal.

The definitions preceding the 18th might stand as they do at present, if instead of the first definition, that which we have proposed (see Art. 52.) were adopted.

p The postulates prefixed to the Elements are in number (as they ought to be) the fewest possible; for, as Sir Isaac Newton observes, "postulates are principles which Geometry borrows from the arts, and its excellence consists in the paucity of them." The postulates of Euclid are all problems derived from the mechanics. *Ingram.*

matical instruments, to afford the student an opportunity of practical as well as mental improvement.

83. *Postulate* 1. If it be granted, that "a straight line may be drawn from *any* one point to *any* other point," it follows as an evident consequence, that a straight line can be drawn from the point *A* to the point *B*. Lay a straight scale or ruler, so that its edge may touch the two proposed points *A* and *B*, then with a pen or pencil draw along the edge of the scale or ruler a line from *A* to *B*, and what was granted in general will in this particular instance be performed.

84. *Post.* 2. To *produce* a line means to lengthen it. A straight line of two inches in length, may according to this postulate be produced until it is three, four, five, or more inches in length. Lay the edge of your scale touching every point of the given line, and with the pencil or pen, as before, draw the line to the length proposed.

85. *Post.* 3. Extend the points of the compasses to the required distance, then with one foot fixed on the given point as a centre, let the other be turned completely round on the paper, and it will describe the circle required.

On the Axioms.

86. An axiom is a self-evident theoretical proposition, which neither admits of, nor requires proof. Axioms evidently depend in the first instance on particular observation, from whence the mind intuitively perceives their truth in general: like the postulates, these general truths being previously laid down and acknowledged, are applied to the proof of the demonstrable propositions which follow.

87. *Axioms* 1, 2, 3, 4, 5, 6, 7, 9, *and* 10, are too plain to require illustration; the 10th is what is usually called an identical proposition, amounting to no more than this, namely, that "all right angles are right angles."

88. *Ax.* 8. Should the learner feel disposed to hesitate at this axiom, he may be informed, that every one readily admits its truth in practical matters; a farmer who has two quantities of corn, each of which exactly fills his bushel, would be surprised if any one should deny that these two quantities are equal to each other.

89. The 12th axiom, as it is called, is not properly an axiom, but a proposition which requires proof; the learner, if he can-

not readily understand its import, may pass on until he has read the 28th proposition: it must then be resumed as necessary to the demonstration of the 29th.

On the Propositions.

90. The propositions in Euclid, we have before shewn, are either problems or theorems; the problems shew how to perform certain things proposed, and the theorems to establish and confirm proposed truths: both require demonstration, and the process is nearly the same in both; indeed problems may be changed into theorems, and theorems into problems, by a slight alteration in the wording. The demonstration of the first proposition depends solely on the definitions, postulates, and axioms; that of the second proposition on these and the first, and so on: the truths obtained by the proof of propositions being always employed, where necessary, in succeeding demonstrations.

91. Every geometrical proposition may be considered as comprehending three particulars, viz. the enunciation, the construction, and the demonstration. The enunciation declares in general terms what is intended to be done or proved. The construction teaches to draw the necessary lines, circles, &c. and applies the enunciation to the figure thus constructed. The demonstration is the system of reasoning which follows, whereby what was enunciated is clearly and fully made out and proved.

92. The numbers and letters in the margin are references to the proposition, axiom, postulate, or definition, where the particular cited in the corresponding part of the demonstration is to be found, or is proved; thus 1 post. means the first postulate; 15 def. the 15th definition; 3 ax. the third axiom; 2. 1. means the second proposition of the first book, &c. the first number always referring to the proposition, and the second to the book.

93. Before the student begins to learn the demonstration, he must be able to define *accurately* all the terms of science which occur in the proposition, and to repeat the postulates, axioms, enunciations, &c. referred to in the margin; next, the enunciation of the proposition must be well understood and learned by heart: all this will, in a very short time, become perfectly easy. The construction of the figure comes next: the figure should be made solely from the directions which immediately follow the enunciation; if this be thought difficult at first, the figure in

Euclid may be taken as a guide: every part of the figure may be drawn by hand, and the more accurately this is done, the better will it assist the recollection; the instruments *may* be employed for this purpose, but they are not *absolutely* necessary, as the truth of any proposition does not in the least depend on the accuracy of the construction: letters must be made at the angles and other prominent parts of the figure; these may (at first) be copied from the figure in Euclid. Lastly, in order to prepare the way for demonstrating the first proposition, as well as some of the following ones, in a complete and satisfactory manner, it will be necessary to premise the three following axioms:

94. *Axiom* 1. If a point be taken *nearer* the centre than the circumference is, that point is *within* the circle.

95. *Axiom* 2. If a point be taken *more distant* from the centre than the circumference is, that point is *without* the circle.

96. *Axiom* 3. If a point be taken *within* the circle, and another point *without* it, any line which joins these two points will cut the circumference.

97. Previous to attempting the first proposition, the student must be prepared (agreeably to what has been said in Art. 93.) to answer the following questions: viz. what is a proposition? (for the answer, see Art. 31.) what is a problem? (see Art. 38.) what is a point? (see Art. 52.) what is a line? (see Def. 2.) what is a straight line? (see Def. 4.) what is a triangle? (see Def. 21.) what is an equilateral triangle? (see Def. 24.) what is a circle? (see Def. 15.) what is the first postulate? what is the third postulate? what is Euclid's first axiom?—We will now shew how the first proposition ought to be demonstrated.

Enunciation.

98. PROPOSITION 1. PROBLEM. To describe an equilateral triangle upon a given [q] straight line. (*See the figure in Euclid.*)

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.

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[q] In Euclid it is "a given *finite* straight line;" here the word "finite" is superfluous, for whatever is *given* must of necessity be *finite*; a line is said to be "given," when another line equal to it can be actually drawn; (see Euclid's Data, def. 1.) but who can draw a line equal to an infinite line?

Construction.

From the centre *A*, at the distance *AB*, describe the circle *BCD*, *by the 3d postulate*[*]; and from the centre *B*, at the distance *BA*, describe the circle *ACE by the 3d postulate; these circles will cut one another, by Art.* 94, 95, 96; then from the point *C*, where they cut one another, draw the straight lines *CA*, *CB* to the points *A* and *B*, *by the 1st postulate*; *ABC* shall be an equilateral triangle.

Demonstration.

Because the point *A* is the centre of the circle *BCD*, *AC* is equal to *AB*, *by the 15th definition*; and because the point *B* is the centre of the circle *ACE*, *BC* is equal to *BA*, *by the 15th definition*: therefore *CA*, *CB* are each of them equal to *AB*; but things which are equal to the same are equal to one another, *by the 1st axiom; wherefore CA and CB are equal to one another, being each equal to AB*; consequently the three straight lines *CA*, *AB*, and *BC* are equal to one another, *and form a triangle ABC, by the 21st definition*, which is therefore equilateral, *by the 24th definition*, and it is described upon the given straight line *AB*, *because AB is one of its sides.* Which was required to be done.

99. With similar accuracy every proposition in the Elements ought to be demonstrated; the difficulty of acquiring a habit of strict and close reasoning would by this practice very soon be surmounted, and the powers of the mind gradually strengthened and enlarged.

100. *Prop.* 2. Having read over attentively the demonstration, it may perhaps be objected, that in drawing the straight line from *A*, we are confined by Euclid's figure to one particular direction *AL*; the proposition seems at first sight to be limited in this respect, but it is not so; for if from *A* as a centre, with the distance *AL*, a circle be described, straight lines may be drawn from the centre *A* to the circumference in *every* direction by the 1st postulate, and each of these lines will be equal to *AL* by the 15th definition.

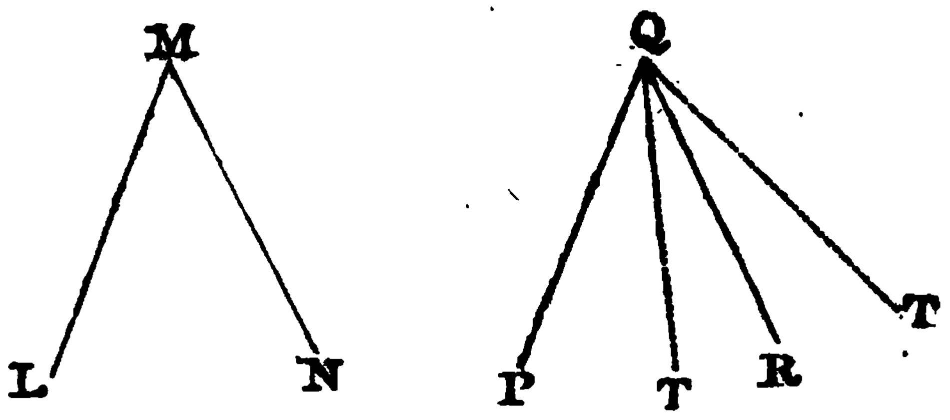
101. *Prop.* 2. and 3. have been objected to as sufficiently evi-

* The sentences in Italics are not in Euclid, but they are necessary, and should be supplied by the student as a proof that he understands his subject.

dent without proof; but it appears to have been the design of the ancient geometers to erect a complete system of science on as narrow a basis as possible: hence Euclid lays down self-evident principles which *admit* of no demonstration, and of these the fewest number possible that can be taken to effect his purpose; by means of which and the definitions he demonstrates *all such of his propositions as are susceptible of proof*, without regard to their being easy or difficult, or to the degree of evidence with which their truth may at first sight appear.

102. The third proposition being much less difficult than either the first or second, it may be asked, why was it not put first? The answer is, the proof of this proposition depends on the second, and that of the second depends on the first, and *successive dependence* is the only order that can possibly be attended to in any connected system of reasoning.

103. The following lemma should be understood before the fourth proposition is attempted. *Lemma.* Let *LMN*, *PQR* be two equal angles, and let them be applied to (laid upon) each other, so that the point *M* may coincide with the point *Q*, and the straight line *ML* with the straight line *QP*; then will *MN* fall upon *QR*. For if *LMN* be applied to *PQR* as



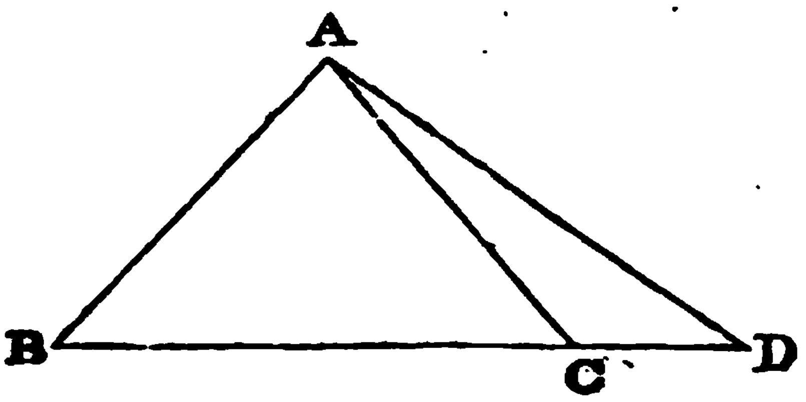
above, and *MN* do not fall upon *QR*, let it fall otherwise as *QT*, then the angle *LMN* becomes *PQT*; but *LMN* is by hypothesis equal to *PQR*, therefore the angles *PQT* and *PQR* are equal to each other, the greater equal to the less, which is absurd; therefore *MN* cannot fall otherwise than on *QR*, which was to be shewn.

104. This kind of proof, we have already observed, is what is called "reductio ad absurdum." The method of proving the equality of two figures by laying them one on the other, and shewing that their corresponding parts coincide, is called *supraposition*, and has been objected to, not from its want of evidence, but because it has been considered ungeometrical, as depending on no postulate: indeed we are no more bound to admit the

possibility of applying one figure to another, than we are to admit the possibility of joining two points, producing a straight line, or describing a circle: hence a postulate to that effect seems necessary *.

105. *Prop.* 4. This and the eighth are important propositions, as on them depends the whole doctrine of triangles; they are both proved by supraposition, which has been explained above. "It is worth while to remark," says Mr. Ludlam, "with what caution and accuracy all Euclid's propositions are worded. A careless writer might say, *if two triangles have two sides and an angle equal, then the third side of the one will be equal to the third side of the other, &c.* But Euclid cautions you not only that the sides must be equal *each to each*, but also that the angle spoken of must be that which is *contained by* the respectively equal sides. We will shew that two triangles may have (as was said) two sides respectively equal, and also one angle, yet neither their third sides nor the figures themselves will be equal."

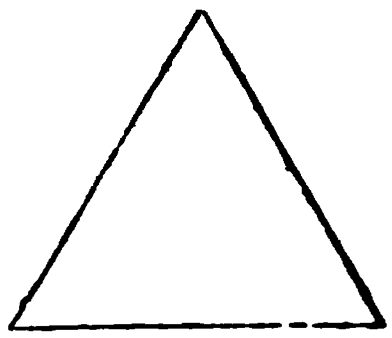
106. "Let *ABC* be an isosceles triangle, *A* the vertex, *BC* the base. Produce the base *BC* to *D*, and join *AD*; then we shall have two triangles formed, viz. *ABD* and *ACD*, having two sides and an angle respectively equal; that is, the side *AB* in the triangle *ABD*, equal to the side *AC* in the triangle *ACD*; also the side *AD* common to both triangles. The angle *ADC* is also common to both triangles; yet



* "Euclid," says Mr. Ingram, "*never* supposed any thing to be possible which he has not before shewn to be possible; this was not merely to avoid impossibilities, as some allege, but to secure evidence, and to make his reader as certain of his conclusions as he himself was." *Elem. of Euclid*, p. 281. It must be confessed this is Euclid's general rule, to which the instance in question is undoubtedly an exception, notwithstanding the great difficulty Mr. Playfair finds in admitting the fact: to avoid it, the learned Professor has shewn how the fourth and eighth propositions may be proved without the aid of supraposition; but the postulate he requires for that purpose cannot consistently with geometrical correctness be granted, because it is a demonstrable proposition. Compare his postulate (*Elem. of Geom.* p. 355.) with the 18th proposition of the 6th book of Euclid, and they will be found to be the same.

the third side *BD*, in the former triangle, is not equal to the third side *CD* in the latter; for *CD* by the construction is only a part of *BD*: nor are the figures *ABD* and *ACD* equal, for the former contains the latter, as appears from the figure [t]."

107. *Prop.* 5. *Cor.* Every equilateral triangle may be considered as isosceles. Let *ABC* be such a triangle; and since $AB = AC$, the angle $B =$ the angle C; and since $BA = BC$, the angle $A =$ the angle C, both by the proposition; wherefore, since $B = C$ and $A = C$, it follows (from axiom 1.) that $B = A$; wherefore the three angles A, B, and C are equal to each other, that is, the equilateral triangle *ABC* is also equiangular.



108. The enunciation of every theorem consists of two parts, viz. the SUBJECT and the PREDICATE. The *subject* is that of which something is affirmed or denied, and the *predicate* is that which is affirmed or denied of the subject: thus, in prop. 4. *two triangles having two sides of the one equal to two sides of the other, each to each, and the included angles equal,* is the subject; and that such triangles *will have their bases equal, their other angles equal, and be equal in all respects,* is the predicate. The *subject* of prop. 5. is, *an isosceles triangle*, and the *predicate* is, that *the angles at its base are equal to each other, and likewise the angles under the base.*

109. Two propositions are said to be the CONVERSE of each other, when the *subject* of one is made the *predicate* of the other, and the *subject* of the latter the *predicate* of the former. Propositions wherein the subject and predicate thus change places, are called CONVERSE PROPOSITIONS [u].

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[t] Ludlam's *Rudiments of Mathematics*, 5th Ed. p. 183, 184.

[u] Two converse propositions, although in most cases *both* true, are not in all cases so; one may be true, and the other false: thus, the proposition, "If two triangles have the three sides of the one respectively equal to the three sides of the other, the three angles of the one will be respectively equal to the three angles of the other," may be proved to be true; but its converse, viz. "If the three angles of one triangle be respectively equal to the three angles of another, then will the sides of the first triangle be respectively equal to those of the other," is not necessarily true; there may be a million triangles circum-

110. Prop. 6. is the converse of prop. 5. and its proof is by reductio ad absurdum; the words "the base *DC* is equal to the base *AB*, and" may be left out as unnecessary, and instead of "therefore *AB* is not unequal to *AC*, &c." it will be more proper to read, "therefore *DB* is not equal to *AC*; and in the same manner it may be proved, that no straight line, either greater or less than *AB*, can be equal to *AC*, wherefore *AB* is equal to *AC*, which was to be demonstrated."

111. The corollary to prop. 6. may be thus proved: (see the fig. to Art. 107.) because the angle B=the angle C, $\because$ the side AC=the side AB, (by the prop.) and because the angle A= the angle C, $\because$ the side BC=the side AB, $\therefore$ $AC=AB=BC$, which was to be shewn. This and the corollary to prop. 5. are the converse of each other.

112. *Prop.* 7. Many of the propositions in Euclid are merely *subsidiary*, that is, they are in themselves of no other use, than as necessary to the proof of other propositions that *are* useful; of this kind are prop. 7, 16, and 17, of the first book. The demonstration of this proposition is another instance of reductio ad absurdum; we here suppose an impossibility to be possible, in order to shew the absurdity of that supposition: a figure is here *made* to represent what no figure *can* represent, i. e. an impossibility; for we suppose not only that the lines *AC* and *AD* are equal to one another, but also that *CB* and *DB* are equal to one another, which the demonstration shews cannot be true, unless the points *C* and *D* coincide, and then the two triangles

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scribed about one another, which have their corresponding angles *all* equal to each other, but it is plain that the corresponding sides of *no two* of the triangles can possibly be equal, since one of these triangles always contains the other.

Converse and *contrary* propositions are not to be confounded, they are altogether different; the former we have explained above: two propositions are *contrary* to one another, when one affirms what the other denies, or denies what it affirms; thus, if it be affirmed that "two and three *are* five," the *contrary* proposition is, that two and three *are not* five. Again, "two straight lines *cannot* inclose a space," and "two straight lines *can* inclose a space," are *contrary* propositions. Two contrary propositions cannot be both true or false: thus, *A is* equal to *B*, and *A is not* equal to *B*, are contrary propositions; now it is evident, that if the former of these be true, the latter cannot; and if the latter be true, the former cannot; in the same manner it may be shewn that they cannot be both false.

will altogether coincide and form but one triangle. It is possible that AC and AD terminated at the extremity A *may* be equal, but then CB and DB terminated at the extremity B *cannot* be equal: in like manner CB and DB may be equal, but if they are, AC and AD cannot; and this is all that was required to be proved. The enunciation of prop. 7. which in the original is awkward and unintelligible, has been improved by Dr. Simson; he has likewise added the second case, which is not in the Greek text of Euclid, although it is found in the Arabic version; this case is demonstrated by means of the latter part of prop. 5. which is cited in no other part of the Elements.

113. *Prop.* 8. The 7th proposition is of no other use than as it serves to demonstrate this: we have here a second instance of a proof by supraposition; and since it is shewn that the triangles so applied completely coincide, it follows from the 8th axiom, that the triangles are equal; that the sides of the one are respectively equal to the sides of the other; and the angles of the one, to the angles of the other.

114. *Cor.* Hence, if the three sides of one triangle be respectively equal to the three sides of another, the two triangles will be both equal, and equiangular to each other [x].

115. *Prop.* 9. If the angles BAF, CAF be bisected, the whole angle BAC will be divided into *four* equal parts; and if each of these parts be bisected, the angle BAC will be divided into *eight* equal parts; again, if each of these parts be bisected, the whole angle BAC will be divided into *sixteen* equal parts, and so on. Hence by this proposition, an angle may be divided into *any number* of equal parts, provided that number be some power of the number 2.

116. *Cor.* Hence, if a straight line bisect an angle of an equilateral triangle, or if it bisect the angle included by the

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[x] The terms *equiangular* and *equiangular to each other*, must not be misunderstood or confounded; a figure is said to be *equiangular*, when all its angles are equal; and *two* figures are said to be *equiangular to each other*, when each of the angles in one figure is equal to its corresponding angle in the other, although neither of these figures may be equiangular in the former sense: a similar observation applies to the terms *equilateral* and *equilateral to each other*.

The converse of prop. 8. is not necessarily true, as is shewn in the note on Art. 109.

equal sides of an isosceles triangle, it shall likewise bisect the base. (See the fig. in Euclid.)

For $AC = BC$, and CD is common; also the angle $ACD =$ the angle BCD, therefore (prop. 4.) the base $AD =$ the base BD.

117. It has been shewn in prop. 9. and Art. 115. that any angle may be bisected geometrically, but the geometrical trisection of an angle (except in one particular case, see the note on Art. 140.) still remains among the desiderata in science; no method having yet been discovered whereby any section, except the bisection, can be performed by the Elements of Geometry [y].

118. *Prop.* 10. The word "finite," as used in this place, is redundant. See the note on Art. 98. The method of bisecting a given straight line with instruments will be shewn hereafter.

119. *Prop.* 11. Drawing a straight line perpendicular to a given straight line from a given point *in* the latter, is called "erecting a perpendicular."

120. From the corollary to this proposition it appears, that two straight lines can meet one another in only one point; for if they meet each other in two points A and B, (see the figure in Euclid,) the parts intercepted between A and B must either coincide or inclose a space; but they cannot coincide, otherwise the two straight lines would have a common segment, which by

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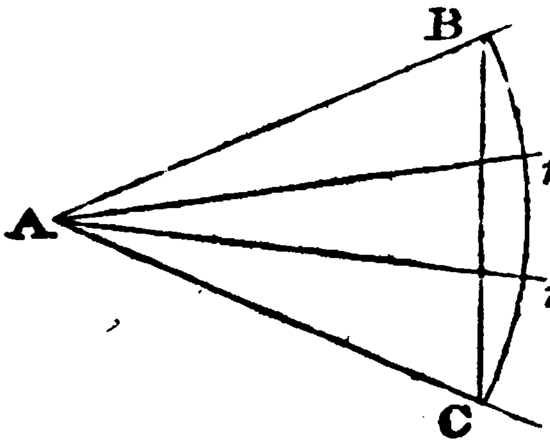
[y] Any angle may be trisected *algebraically* as follows:

From the angular point A as a centre, with *unity* for radius, describe the arc BC, draw the chord $BC = c$, and let $x =$ the chord of Br, one third the arc BC; then will $x^3 - 3x = -c$, which solved by Cardan's rule, gives

$$x = \sqrt[3]{-\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}cc - 1}} - \frac{-1}{\sqrt[3]{-\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}cc - 1}}};$$

if this answer be turned into a number, (by restoring the value of c, &c.) and chords be drawn from B and C to the points r and n, and Ar An be joined, these lines will trisect the given angle BAC, as was required.



[z] Several methods of trisecting an angle may be found in the works of those who have written of the higher Geometry, as Pappus, Vieta, Guisnée, L'Hôpital, Simpson, Maclaurin, Emerson, D'Omerique, Waring, &c.

the corollary is impossible; neither can they inclose a space, (axiom 10.) therefore they cannot meet each other in more than one point.

121. *Prop.* 12. Drawing a perpendicular to a given straight line, from a given point *without* it, is called "letting fall a perpendicular." We are told in the proposition to "take any point D upon the *other side* of AB;" by "other side," we are to understand the side opposite to that on which C stands.

122. *Prop.* 13. Learners are generally perplexed with demonstrations of which they cannot previously understand something of the plan and scope, and with none more frequently than that of prop. 13. Let such as find it difficult observe, *first*, that CBE, EBD are *by construction* two right angles; *secondly*, that the three angles CBA, ABE, EBD, are equal to the above two, consequently to two right angles; and *thirdly*, that the two given angles DBA, ABC are equal to the last-mentioned three, consequently to the fore-mentioned two, and consequently to two right angles, which was proposed to be proved.

123. *Cor.* Hence, if the angles ABD, ABC be unequal, the greater is obtuse, and the less acute; the former being as much greater than a right angle, as the latter is less, as is evident from the proposition.

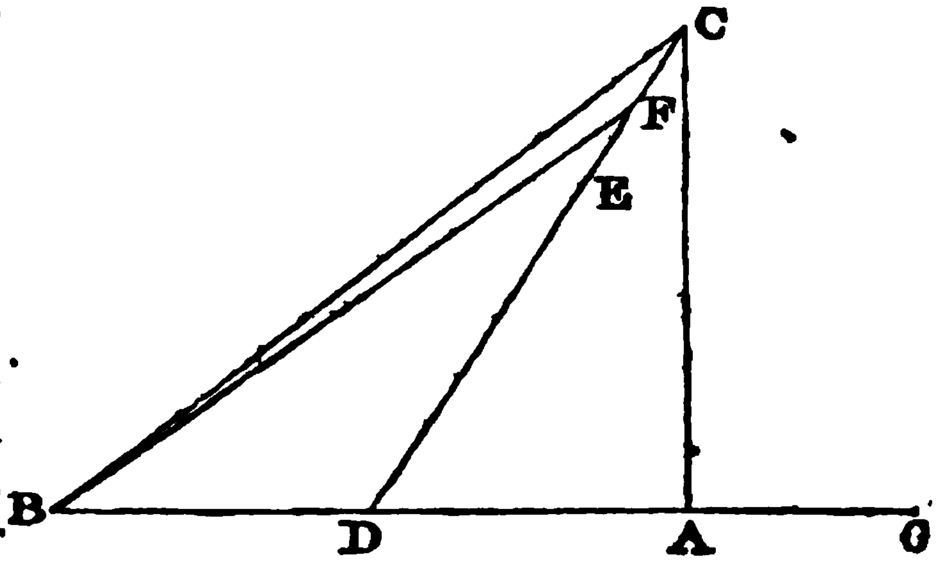
124. The 13th and 14th, the 18th and 19th, and the 24th and 25th, are converse propositions; the 29th is the converse of the 27th and 28th, and the 48th of the 47th.

125. The following is not completely the converse of prop. 15, but it is partly so. If two straight lines AE, EB, (see Euclid's fig. pr. 15.) on the opposite sides of CD, meet CD in any point E, so as to make the vertical angles AEC, DEB equal, then will AE and EB be in the same straight line. For the four angles at E being equal to four right angles by cor. 2, and the two CEA, AED = the two DEB, BEC, each of these equals will be the half of four right angles, that is, equal to two right angles; whence (prop. 14.) AE and EB are in the same straight line.

126. *Prop.* 20. Dr. Simson remarks, (from Proclus,) that "the Epicureans derided this proposition as being manifest to asses;" some of the moderns have done the same, but equally without reason: according to Euclid's plan, a demonstration was necessary, as will appear by referring to Art. 101.

127. *Prop.* 21. "It is essential to the truth of this proposition, that the straight lines drawn to the point within the triangle, be drawn from *the two extremities* of the base;" omitting this limitation, there are cases in which the sum of the two lines drawn from the base to a point within the triangle, will *exceed* the sum of the two sides of the triangle, which may be shewn as follows:

Let ABC be a triangle, right angled at A, D any point in AB, let CD be joined, and BA produced to G; then since CAD is a right angle, CAG is also a right angle, (prop. 13.) but CAG is greater than CDA, (prop. 16.) ∴ CAD is likewise greater than CDA, ∴ CD is greater than CA, (prop. 19.) From CD cut off $DE = AC$, (prop. 3.) bisect CE in F, (prop. 10.) and join BF; then will the sum of the two straight lines BF and FD be greater than the sum of BC and CA, the sides of the triangle.



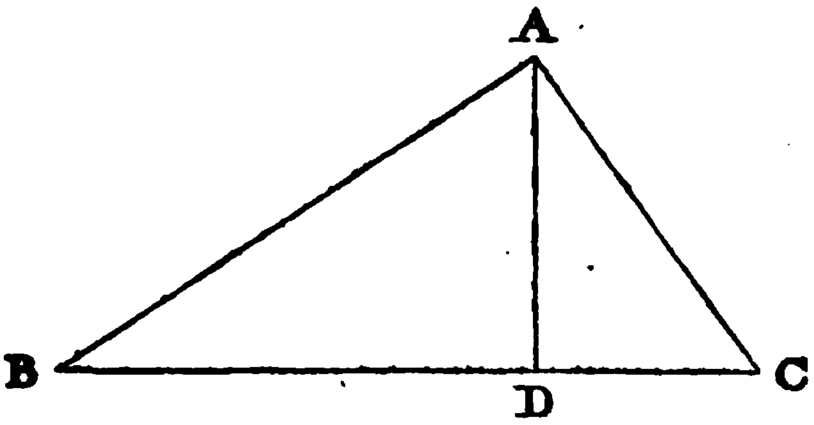
Because $CF = FE$ by construction, ∴ $CF + FB = EF + FB$, but $CF + FB > BC$, (prop. 17.) ∴ $EF + FB > BC$; to these unequals, let there be added the equals . . . $ED = AC$

and we shall have (by axiom 4.) . . . . $EF + FB + ED > BC + AC$, but $EF + ED = FD$ ∴ $BF + FD > BC + AC$. Q. E. D. and the same may be proved if the angle CAB be obtuse.

128. *Prop.* 22. To invalidate the force of an objection which has been made to the demonstration of this proposition, it will be necessary to prove that the two circles (see Simson's figure) *must* cut each other: thus, because any two of the straight lines DF, FG, GH, are together greater than the third (by hypothesis), ∴ $FD < (FG + GH$, or) FH, ∴ the circle DKL must meet the line FE somewhere between F and H, (see Art. 95.) for the like reason, the circle KHL *must* meet DG between D and G; consequently these circumferences will pass both *without* and *within* each other, and therefore must cut each other. See Art. 96.

129. *Prop.* 26. It must be observed, that the two *equal* sides (viz. one in each triangle) must be alike situated in the triangles; both must be either *between* the given angles, or opposite *equal* angles, otherwise the triangles will not necessarily be equal.

Let *ABC* be a triangle, right angled at *A*, from whence let *AD* be drawn perpendicular to the base *BC*, (12. 1.) this will divide the triangle into two others, *ADB* and *ADC*, having a right angle in each, (viz. at *D*,) and the angles *ABD*, *CAD* equal [a], and also the side *AD* common; these triangles therefore have two angles of the one equal to two angles of the other, each to each, but the *common* side *AD* not lying either *between* given, or *opposite* equal angles, the triangles are therefore not necessarily equal.



129. *Prop.* 29. We have before remarked, that this proposition is the converse of the 27th and 28th. It has given the geometers of both ancient and modern times more trouble than all the rest of Euclid's propositions put together: to demonstrate it the 12th axiom was assumed; but this axiom is by no means self-evident, and therefore the 29th, which depends on it, cannot be said to be proved, unless the axiom itself be previously proved, which cannot easily be done, but by introducing an axiom scarcely less exceptionable than that which was to be demonstrated. "This defect in Euclid," says an ingenuous commentator, " is therefore abundantly evident, but the manner of correcting it is by no means obvious;" the methods chiefly employed for that purpose are the following three; 1. "A new definition of parallel lines:" 2. " A new manner of reasoning on the properties of straight lines without a new axiom:" and 3. " The introduction of a new axiom less objectionable than Eu-

[a] See the 8th prop. b. 6. also Ludlam's Rudiments, p. 186.

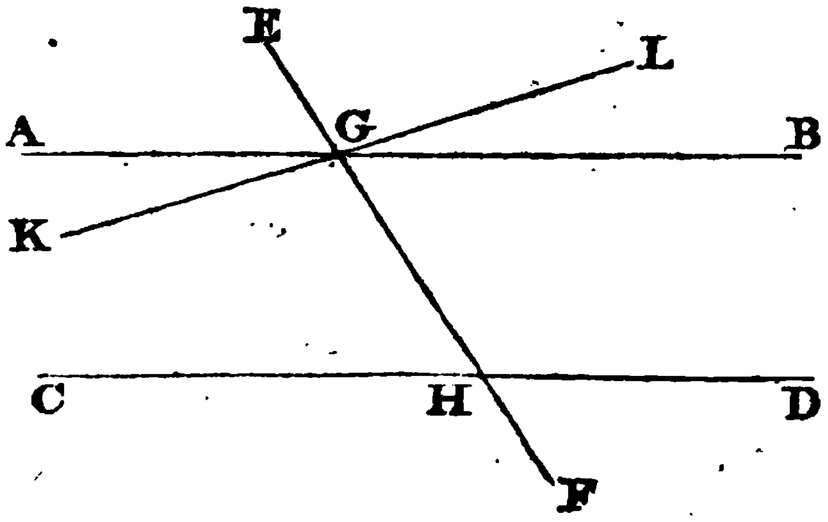
Where two numbers are placed, as (12. 1.) in the above article, the first number refers to the proposition, and the second to the book in Euclid; also if no figure be mentioned, that belonging to the proposition in Euclid which is under consideration, is always meant.

clid's 12th *." Omitting the two former methods, we shall avail ourselves of the latter, by introducing an axiom which Euclid himself seems to have tacitly admitted, (see prop. 35, 36, 37, and 38, book 1.) although he has not formally proposed it. The axiom is as follows:

130. *Axiom.* If two straight lines be drawn through *the same* point, they are not *both* parallel to the same straight line.

By the help of this axiom (if it be admitted as such) we may demonstrate the 29th proposition in the following manner, without the aid of Euclid's 12th axiom.

131. If *AGH* be not equal to *GHD*, one of them must be greater than the other; let *AGH* be the greater, and at the point *G* in the straight line *GH* make the angle *KGH* = *GHD*, (23. 1.) and produce *KG* to *L*; then will *KL* be parallel to *CD*, (27. 1.) ∵ two straight lines passing through the same point *G* are both parallel to *CD*, which by our axiom is impossible. The angles *AGH* and *GHD* are therefore not unequal, that is, they are equal. The latter part of the demonstration may proceed as in Simson, beginning at the words, *but the angle AGH is equal to the angle EGB*, &c.



132. *Cor.* Hence, if two straight lines *KL* and *CD* make

* Boscovich, Thomas Simpson, Bezout, Wolfius, D'Alembert, Sturmius, Varignon, and several others, are for adopting a new definition of parallel lines; Ptolemy, Franceschinis, &c. have endeavoured to demonstrate the properties of parallel lines without the help of either a new definition or a new axiom, but have failed: Professor Playfair introduces the axiom we have adopted above, which on the whole seems to be the best, and preferable in several respects to Euclid's. Clavius has bestowed greater attention on the subject than any modern geometer: whether he considered his demonstration as founded on a new axiom or not, it is not quite certain, but it appears that his reasoning depends on a proposition which ought not to be admitted as self-evident. A further elucidation of this subject may be found in the notes on the 29th prop. in *Simson's Euclid*, *Ingram's Euclid*, *Playfair's Elements of Geometry*, *Simpson's Elements of Geometry*, &c.

with another straight line EF the angles KGH, GHC together less than two right angles, KL and CD will meet towards K and C, or on that side of EF on which are the angles which are less than two right angles.

For if not, KL and CD are either parallel, or meet towards L and D; but they are not parallel, for if they were, the angles KGH, GHC would be equal to two right angles (by prop. 29.) which they are not: neither do KL and CD meet towards L and D, for if they did, the angles LGH, GHD, being in that case two angles of a triangle, (17. 1.) would be less than two right angles; but this is impossible, for the four angles KGH, LGH, CHG, DHG, are together equal to four right angles, (13. 1.) of which the two KGH, CHG are by hypothesis less than two right angles; therefore the remaining two LGH, DHG are greater than two right angles. Therefore, since KL and CD are in the same plane and not parallel, they must meet somewhere; but it has been shewn that they cannot meet towards L and D, wherefore they must meet towards K and C, or on that side of EF on which are the angles KGH, GHC, which are together less than two right angles. Q. E. D. Thus, by the assistance of our axiom, we have demonstrated Euclid's 12th, which is neither self-evident, nor easily understood by a beginner.

133. *Prop.* 32. This proposition, which is ascribed to Pythagoras, is one of the most useful in the whole Elements, as will be evident in some sort from the following corollaries derived immediately from it, viz.

134. *Cor.* 1. The exterior angle is the difference between the interior and adjacent angle and two right angles, and each of the interior angles is equal to the difference between the two remaining interior angles and two right angles.

Thus, let R represent a right angle, A, B, and C the interior angles of the triangle: (see Euclid's figure:) *then will the exterior angle $ACD=2R-C$, also $A=2R-B-C$, $B=2R-A-C$, and $C=2R-A-B$.*

135. *Cor.* 2. The difference between the exterior angle and either of the two interior opposite angles, is equal to the other interior opposite angle.

Thus, $ACD-A=B$, and $ACD-B=A$.

136. *Cor.* 3. If one angle of a triangle be a right angle, the

other two angles taken together make a right angle, consequently each of them is acute: these acute angles are called *complements* of one another to a right angle.

Thus, if C be a right angle, then will A be the complement of B, and B the complement of A.

137. *Cor.* 4. If one angle be obtuse, the remaining two will be together less than a right angle, and consequently both acute.

138. *Cor.* 5. If the sum of two angles in one triangle be equal to the sum of two angles in another, the remaining angle in the one will be equal to the remaining angle in the other; and if one angle in one triangle be equal to one angle in another, the sum of the two remaining angles in the former will be equal to the sum of the two remaining angles in the latter.

139. *Cor.* 6. If one angle at the base of an isosceles triangle be equal to one angle at the base of another isosceles triangle, the two remaining angles in the one will be equal to the two remaining angles in the other, each to each; and if the vertical angle of one isosceles triangle be equal to the vertical angle of another, each of the angles at the base of the one will be equal to each of the angles at the base of the other.

140. *Cor.* 7. Each angle of an equilateral triangle is one-third of two right angles, or two-thirds of one right angle [b].

141. *Cor.* 8. "All the interior angles," &c. as Cor. 1. in Simson.

142. *Cor.* 9. All the interior angles of any rectilineal figure, are equal to twice as many right angles, except four, as the figure has sides.

Thus, let n=the number of sides, S=the sum of the interior angles in any rectilineal figure, then will

Cor. 8. *stand thus,* $S+4R=2n.R.$

and Cor. 9. *thus,* $S=\overline{2n-4}.R.$

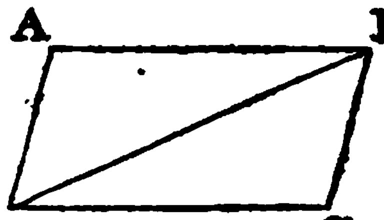
[b] Hence, if the angle of an equilateral triangle be bisected, (9. 1.) each of the parts will be one-third of a right angle, which is the only angle that can be geometrically trisected.

143. *Cor*. 10. **Hence, the interior angles of the following rectilineal figures will be as below: if the figure have**

Three	sides, the sum of its interior angles will=	$6-4=2$	right angles.
Four		$8-4=4$	
Five		$10-4=6$	
Six		$12-4=8$	
Seven		$14-4=10$	
Eight		$16-4=12$	
Nine		$18-4=14$	
Ten		$20-4=16$	
Eleven		$22-4=18$	
Twelve		$24-4=20$	

144. The converse of the former part of prop. 34. is as follows: "If the opposite sides of a quadrilateral figure be equal, the figure will be a parallelogram."

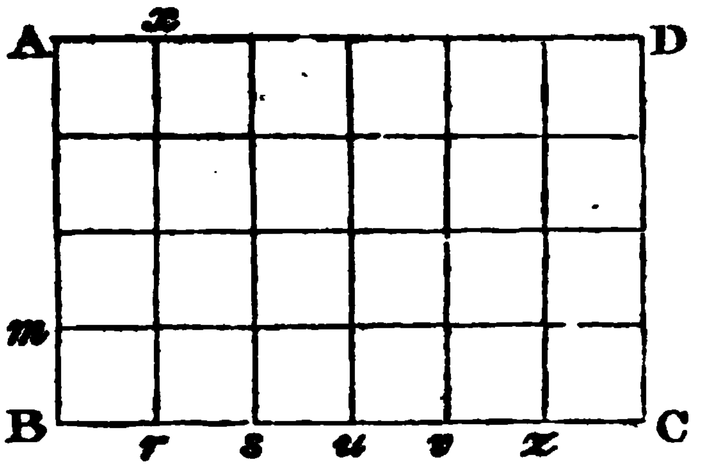
Let *ABCD* be a quadrilateral figure, having its opposite sides equal, viz. $AD=BC$, and $AB=DC$, then will *AD* be parallel to *BC*, and *AB* to *DC*. Join *BD*, then because $AD=BC$, and $AB=DC$, also *BD* common, ∵ the angle $ADB=$ the angle *DBC*, and $ABD=BDC$, (8. 1. and Art. 113.) ∵ *AD* is parallel to *BC*, and *AB* to *DC* (27. 1.) ∵ *ABCD* is a parallelogram, according to the definition, prop. 34.



145. *Cor*. Hence, if the opposite sides of a quadrilateral figure be equal, its opposite angles will likewise be equal by prop. 34.

146. The converse of the second part of prop. 34. is this: "If the opposite angles of a quadrilateral figure be equal, the figure will be a parallelogram." Let the angle $BAD=BCD$, (see the above figure,) and $ADC=ABC$; and since these four angles are the interior angles of a quadrilateral figure, they are together equal to four right angles; (by Art. 143.) let now the above equals be added and the wholes will be equal, (Ax. 2.) that is, $BAD+ADC=BCD+ABC$, ∵ the former two angles, as well as the latter two, will be (half of four right angles, or) two right angles, ∵ (by prop. 29.) *AD* is parallel to *BC*, and *AB* to *DC*; that is, *ABCD* is a parallelogram.

146. In the right angled parallelogram *ABCD*, if the side *AB* be supposed to move along the line *BC*, and perpendicular



to it, when AB arrives at C, it will coincide with DC, and by its motion it will have described or generated the parallelogram $ABCD$; let AB consist of suppose 4 equal parts, each of which we will call unity, (or 1.) let Bm=one of those parts, and Br, rs, su, &c. each$=Bm$; now it is plain, that when AB arrives at r, it will by its motion have described the four rectangles between AB and xr, each of which will be the square of (Bm, that is of) unity; in like manner, when AB arrives at s, u, v, z, C, it will have described 8, 12, 16, 20, 24 squares of (Bm, or) unity: whence it appears, that the area $ABCD$ or 24, is found by multiplying[c] the number of equal parts (called units) contained in AB, or 4, by the number of like parts in BC, or 6. In like manner, if AB contain n units, and BC m units, the area $ABCD$ will contain $n\times m=nm$ units: if $n=m$, the figure $ABCD$ will be a square, and nm will become n^2 or m^2. Hence the area of a rectangle is found by multiplying the two sides about one of its angles into each other, and the area of a square by multiplying the side into itself.

147. *Prop.* 35. From this proposition, and the preceding article, we derive a method of finding the area of any parallelogram whatever: for let $ABCD$ (see Simson's first figure) be supposed to be a right angled parallelogram, its area will be $AB\times BC$, (by Art. 146.) or the perpendicular AB, drawn into (or multiplied by) the base BC; but $DBCF=ABCD$ by the proposition, $\because$ $DBCF$=perp. $AB\times$ base BC.

148. Hence we have the following practical rule for finding

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[c] The terms *multiplying* and *dividing* do not occur in geometrical language; thus, in the expression $AB\times BC=ABCD$, AB is said to be *drawn into* BC, and $ABCD$ is not called the *product* of AB and BC, but their *rectangle*; and in expressions like the following $\frac{AB}{C}$, AB is not said to be *divided by* C, but C is said to be *applied to* AB. The old writers are very particular in this respect, but the moderns are less so, as we frequently find arithmetical terms made use of in their geometrical problems; but this abuse should as much as possible be avoided.

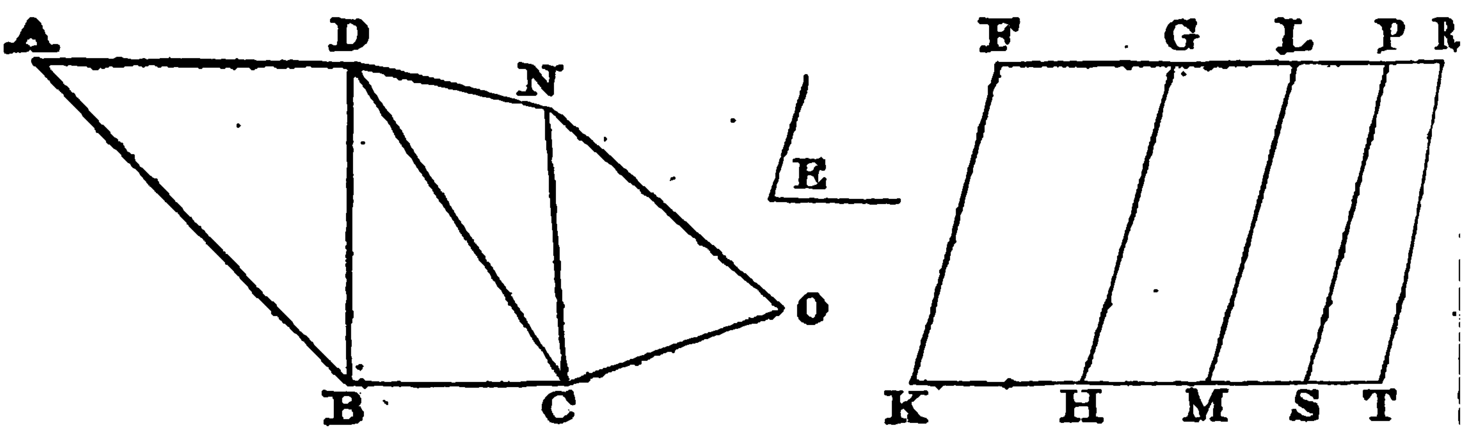
the area of a parallelogram. 1. Let fall a perpendicular on the base from any point in the opposite side. 2. Multiply the base and perpendicular together, and the product will be the area required.

149. *Prop.* 37. Since every triangle is half of the parallelogram described upon the same base, and between the same parallels, (see also prop. 41.) and the area of the parallelogram is $= perp. \times base$, (by the last article,) $\because$ the area of the triangle will be $\frac{perp. \times base}{2}$; that is, *half the perpendicular multiplied into the base, or half the base multiplied into the perpendicular,* will give the area of the triangle.

150. *Prop.* 38. *Cor.* Hence, if the base *BC* be greater than the base *EF*, the triangle *ABC* will be greater than the triangle *EDF*; and if *BC* be less than *EF*, the triangle *ABC* will be less than the triangle *EDF*. Also, if *ABC* be greater than *EDF*, then is *BC* greater than *EF*; and if less, less.

151. In prop. 42. we are taught how "to describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle." In prop. 44. we are to describe a parallelogram with the two former conditions, and also one more: we are "to apply a parallelogram *to a given straight line,* which parallelogram shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle;" to "apply a parallelogram to a straight line," means to make it on that straight line, or so that the said line may be one of its sides.

152. *Prop* 45. The enunciation of this proposition is *general,* if by "a given rectilineal figure" we are to understand "*any* given rectilineal figure:" but the demonstration applies to only a *particular case;* for it extends no further than to four-sided figures, and does not even hint at any thing beyond; but the defect is easily supplied as follows:



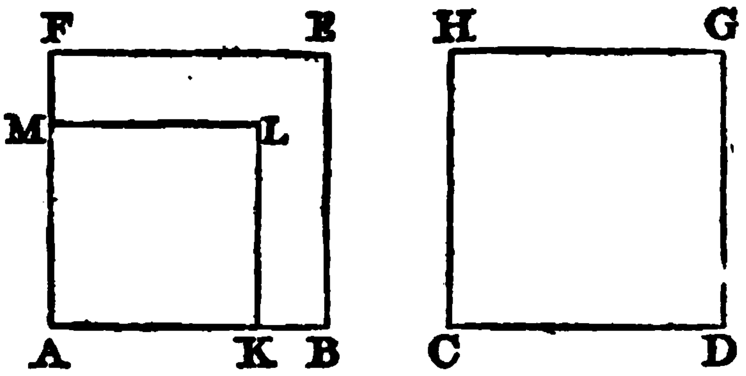
Let *ABCOND* be any rectilineal figure; join *DB*, *DC*, *CN*, then having made the parallelogram *FKML* equal to the quadrilateral figure *ABCD*, as in the proposition, apply the parallelogram *LS*=*DCN* to the straight line *LM*, having an angle *LMS*=*E*, then it may be proved as before, that *FL* and *LP* are in the same straight line as are *KM* and *MS*; also that *PS* is parallel to *FK*, and consequently that *FKSP* is a parallelogram and equal to *ABCND*; and applying as before a parallelogram *PT*=*NCO*, having the angle *PST*=*E*, to the straight line *PS*, *FKTR* may in like manner be proved to be a parallelogram equal to *ABCOND*, and having an angle *FKT*=*E*; and by a similar process a parallelogram may be made equal to any given rectilineal figure whatever, and having an angle equal to any given rectilineal angle. The foregoing illustration being understood, the corollary to this proposition will be evident.

Cor. Hence we have a method of determining the difference of any two rectilineal figures. Thus *ABCOND* exceeds *DCON* by the parallelogram *FM*.

153. *Prop*. 46. *Cor*. In a similar manner the rectangle contained by any two given straight lines may be described.

154. The squares of equal straight lines are equal to one another.

Let the straight lines *AB* and *CD* be equal, then will the squares *ABEF*, *CDGH* described on them be equal. For since *AB*=*CD* by hypothesis, and *HC*=*CD* (Def. 30.) ∵ *HC*=*AB*, but *FA*=*AB*, (Def. 30.) ∵ *HC*=*FA*;



wherefore if the square *FB* be applied to the square *HD*, so that *A* may be on *C*, and *AB* on *CD*, *B* shall coincide with *D*

because $AB=CD$; and AB coinciding with CD, AF shall coincide with CH because the angle $BAF=DBH$, (Def. 30. and Ax. 11.) also A coinciding with C, and AF with CH, the point F shall coincide with H, because $AF=CH$; in the same manner it may be shewn, that FE and EB coincide respectively with HG and GD, therefore the two figures coincide, and consequently are equal by Ax. 8. Q. E. D.

Cor. 1. Hence two squares cannot be described on the same straight line and on the same side of it.

Cor. 2. Hence two rectangles which are equilateral to one another will likewise be equal.

155. If two squares be equal, the straight lines on which they stand will also be equal.

Let $ABEF=CDGH$, (see the preceding figure) then will $AB=CD$; for if not, let AB be the greater, and from it cut off $AK=CD$ (3.1) and on AK describe the square $AKLM$, (46.1) then since $AK=CD$, the square $AL=$the square CG, (Art. 154.) but $AE=CG$ by hypothesis, ∴ $AL=AE$ the greater to the less which is impossible, ∴ AK is not equal to CD, and in like manner it may be shewn that no straight line, either greater or less than AB, can be equal to CD, ∴ $AB=CD$. Q. E. D.

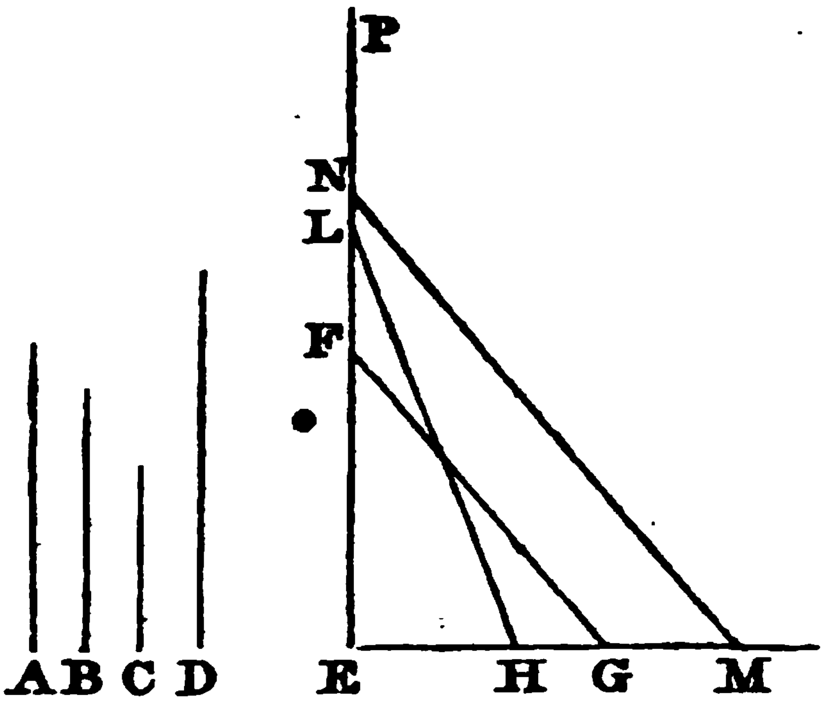
156. *Prop.* 47. This proposition, which is known by the name of the *Pythagorean Theorem*, because the philosopher Pythagoras was the inventor of it, is of very extensive application; its primary and obvious use is to find the sum and difference of given squares, the sides of right angled triangles, &c. as is shewn in the following articles [d].

157. To find a square equal to the sum of any number of given squares. Let A, B, C, D, &c. be any number of given straight lines; it is required to find a square equal to the sum of the squares described on A, B, C, D, &c.

Take any straight line EM, and from any point E in it draw EP perpendicular to EM (11.1); take $EF=A$, $EG=B$

[d] This proposition has been proved in a variety of ways by Ozanam, Tacquet, Sturmius, Ludlam, Mole, and others; it supplies the foundation for computing the tables of sines, tangents, &c. on which the practice of Trigonometry chiefly depends, and was considered by Pythagoras of such prime importance, that (as we are told) he offered a hecatomb, or sacrifice of 100 oxen, to the gods for inspiring him with the discovery of so remarkable and useful a property.

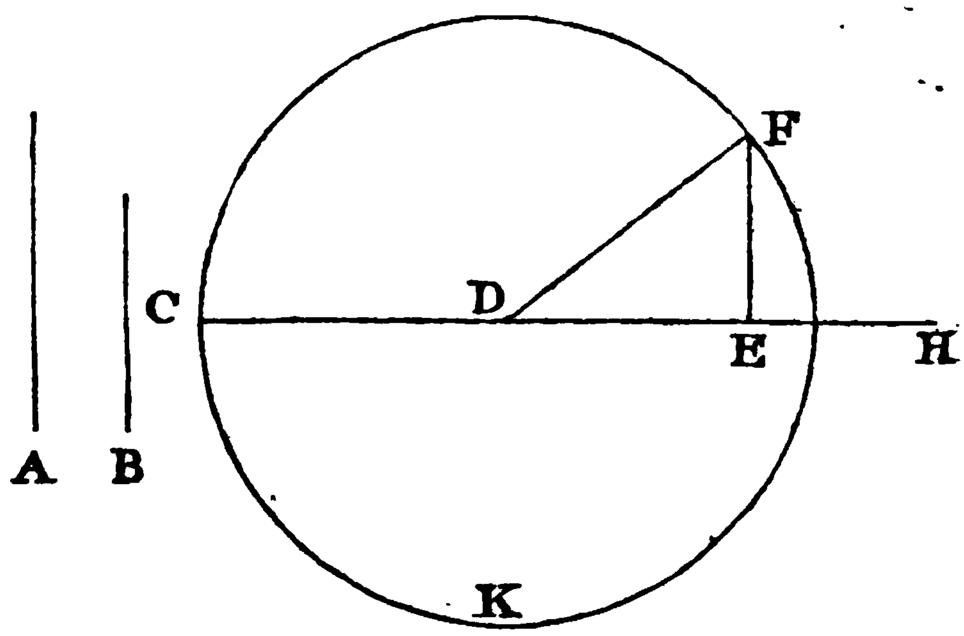
(3.1), join FG, make $EL=FG$, $EH=C$, join HL, take $EN=HL$, $EM=D$, and join MN; the square of MN will be equal to the sum of the squares of A, B, C and D.



Because $EF=A$, and $EG=B$, $\because \overline{FG}|^2=(\overline{FE}|^2+\overline{EG}|^2$ (47. 1.) $=)$ A^2+B^2; and because $EL=FG$, and $EH=C$, $\because \overline{LH}|^2=(\overline{EL}|^2+\overline{EH}|^2=\overline{FG}|^2+C^2=)$ $A^2+B^2+C^2$; and because $EN=LH$, and $EM=D$, $\because \overline{MN}|^2=(\overline{EN}|^2+\overline{EM}|^2=\overline{LH}|^2+D^2=)$ $A^2+B^2+C^2+D^2$, which was to be shewn; and in the same manner any number of squares may be added together, that is, a square may be found equal to their sum.

158. To find a square equal to the difference of the squares of two given unequal straight lines.

Let A and B be two unequal straight lines, whereof A is the greater; it is required to find a square equal to the excess of the square of A above the square of B.



In any straight line CH take $CD=A$, $DE=C$, (3. 1.) from D as a centre with the distance DC describe the circle CKF, from E draw EF perpendicular to CH (11.1), and join DF; EF will be the side of the square required.

Because $FD=(DC=)$ A, $DE=B$, and DEF is a right angle, $\because$ (47. 1.) $\overline{FD}|^2=(\overline{DE}|^2+\overline{EF}|^2=)$ $B^2+\overline{EF}|^2$, that is $A^2=B^2+\overline{EF}|^2$; take B^2 from each of these equals, and $A^2-B^2=\overline{EF}|^2$; that is, EF is the side of the square, which is the difference required.

159. Hence, if any two sides of a right angled triangle be given, the third side may be found. (See the preceding figure.)

For since $\overline{DE}|^2+\overline{EF}|^2=\overline{DF}|^2$, $\because$ $\sqrt{\overline{DE}|^2+\overline{EF}|^2}=DF$.

$\overline{DF}|^2-\overline{DE}|^2=\overline{EF}|^2$, $\because$ $\sqrt{\overline{DF}|^2-\overline{DE}|^2}=EF$.

$\overline{DF}|^2-\overline{EF}|^2=\overline{DE}|^2$, $\because$ $\sqrt{\overline{DF}|^2-\overline{EF}|^2}=DE$.

EXAMPLES.—1. If the base DE of a right angled triangle be 6 inches, and the perpendicular EF 8 inches, required the longest side, or hypothenuse DF *?

Here $\sqrt{\overline{DE}|^2+\overline{EF}|^2}=\sqrt{6^2+8^2}=\sqrt{36+64}=\sqrt{100}=10=$ DF.

2. Given the hypothenuse $=20$, and the base $=11$, to find the perpendicular?

Thus $\sqrt{\overline{20}|^2-\overline{11}|^2}=\sqrt{400-121}=\sqrt{279}=16.703293=$ *the perpendicular required.*

3. Given the hypothenuse 13, and the perpendicular 10, to find the base?

Thus $\sqrt{\overline{13}|^2-\overline{10}|^2}=\sqrt{169-100}=\sqrt{69}=8.3066239=$ *the base required.*

4. Given the base 7, and the perpendicular 4, to find the hypothenuse? *Ans.* 8.0622577.

5. Given the hypothenuse 12, and perpendicular 10, to find the base? *Ans.* 6.6332496.

6. Given the hypothenuse 123, the base 99, to find the perpendicular?

ON THE SECOND BOOK OF EUCLID'S ELEMENTS.

160. The second Book of Euclid treats wholly of rectangles and squares, shewing that the squares or rectangles of the parts of a line, divided in a specified manner, are equal to other rectangles or squares of the parts of the same line, differently divided: by what rectangle the square of any side of a triangle exceeds,

* In a right angled triangle the longest side, (viz. that opposite the right angle) is called the *hypothenuse*, the other two sides are called *legs*, that on which the figure stands is called the *base*, and the remaining leg the *perpendicular*.

or falls short of the sum of the squares of the other two sides, &c.

161. Rectangles and squares may in every case be represented by numbers or letters, as well as by geometrical figures, and frequently with greater convenience; thus, one side of a rectangle may be called a, and its adjacent side b, and then the rectangle itself will be expressed by ab; if the side of a square be represented by a, the square itself will be represented by aa or a^2; and since in this book, the magnitudes and comparisons only, of *rectilineal* figures are considered, its object may be attained by algebraic reasoning with no less certainty and with much greater facility than by the geometrical method employed by Euclid; we will therefore shew, how the propositions may be algebraically demonstrated.

162. *Def.* 1. Euclid tells us what "every right angled parallelogram is said to be contained by," but he has not informed us either here, or in any other part of the Elements, what we are to understand by the word *rectangle*, although this seems to be the sole object of the definition; instead then of Euclid's definition, let the following be substituted.

"Every right angled parallelogram is called a rectangle; and this rectangle is said to be contained by any two of the straight lines which contain one of its angles [f]."

163. *Prop* 1. Let the divided line $BC=s$, its parts $BD=a$, $DE=b$, and $EC=c$; then will $s=a+b+c$. Let the undivided line $A=x$, then if the above equation be multiplied by x, we shall have $sx=(\overline{a+b+c}.x=)\ ax+bx+cx$; "that is, the rectangle sx contained by the entire lines s and x, is equal to the several rectangles ax, bx, and cx, contained by the undivided line x, and the several parts a, b, and c, of the divided line s." Q. E. D.

Cor. Hence, if two given straight lines be each divided into any number of parts, the rectangle contained by the two straight lines will be equal to the sum of the rectangles contained by each of the parts of the one, and each of the parts of the other.

Thus, let $s=a+b+c$, *as before.*

And $x=y+z$.

Then $sx=(\overline{a+b+c}.\overline{y+z}=)ay+by+cy+az+bz+cz$.

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[f] The rectangle *contained* by two straight lines AB, BC, is frequently called "the rectangle *under* AB, BC;" or simply "the rectangle AB, BC."

164. *Prop.* 2. Let $AB=s$, $AC=a$, and $CB=b$.

Then $a+b=s$, multiply these equals by s, and $as+bs=ss$; that is, the rectangle contained by the whole line s and the part a, together with that contained by the whole line s and the other part b, are equal to the square of the whole line s. Q. E. D.

This proposition is merely a particular case of the former, in which if the line s be divided into the parts a and b, and the undivided line $x=s$, we shall have $sx=ax+bx$, become $ss=as+bs$, as in this proposition.

165. *Prop.* 3. Let $AB=s$, $AC=a$, and $CB=b$, then will $s=a+b$, and $sb=(\overline{a+b}.b=)\ ab+bb$; in like manner $sa=(\overline{a+b}.a=)\ aa+ab$; that is, in either case the rectangle contained by the whole s, and either of the parts a or b, is equal to the rectangle ab contained by the two parts a and b, together with the square of the aforesaid part a, or b as the case may be. Q. E. D.

This proposition is likewise a particular case of the first, in which the undivided line is equal to one of the parts of the divided line.

166. *Prop.* 4. [*] Let $AB=s$, $AC=a$, and $BC=b$, then will $s=a+b$; square both sides, and $ss=(\overline{a+b}|^2=)\ aa+2ab+bb$; that is, the square of the whole line s, (viz. ss) is equal to the sum of the squares of the parts a and b, (viz. $aa+bb$) and twice the rectangle or product of the said parts, (viz. $2ab$.) Q. E. D.

167. *Prop.* 5. Let $AC=CB=a$, $CD=x$, then will $AD=a+x$, and $DB=a-x$, and their rectangle or product $\overline{a+x}.\overline{a-x}=aa-xx$; to each of these equals add xx, and $\overline{a+x}.\overline{a-x}+xx=aa$, that is, the rectangle contained by the unequal parts, together with the square of (x) the line between the points of section is equal to the square of (a) half the line. Q. E. D.

In the corollary, it is evident that $CMG=$the difference or excess of CF above LG, that is, of the square of (CB, or) AC above the square of CD; but CMG is $=AH=(\overline{AC+CD}\times\overline{AC-CD}=)\ AD\times DB$, therefore ($\overline{CB}|^2-\overline{CD}|^2$, that is) $\overline{AC}|^2-\overline{CD}|^2=AD\times DB$, or as we have shewn above $au-xx=\overline{a+x}.\overline{a-x}$.

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[*] In Euclid's demonstration there is no necessity to prove the figure $CGKB$ rectangular in the manner he has done; it may be shewn thus, "because $CGKB$ is a parallelogram, and the angle CBK (the angle of a square) a right angle, therefore all the angles of $CGKB$ are right angles by Cor. 46. 1.

168. *Prop.* 6. Let $AC=CB=a$, $BD=x$, then will $AB=2a$, and $AD=2a+x$; then the rectangle contained by AD and DB will be $\overline{2a+x}.x=2ax+xx$. to these equals let aa (the square of half AB) be added, and $\overline{2a+x}.x+aa=(aa+2ax+xx=)\overline{a+x}|^2$; that is, the rectangle contained by the line produced and part produced, together with the square of half the line bisected, is equal to the square of the line made up of the half, and part produced. Q. E. D.

Cor. Hence, if three lines x, $a+x$, and $2a+x$ be arithmetically proportional, the rectangle contained by the extremes $(x.\overline{2a+x})$ together with the square of the common difference a, (or aa) is equal to $(\overline{a+x}|^2)$ the square of the middle term.

169. *Prop.* 7. Let $AB=s$, $AC=a$, $CB=b$, then $s=a+b$, and $ss=(\overline{a+b}|^2=aa+2ab+bb=)\ 2ab+bb+aa$, to these equals add bb, and $ss+bb=(2ab+2bb+aa=2.\overline{a+b}.b+aa=)\ 2sb+aa$; that is, the square of the whole line, (or ss) and the square of one part b (or bb,) is equal to twice the rectangle contained by the whole s, and that part b, (or $2sb$,) together with (aa) the square of the other part. Q. E. D.

Cor. Hence, because $2sb+aa=ss+bb$, by taking $2sb$ from both, we have $aa=ss-2sb+bb$; that is, the square of the difference of two lines (s) AB and (b) CB, is less than the sum of the squares of (s) AB and (b) CB, by twice the rectangle $(2sb)$ $2.AB.CB$ contained by those lines.

170. *Prop.* 8. Let $AB=s$, $AC=a$, $CB=b$, then $s=a+b$, or $a=s-b$, $\because$ $aa=(\overline{s-b}|^2=)\ ss-2sb+bb$, to each of these equals add $4sb$, and $4sb+aa=ss+2sb+bb=\overline{s+b}|^2$; that is, ($4sb$, or) four times the rectangle contained by the whole s, and one part b, together with (aa) the square of the other part a, is equal to $(\overline{s+b}|^2$ or) the square of the straight line made up of the whole s, and the part b. Q. E. D.

171. *Prop.* 9. Let $AC=CB=a$, $CD=x$, then will the greater segment $AD=a+x$, and the less segment $DB=a-x$.

$$\text{Then } \overline{a+x}|^2=aa+2ax+xx$$
$$\text{And } \overline{a-x}|^2=aa-2ax+xx$$
$$\text{The sum of both}=2aa+2xx=2.\overline{aa+xx}$$

That is, $\overline{a+x}|^2+\overline{a-x}|^2=2.\overline{aa+xx}$, or the sum of the squares of the unequal parts ($a+x$ and $a-x$) is equal to double the square of the half a, and of the part x between the points

of section; or, which is the same thing, "the aggregate of the squares of the sum and difference of two straight lines a and x is equal to double the squares of those lines." Q. E. D.

172. *Prop.* 10. Let $AC=CB=a$, $BD=x$, then will $AD=2a+x$, and $CD=a+x$.

Now $\overline{2a+x}|^2=4aa+4ax+xx$

Add xx to this, and the sum is $4aa+4ax+2xx$

Also $\overline{a+x}|^2=aa+2ax+xx$, add aa to this, and it becomes $2aa+2ax+xx$; now the former of these sums is double of the latter, that is $4aa+4ax+2xx=2.\overline{2aa+2ax+xx}$; or, the square of the produced line $2a+x$, together with the square of the part produced x, is double the square of a half the line, and the square of $a+d$ the line made up of the half and the part produced. Q. E. D.

173. *Prop.* 11. This proposition is impossible by numbers, for there is no number that can be so divided, that the product of the whole into one part, shall equal the square of the other part; the solution may however be approximated to as follows:

Let $AB=2a$, $AH=x$, $HB=y$, then by the problem $x+y=2a$, and $2ay=xx$; from the first equation $y=2a-x$, this value being substituted for y in the latter equation, we shall have $4aa-2ax=xx$, or $xx+2ax=4aa$, this solved (by Art. 97. part. 3.) gives $x=\pm\sqrt{5aa}-a$, and $y=(2a-x=2a-\sqrt{5aa}-a=)\,3a-\sqrt{5aa}$, or which is the same $x=1.236068$, &c. $\times a$, and $y=.763931$, &c. $\times a$.

174. *Prop.* 12. Let $AB=a$, $BC=b$, $CD=x$, and $AD=z$;

$$\begin{array}{lr}
\text{Then (47. 1.) } \overline{AB}|^2=\overline{BD}|^2+\overline{DA}|^2=\overline{b+x}|^2+zz= & \\
 & bb+2bx+xx+zz \\
\text{And } \overline{CB}|^2+\overline{AC}|^2= & bb\quad *\quad +xx+zz \\
\text{(Subtract the latter from the former,)} & \overline{\hphantom{bb+2bx+xx+zz}} \\
\text{Therefore } \overline{AB}|^2-\overline{\overline{CB}|^2+\overline{AC}|^2}=\ldots\ldots & 2bx\quad *\quad *
\end{array}$$

That is, the square of AB, the side subtending the obtuse angle, is greater than the sum of the squares of CB and AC, the sides containing the obtuse angle, by ($2bx$) twice the rectangle BC, CD. Q. E. D.

175. *Prop.* 13. Let $AB=a$, $CB=b$, $AC=c$, $AD=d$, $BD=m$, $DC=n$; then the first case of this proposition is proved as follows:

First, $bb+mm=2bm+nn$ (7. 2.) To each of these equals add

dd, and $bb+mm+dd=2\,bm+dd+nn$. But $aa=mm+dd$, and $cc=dd+nn$ (47. 1.) $\because$ if aa and cc be substituted for their equals in the preceding equation, we shall have $bb+aa=2\,bm+cc$, or $cc=bb+aa-2\,bm$.

Second case. Because $aa=cc+bb+2\,bn$ (12. 2.) add bb to both sides, and $aa+bb=cc+2\,bb+2\,bn$, but $bm=bn+bb$ (3. 2.) $\because 2\,bm=2\,bn+2\,bb$; substitute $2\,bm$ for its equal in the preceding equation, and $aa+bb=cc+2\,bm$, or $cc=aa+bb-2\,bm$.

Third case. Here the points C and D coincide, $\because b=m$; wherefore since $cc+bb=aa$ (47. 1.) to each of these equals add bb, and $cc+2\,bb=aa+bb$, or $cc=aa+bb-2\,bb$, which corresponds with the former cases since $2\,bb$ here answers to $2\,bm$ there. Wherefore cc is less than $aa+bb$ by $2\,bm$, or $\overline{AC}|^2 < \overline{AB}|^2+\overline{BC}|^2$ by 2, CB, BD. Q. E. D.

176. *Prop.* 14. By help of this problem any pure quadratic equation may be geometrically constructed. To construct an equation is to exhibit it by means of a geometrical figure, in such a manner, that some of the lines may express the conditions, and others the roots of the given equation.

EXAMPLES.—1. Let $x^2=ab$ be given to find x by a geometrical construction. *See Euclid's figure.*

Make $BE=a$, $EF=b$, then if BF be bisected in the point G, (10. 1.) *and from G, as a centre, with the distance GF, a circle be described, and EH be drawn perpendicular to BF from the point E,* (11. 1.) *it is plain that EH will be the value of x, For by the proposition $\overline{EH}|^2=BE\times EF=ab$, but by hypothesis $x^2=ab$, $\because \overline{EH}|^2=x^2$, and $EH=x$; which was to be shewn.*

But the root of x^2 is either $+x$ or $-x$, now both these roots may be shewn by the figure, for if $EH=+x$, and EH be produced through D till it meet the circumference below BF, the line intercepted between E and the circumference will $=-x$, for in this case $BE\times EF=-x\times -x=+x^2$, as before.

2. Let $x^2=36$ be given, to find the value of x.

Here, because $36=9\times 4$, make $BE=9$, $EF=4$; then proceeding as before, $\overline{EH}|^2=9\times 4=36$, and $EH=6$.

3. Let $x^2=120=12\times 10$ be given.

Make $BE=12$, $EF=10$, then $\overline{EH}|^2=120$, and $EH=$ $(\sqrt{120}=)$ $10.95445=x$.

4. Let $x^2=3$ be given.

Here $3=3\times1$; *make* $BE=3$, $EF=1$, *then* $\overline{EH}|^2=3$, *and* $EH=1.73205=$

ON THE THIRD BOOK OF EUCLID'S ELEMENTS.

177. This book demonstrates the fundamental properties of circles; teaching many particulars relating to lines, angles, and figures inscribed; lines cutting them; how to draw tangents; describe or cut off proposed segments, &c.

178. *Def.* 1. "This," as Dr. Simson remarks, "is not a definition, but a theorem;" he has shewn how it may be proved: and it may be added, that the converse of this theorem is proved in the same manner.

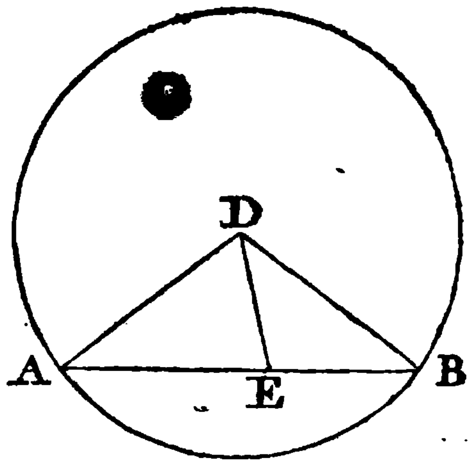
179. Def. 6 has been already given in the first book, and might have been omitted here, (see Art. 74.) Def. 7 is of no use in the Elements, and might likewise have been omitted. In the figure to def. 10 there is a line drawn from one radius to the other, by which the figure intended to represent a sector of a circle is redundant: that line should be taken out.

180. *Prop.* 1. *Cor.* To this corollary we may add, that if the bisecting line itself be bisected, the point of bisection will be the centre of the circle.

181. *Prop.* 2. This proposition is proved by reductio ad absurdum. The figure intended to represent a circle is so very unlike one, that it will hardly be understood, the part *AFB* of the circumference being *bent in*, in order that the line which joins the points *A* and *B* may fall (where it is impossible for that line to fall) without the circle.

The demonstration given by Euclid is by reductio ad absurdum. Commandine has proved the proposition *directly*; his proof depends on the following axiom which we have already given, viz. "If a point be taken nearer the centre than the circumference is, that point is within the circle." Thus,

182. Let *AB* be two points in the circumference *ACB*, join *AB*, this line will fall wholly within the circle. Find the centre

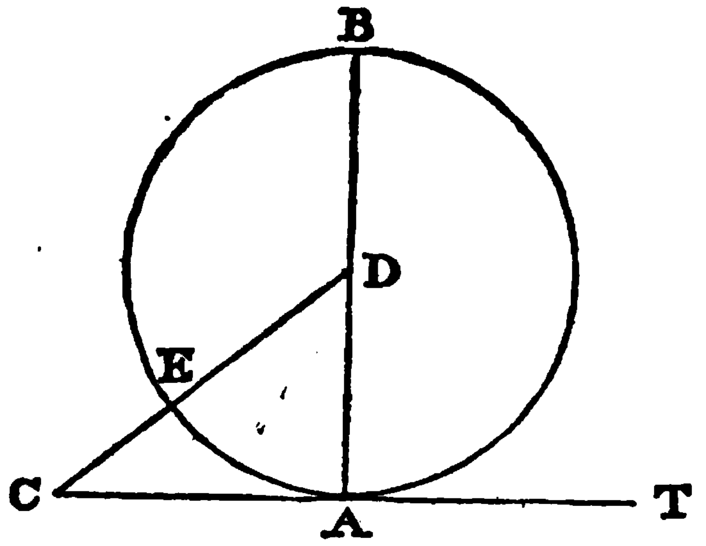


D, (Art. 179.) in *AB* take any point *E*, and join *DA*, *DE*, and *DB*. Because *DA*=*DB*, ∵ the angles *DAB* *DBA* are equal, (5. 1.) but *DEB* ≻ than *DAB* (16. 1.) consequently ≻ than *DBA*; ∵ *DB* ≻ *DE* (19. 1.) ∵ by the axiom the point *E* is within the circle, and the same may be proved of every point in *AB*, ∵ *AB* falls within the circle. Q. E. D.

183. *Prop.* 4. It is shewn in prop. 3. that one line passing through the centre *may* bisect another which does not pass through the centre; but it is plain that the latter cannot bisect the former, since it does not pass through the centre, which is the only point in which the former can be bisected.

184. *Prop.* 16. A direct proof may here be given as in Art. 181. prop. 2. provided the corresponding axiom be admitted, namely, "If a point be taken farther from the centre than the circumference is, that point is without the circle." Thus,

Let *BEA* be a circle, *D* its centre, *BA* a diameter, and *CAT* a straight line at right angles to the diameter *BA* at the extremity *A*, the line *CAT* shall touch the circle in *A*. In *CT* take any point *C*, and join *DC* cutting the circle in *E*, then because *DAC* is a right angle, *DCA* is less than a right angle (17. 1.) ∵ *DC* ≻ *DA* (19. 1.) ∵ *D* is farther from the centre than *A*, consequently by the axiom *C* is without the circle, and the same may be shewn of every point in *CT*, ∵ *CT* is without the circle. Q. E. D.



Cor. Hence it appears that the shortest line that can be drawn from a given point to a given straight line, is that which is perpendicular to the latter.

185. In the enunciation of this proposition we read, that "no straight line can be drawn *between* that straight line (*i. e.* the touching line, or *tangent*) and the circumference from the ex-

tremity (of the diameter) *so as not to cut the circle*;" this appears to be an absurdity, for how can a line be said to be *between* the tangent and circumference, if it cut the latter? and how can a line which cuts the circumference be *between* it and the tangent? The like may be observed of the sentence, "therefore no straight line can be drawn from the point *A between AE and the circumference*, which does not *cut* the circle." It was for the sake of the latter part of the demonstration that the seventh definition of this book was introduced; both may be passed over, as they do not properly belong to the Elements.

186. *Prop.* 24. The demonstration of this proposition is manifestly imperfect; after the words "the segment *AEB* must coincide with the segment *CFD*," let there be added, "for if *AEB* do not coincide with *CFD*, it must fall otherwise (as in the figure to prop. 23.) then upon the same base, and on the same side of it, there *will* be two similar segments of circles not coinciding with one another, but this has been shewn (in prop. 23.) to be impossible; wherefore, &c." Without this addition, the proposition cannot be said to be fairly proved.

187. *Prop.* 30. It is of importance to shew that *DC* falls *without* each of the segments *AD* and *DB*, and since the centre is somewhere in *DC* (cor. 1. 3.) it must be likewise without each of those segments; wherefore (by the latter part of 25. 3.) each of the segments *AD* and *DB* is *less* than a semicircle.

188. By means of prop. 35. and 36. the geometrical construction of the three forms of affected quadratic equations may be performed.

The first and second forms are thus constructed [h].

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[h] The geometrical construction of an equation is the reducing it to a geometrical figure, wherein the conditions of the proposed equation being exhibited by certain lines in the figure, the roots are determined by the intersections which necessarily take place in consequence of the construction.

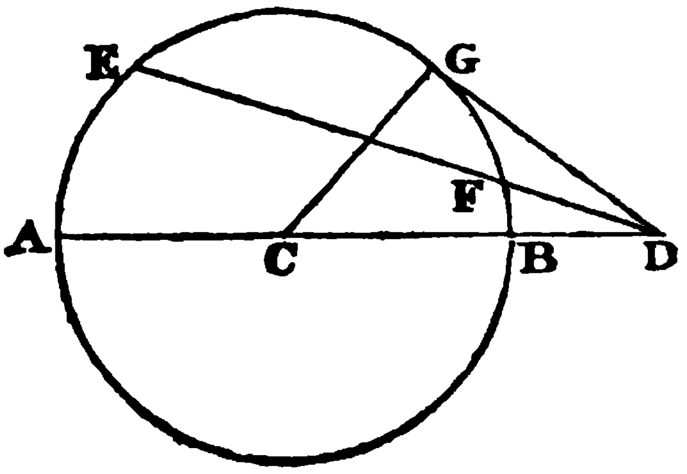
The ancients made great use of geometrical constructions, which is probably owing to the imperfect state of their analysis; but the improvements of the moderns, particularly of Mercator, Newton, Leibnitz, Wallis, Sterling, Demoivre, Taylor, Cramer, Euler, Maclaurin, and others, have in a great measure superseded the ancient methods.

Simple equations are constructed by the intersection of right lines, quadratics by means of right lines and the circle, but equations of higher dimensions require the conic sections, or curves of superior kinds, for their construction;

First form $xx+ax=bc$.

Second form $xx-ax=bc$.

From C as a centre with a distance $=\frac{1}{2}a$ describe the circle AGB, then (supposing $b \succ c$,) with the distance $b-c$ in the compasses (taken from any convenient scale) from any point E in the circumference, describe a small *arc* cutting the circumference GB in F, join EF, and produce it to D, making $FD=c$, and from D draw $DBCA$ passing through the centre C, then will DB and DA be the values of x in both the first and second forms, viz. $x=+DB$ or $-DA$ in the first form, and $x=+DA$ or $-DB$* in the second form. For since $AB=a$ by construction, if $DB=x$, DA will be $x+a$, but if $DA=x$, then $DB=x-a$; but $DA.DB=DE.DF$ (37. 3.) or ($\overline{x+a}.x=$) $xx+ax=bc$ in the first form, and ($x.\overline{x-a}=$) $xx-ax=bc$, in the second, and since the two proposed equations differ only in the sign of the second term, it is plain that they will have the same roots with contrary signs, (see Art. 30. part 5.)



189. If we suppose $b=c$, the construction will be still more simple, for ($b-c=$) $EF=o$, that is EF will vanish, and DF will consequently touch the circle in G, and become DG, and we shall then have $DA.DB=\overline{DG}|^2$; wherefore if a right angled triangle DGC be constructed having $GC=\frac{1}{2}a$, and $DG=b$,

then will $x=\begin{cases} BD=DC-CG \text{ in the } \textit{first form}, \text{ and its negative} \\ \text{value } -\overline{DC+CG}. \\ DA=DC+CG \text{ in the } \textit{second}, \text{ and its negative} \\ \text{value } -\overline{DC-CG}. \end{cases}$

190. To construct the third form of affected quadratic equations, or $xx-ax=-ab$.

From the centre C with the distance $CB=\frac{1}{2}a$, describe the circle AEF as before, from any point E draw $EF=b+c$, make

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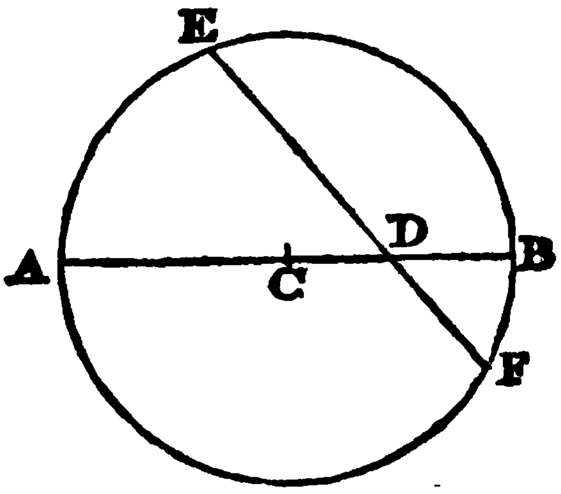
various methods of constructing equations may be seen in the writings of Slusius, Vieta, Albert Girard, Schooten, Fermat, Des Cartes, Ghetaldus, De la Hire, Barrow, Roberval, Halley, Newton, Gregory, Baker, Hyac, Starmius, De l'Hôpital, Sterling, Maclaurin, Simpson, Emerson, and others.

$ED=b$, then $DF=c$, join DC and produce it both ways to A and B.

Since $AB=a$, if AD be called x, then will $DB=a-x$, but $AD.DB=ED.DF$ (35. 3.) that is, $(x.\overline{a-x}=)$ $ax-xx=bc$, or which is the sum $xx-ax=-bc$ as was proposed to be shewn. The like conclusion will follow by supposing $DF=x$, whence the two roots of the given equation are AD and DB.



191. If $b=c$, then will $ED=DF$, and AB will be perpendicular to EF (3. 3.) and EC being joined, we shall in that case have a right angled triangle, the hypothenuse of which will $=\frac{1}{2}a$, and one of its sides $=b$, wherefore the sum and difference of the hypothenuse and the other side will be the two roots of the equation as is manifest.

ON THE FOURTH BOOK OF EUCLID'S ELEMENTS.

192. This book will be found of great use to the practical geometrician, it treats solely on the inscription of regular rectilineal figures in, and their circumscription about a circle; and of the description of a circle in and about such rectilineal figures.

193. *Prop.* 1. The reason why the straight line required to be placed in the given circle must not be greater than the diameter, appears from the 15th proposition of the 3rd book, where it is proved, that the diameter is the greatest straight line that can be placed in a circle.

194. *Prop.* 4. From this proposition it appears, that the three lines which bisect the three angles of a triangle, will all meet in *the same* point within the triangle. Also the sides of any triangle being known, the segments intercepted between their extremes, and the points of contact, may be found [1].

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[1] Thus, let $AB=40$, $AC=30$, and $BC=20$, then will $AB+BC=60$; from this subtract $AC=AE+FC=30$, and the remainder is $BE+BF=30$; therefore $BE=BF=15$, $FC=CG=(BC-BF=)$ 5, and $AG=AE=(AC-CG=)$ 25.

195. *Prop.* 5. We hence learn that it is possible to describe a circle through any three given points, provided they are not placed in a straight line; for by joining every two points, a triangle will be formed, and the proof will be the same as in the proposition. Also only one circle can pass through the same three points. (10. 3.)

196. "The line DF is called the LOCUS of the centres of all the circles that will pass through A and B. And the line EF is the *locus* of the centres of all the circles that will pass through A and C. And this method of solving geometrical problems, by finding the *locus* of all those points that will answer the several conditions separately, is called *constructing of problems by the intersection of* GEOMETRIC LOCI [k]."

197. *Prop.* 6. Hence the diameters of a square (being each the diameter of its circumscribing circle) are equal to each other; they also bisect the angles of the square, and divide it into four triangles, which are equal and alike in all respects: and since the square of BD = the sum of the squares of BA and AD (47. 1.) $=2.\overline{AB}|^2$, it follows that $\overline{BD}|^2+\overline{AC}|^2=\overline{AB}|^2+\overline{BC}|^2+\overline{CD}|^2+\overline{DA}|^2=4.\overline{AB}|^2$.

198. *Prop.* 7. Because the side of a square is equal to the diameter of its inscribed circle (for $GF=BD$,) and the square of the diameter is equal to twice the inscribed square, (see the preceding article); therefore a square circumscribed about a circle is double the square inscribed in it.

199. *Prop.* 10. Since the interior angles of $ABD=2$ right angles (32. 1.) and the angle $B=D=2A$, ∵ the angles at A, B, and D, are together equal to $(A+2A+2A=)\ 5A$, that

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[k] *Ludlam's Rudiments*, p. 207, Loci are expressed by algebraic equations of different orders, according to the nature of the locus. If the equation be constructed by a right line, it is called *locus ad rectum;* if by a circle, *locus ad circulum;* if by a parabala, *locus ad parabolam;* if by an ellipsis, *locus ad ellipsim.* The loci of such equations as are right lines or circles the ancients called *plane loci;* of those that are conic sections, *solid loci;* and of those that are of curves of a higher order, *sursolid loci.* But the moderns distinguish the loci into orders, according to the dimensions of the equations by which they are expressed.—*Hutton.* The following authors, among many others, have treated of this subject, viz. Euclid, Apollonius, Pappus, Aristæus, Viviani, Fermat, Des Cartes, Slusius, Baker, De Witt, Craig, L'Hôpital, Sterling, Maclaurin, Emerson, and Euler.

is $5A=2$ right angles, and $A=\frac{1}{5}$ of 2 right angles; wherefore if A be bisected, each of the parts will be $\frac{1}{5}$ of one right angle. Hence by this proposition a right angle is divided into five equal parts, and if each of these parts be bisected, and the latter again bisected, and so on, the right angle will be divided into 10, 20, 40, 60, &c. equal parts; and since the whole circumference subtends four right angles (at its centre), the circumference will, by these sections, be divided into (4×5, 4×10, 4×20, &c. or) 20, 40, 80, &c. equal parts; and by joining the points of section, polygons of the same number of sides will be inscribed in the circle.

200. *Prop.* 11. Because by the preceding article, $CAD=\frac{1}{5}$ of two right angles, and the three angles at A, which form the angle BAE of the pentagon, are equal to one another (being in equal segments 21. 3.) $\because$ $BAE=\frac{3}{5}$ of two right angles or $\frac{6}{5}$ of one right angle.

201. *Prop.* 13. It follows, that if any two angles of an equilateral and equiangular figure be bisected, and straight lines be drawn from the point of bisection to the remaining angles, these shall likewise be bisected; and if, from this point as a centre, with the distance from it to either of the angles, a circle be described, this circle shall pass through all the angles, and consequently circumscribe the given equilateral and equiangular figure. See *prop.* 14.

202. *Prop.* 15. Hence the angle of an equilateral and equiangular hexagon, will be double the angle of an equilateral triangle, that is, $\frac{2}{3}$ of 2 right angles, or $\frac{4}{3}$ of one right angle. This proposition is particularly useful in trigonometry.

203. *Prop*: 16. All the angles of a quindecagon (by cor. 1. pr. 32. b. 1.) are equal to ($2\times15-4=$) 26 right angles; wherefore $\frac{26}{15}=1\frac{11}{15}$ right angle $=$ one angle of an equilateral and equiangular quindecagon. If each of the circumferences be bisected, each of the halves bisected, and so on continually, the whole circumference will be divided into 15, 30, 60, 120, &c. equal parts, and these points of bisection being joined as before, equilateral and equiangular polygons of the above numbers of sides, will be inscribed as is manifest.

204. Hence, by inscribing the following equilateral and equiangular figures, and by continual bisection of the circumferences

subtended by their sides, the circle will be divided into the following numbers of equal parts, viz. by the

Triangle, into	3,	6,	12,	24,	48,	96,	192,	384,	&c.	
Square	4,	8,	16,	32,	64,	128,	256,	512,	&c.	equal
Pentagon	5,	10,	20,	40,	80,	160,	320,	640,	&c.	parts.
Quindecagon	15,	30,	60,	120,	240,	480,	960,	1920,	&c.	

The numbers arising from inscribing, bisecting, &c. as before, of the

Hexagon,		Triangle,
Octagon,	are included in those of the	Square,
Decagon,		Pentagon,
Triacontagon,		Quindecagon,

and so on continually: whence it appears that the circle may be geometrically divided into 2, 3, 5, and 15, equal parts, and likewise into a number which is the product of any power of 2 into either of those numbers: but all other equal divisions of the circumference by Geometry, are impossible.

ON THE FIFTH BOOK OF EUCLID'S ELEMENTS.

205. In the fifth book, the doctrine of ratio and proportion is treated of and demonstrated in the most general manner, preparatory to its application in the following books. Some of the leading propositions are of no other use, than merely to furnish the necessary means of proving those of which the use is obvious [1].

206. *Def.* 1. By the word *part* (as it is used here) we are not to understand any portion *whatever* of a magnitude less than

[1] Students accustomed to algebra, will find Professor Playfair's method of demonstrating the propositions of the fifth book, much more convenient and easy, than that of Dr. Simson. There are those who would entirely omit the fifth book, and substitute in its place the doctrine of ratio and proportion as proved algebraically (p. 49—74. of this volume;) which might do very well, if no reference were made to the fifth book; or if the sixth might be allowed to rest its evidence on algebraic, instead of geometrical demonstration; but if this cannot be admitted, it will be advisable to read the fifth book at least once over, in order fully to understand the sixth, where it is referred to not less than 58 times; in that book there are 17 references to the 11th proposition, 10 to the 9th, 8 to the 7th, and 5 to the 22nd; these four may therefore be considered as the most useful propositions in the fifth book.

the whole; it implies that part *only*, which in Arithmetic is called an *aliquot* part. The second definition is the converse of the first.

207. The third definition will be easily understood from what has been said on the subject in part 4. Art. 24. &c.

208. *Def.* 4. The import of this definition is to restrain the magnitudes, which "are said to have a ratio to one another," to such as are *of the same kind*: now of any two magnitudes of the same kind, the less may evidently be multiplied, until the product exceed the greater: thus, a minute may be multiplied till it exceeds a year, a pound weight until it exceeds a ton, a yard until it exceeds a mile, &c. these magnitudes then have respectively a ratio to one another [m]. But since a shilling cannot be multiplied so as to exceed a day, nor a mile so as to exceed a ton weight, these magnitudes *have not* a ratio to each other.

209. *Def.* 5. "One of the chief obstacles to the ready understanding of the 5th book, is the difficulty most people find in reconciling the idea of proportion, which they have already acquired, with that given in the fifth definition;" this obstacle is increased by the unavoidable perplexity of diction, produced by taking the equimultiples of the *alternate* magnitudes, and immediately after, transferring the attention to the multiples of those that are adjacent; operations, which cannot easily be described in a few words with sufficient clearness; besides, the definition is encumbered with some unnecessary repetitions, which might be left out, without endangering its perspicuity or precision. On the subject of this definition, as it appears to me, much more has been said than is necessary. Euclid here lays down a criterion of proportionality, to which we are to appeal in all cases, whenever it is necessary to determine whether mag-

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[m] In order to make the comparison implied here, it is however necessary that the magnitudes compared should be, not only of the same kind, but likewise *of the same denomination*: properly speaking, we cannot compare a minute with a year, a pound weight with a ton, or a yard with a mile; but we can compare a minute with the number of minutes in a year, a pound with the number of pounds in a ton, and a yard with the number of yards in a mile; the ratio of a guinea to a pound can be determined only after they are both reduced to the same denomination; then, and not before, we find that they have a ratio, viz. the former is to the latter as 21 to 20.

nitudes are, or are not proportionals; and he has given us in this book, no less than twelve plain and explicit examples of its application; so that, admitting Euclid's criterion to be just, the mode of reference is, if I am not deceived, as simple, and the evidence as satisfactory, as can be required.

210. But how are we to know, whether Euclid's standard of proportionality be just or not; that is, whether it does or does not agree with our received notions of proportionality, as dictated by common sense? we will compare Euclid's doctrine, as laid down in the fifth definition, with the notion which all persons, whether learned or not, have of proportion, and they will be found to agree.

211. Ask any man what he means by "two things being in the same proportion to one another, that two other things are?" and he will immediately answer, "when the first is as large when compared with the second, as the third is, when compared with the fourth." Now, the obvious method of finding *how large* one magnitude is, when compared with another, is to find how often it contains, or is contained in, the other; or in more correct and scientific language, to find what multiple, part, or parts the former magnitude is of the latter; which is effected, by dividing the number representing the one, by that representing the other. Wherefore, the common notion of proportionality when accurately expressed, will be as follows.

212. "Two magnitudes are proportional to two others, when the first is the same multiple, part, or parts of the second, as the third is of the fourth;" or, when the quotient of the first divided by the second, equals the quotient of the third divided by the fourth: under these circumstances "the four magnitudes are said to be proportionals." This is in substance the same as def. 20. of the 7th book of Euclid's Elements, and Mr. Ludlam has shewn that it agrees with Euclid's doctrine as delivered, in his 5th book, that is, if four magnitudes be proportionals according to def. 5. 5. they are proportionals according to this article; and if they be proportionals according to this article, they are likewise proportionals according to def. 5. 5. First, if $a : b :: c : d$ by 5. def. 5. book, then will $a \times d = b \times c$, and $\frac{a}{b} = \frac{c}{d}$.

213. By hypothesis $a : b :: c : d$
And (15. 5.) .... $a : b :: ad : bd$
Also (11. 5.) .... $c : d :: ad : bd$
And (15. 5.) .... $c : d :: bc : bd$
Wherefore (11. 5.) .. $ad : bd :: bc : bd$
Consequently (9. 5.) . . . $ad = bc$, and the $\frac{1}{bd}$ parts of these equals, will likewise evidently be equal, that is ($ad \times \frac{1}{bd} = bc \times \frac{1}{bd}$ or) $\frac{a}{b} = \frac{c}{d}$, so that if four magnitudes $a : b :: c : d$ be proportionals according to Euclid's 5th definition, they are also proportionals by Art. 211. Q. E. D. See also Art. 56. part 6.

214. It remains to be shewn that "if four magnitudes be proportionals according to Art. 211. they are also proportionals according to def. 5. 5. Euclid."

Let $\frac{a}{b} = \frac{c}{d}$, then will $ad = bc$ agreeably to Art. 211, and if $ad = bc$, then will $a : b :: c : d$ agreeably to def. 5. 5. Euclid.

For let m and n be two multipliers, and let the first and third, (viz. a and c) be multiplied by m, and the second and fourth (or b and d) by n; if ma be greater than nb, then will mc be greater than nd, and if equal equal, and if less less. For since $a \times d = b \times c$, it follows that $ma \times nd = nb \times mc$, $\because$ if ma be greater than nb, it is plain that mc must be greater than nd, if equal equal, and if less; wherefore by def. 5. 5. a, b, c, and d, are proportionals. Q. E. D.

215. It will be readily seen that the definition (Art. 211.), which we derive from the popular notion of proportionals, is restrained to magnitudes which can be expressed by commensurate numbers. Euclid's 5th definition applies equally to commensurate and incommensurate magnitudes; this capacity of universal application gives it a decided preference to the definition in Art. 211. and we have shewn that both agree as far as they can be compared together.

216. Def. 6. and 8. properly form but one definition, which may stand as follows, viz. "magnitudes which have the same ratio are called proportionals, and this identity of ratios is called proportion."

217. The 10th and 11th definitions ought to have been placed

after def. *A*, since duplicate, triplicate, quadruplicate, &c. ratios are particular species of compound ratio; thus, let a, b, c, d, e, &c. be any quantities of the same kind, a has to e the ratio compounded of the ratios of a to b, of b to c, of c to d, and of d to e, (see Art. 40—42. part 4.) and if these ratios be equal to one another, a will have to e the *quadruplicate* ratio of a to b, (or $a^4 : b^4$) that is, the ratio compounded of *four* ratios each of which is equal to that of a to b; in like manner a will have to d the *triplicate* ratio (or $a^3 : b^3$) and to c the *duplicate* ratio (or $a^2 : b^2$) of a to b; wherefore it is plain that each is a particular kind of compound ratio.

218. *Def.* 12. The antecedents of several ratios are said to be *homologous* terms, or *homologous* to one another, likewise the consequents are *homologous* terms, or *homologous* to one another; but an antecedent is not homologous to a consequent, nor a consequent to an antecedent; the word *homologous* is unnecessary, we may use instead of it the word *similar* or *like*, either of these sufficiently expresses its meaning.

ON THE SIXTH BOOK OF EUCLID'S ELEMENTS.

219. The principal object of the sixth book is to apply the doctrine of ratio and proportion (as delivered in the 5th) to lines, angles, and rectilineal figures; we are here taught how to divide a straight line into its aliquot parts; to divide it similarly to another given divided straight line; to find a mean, third and fourth proportional to given straight lines; to determine the relative magnitude of angles by means of their intercepted arcs, and the converse; to determine the ratio of similar rectilineal figures; and to express that ratio by straight lines with many other useful and interesting particulars.

220. *Def.* 1. According to Euclid "similar rectilineal figures are (first,) those which have *their several angles equal, each to each*, and (secondly,) *the sides about the equal angles proportionals*;" now each of these conditions follows from the other, and therefore *both* are not necessary: any two equiangular rectilineal figures will always have the sides about their equal angles proportionals; and if the sides about each of the angles of two rectilineal figures be proportionals, those figures will be equiangular, the one to the other. See prop. 18. book 6.

221. *Def.* 2. Instead of this definition which is of no use,

Dr. Simson has substituted the following. "Two magnitudes are said to be reciprocally proportional to two others, when one of the first is to one of the other magnitudes, as the remaining one of the last two is to the remaining one of the first," (see Simson's note on def. 2. b. 6.) this is perhaps the best definition that can be given for the purpose.

222. *Def.* 3. Thus in prop. 11. b. 2. the line AB is cut in extreme and mean ratio in the point H, for $BA : AH :: AH : HB$ as will be shewn farther on.

223. *Def.* 4. In practical Geometry and other branches depending on it, the line or plane on which a figure is supposed to stand is denominated *the base;* Euclid makes either side indifferently the base, and a perpendicular let fall from the opposite angle (called the vertex) to the base, or the base produced, is called the altitude of the figure (for an example see the three figures to prop. 13. b. 2.)

224. *Prop.* 1. Let A=the altitude, B=the base of one parallelogram or triangle; a=the altitude, b=the base of another; then will AB=the first parallelogram, ab=the second; $\frac{AB}{2}$=the first triangle, and $\frac{ab}{2}$ the second; and if $A=a$, then will $\left.\begin{matrix} AB : ab \\ \frac{AB}{2} : \frac{ab}{2} \end{matrix}\right\} :: B : b;$ and if $B=b$, then will $\left\{\begin{matrix} AB : ab \\ \frac{AB}{2} : \frac{ab}{2} \end{matrix}\right\} :: A : a;$ that is, parallelograms and triangles of equal altitudes are to one another as their bases; and if they have equal bases, they are to one another as their altitudes. Q. E. D.

225. *Prop.* 2. Hence, because the angle $ADE=ABC$, and $AED=ACB$ (29. 1.) and the angle at A common, the triangle ADE will be equiangular to the triangle ABC, (32 1.) And if there be drawn several lines parallel to one side of a triangle, they will in like manner cut the other two sides into proportional segments; and conversely, if several straight lines cut two sides of a triangle proportionally, they will be parallel to the remaining side, and to one another. Hence also if straight lines be drawn parallel to one, two, or three sides of any triangle, another triangle will, in each case, be formed, which is equiangular to the given one.

226. *Prop.* 5. Although in the enunciation it is expressly said, that the equal angles of the two triangles ABC, DEF are

opposite to the homologous sides, yet this circumstance is not once noticed in the demonstration; and hence the learner will be ready to conclude, that the proposition is not completely proved; but let him attentively examine the demonstration, and he will find, that although nothing is expressly affirmed about the equality of the angles which are opposite to the homologous sides, yet the thing itself is incidentally made out; thus AB and DE being the antecedents, it appears by the demonstration that the angle C opposite to AB is equal to the angle F opposite to DE; and BC and EF being the consequents, it is incidentally shewn that the angle A opposite to BC is equal to the angle D opposite to EF; also AC and DF being both antecedents or both consequents, their opposite angles B and E are in like manner shewn to be equal. These observations are likewise applicable to prop. 6.

227. *Prop.* 10. By this proposition a straight line may be divided into any number of equal parts as will be shewn when we treat of the practical part of Geometry.

228. *Prop.* 11. A third proportional to two given straight lines may also be found by the following method, (see the figure to prop. 13.) Let AB and BD be the two given straight lines, draw BD perpendicular to AB (11. 1.) join AD; at the point D draw DC at right angles to AD (11. 1.), and produce AB till it cut DC in C; then will BC be the third proportional to AB and BD. For since ADC is a triangle, right angled at D, from whence DB is drawn perpendicular to the base, by cor. to prop. 8. $AB : BD :: BD : BC$, that is BC is a third proportional to AB and BD. Q. E D.

Let $AB=a$, $AD=b$, then $a : b :: b : \frac{bb}{a}=BC$ which is the same thing performed algebraically.

229. *Prop.* 12. Let a, b, and c, be the three given straight lines, then will $a : b :: c : \frac{bc}{a}=HF$, the fourth proportional required.

230. *Prop.* 13. Let $AB=a$, $BC=b$, and the required mean $=x$, then since $a : x :: x : b$, we have (by multiplying extremes and means) $xx=ab$, and $x=\sqrt{ab}=DB$ [a].

[a] It has been asserted in the introduction to this part, that there is no known geometrical method of finding more than one mean proportional be-

EXAMPLES.—1. To find a mean proportional between 1 and 16.

Here $a=1$, $b=16$, *and* $x=\sqrt{ab}=\sqrt{16}=4$, *the mean required.*

2. To find a mean proportional between 15 and 11.

Here $a=15$, $b=11$, *and* $x=\sqrt{ab}=\sqrt{15\times11}=\sqrt{165}=12.845232578$, *the required mean.*

231. *Prop.* 19. By the help of this useful proposition we are enabled to construct similar triangles, having any given ratio to each other; thus, let it be required to make two similar triangles, one of which shall be to the other as m to n. Make $BC=m$, $BG=n$, and between BC and BG find a mean proportional EF (13. 6.) upon BC and EF make similar triangles ABC, DEF (18. 6.) then by the present proposition $m:n::ABC:DEF$.

EXAMPLES.—1. Let the side of a triangle ABC, viz. $BC=8$, it is required to make a similar triangle, which shall be only half as large as ABC.

Bisect BC in G (10. 1.) *and between BC and BG find a mean proportional EF* (13. 6.); *if a triangle be made on EF similar to ABC, it will be half of ABC. Thus BC being*$=8$, *BG will*$=4$, *and* $\sqrt{BC\times BG}=\sqrt{8\times4}=\sqrt{32}=5.656854=EF$.

2. Let $EF=8$, required the side of a triangle five times as large as DEF, and similar to it? *Ans.* $\sqrt{8\times40}=\sqrt{320}=17.88854382$ *the side required.*

232. *Prop.* 20. Hence, if the homologous sides of any two similar rectilineal figures be known, the ratio of the figures to one another may be readily obtained, namely, by finding a third proportional to the two given sides: for then, the first line will be to the third, as the figure on the first, to the similar and similarly described figure on the second, as is manifest from the

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tween two given straight lines a and b; this may however be done algebraically by the following theorems.

One mean proportional will be $\sqrt{ab}$

Two means .... $\sqrt[3]{a^2b}$, $\sqrt[3]{ab^2}$

Three means .... $\sqrt[4]{a^3b}$, $\sqrt[4]{a^2b^2}$, $\sqrt[4]{ab^3}$

Four means .... $\sqrt[5]{a^4b}$, $\sqrt[5]{a^3b^2}$, $\sqrt[5]{a^2b^3}$, $\sqrt[5]{ab^4}$

n means .... $\sqrt[n+1]{a^nb}$, $\sqrt[n+1]{a^{n-1}b^2}$, $\sqrt[n+1]{a^{n-2}b^3}$
$\sqrt[n+1]{a^3b^{n-2}}$, $\sqrt[n+1]{a^2b^{n-1}}$, $\sqrt[n+1]{ab^n}$.

second cor. to the proposition. Hence also any rectilineal figure may be geometrically increased, or decreased in any assigned ratio. Thus, let it be required to find the side of a pentagon one fifth as large as *ABCDE*, and similar to it; find a mean proportional between *AB* and $\frac{1}{5}$ *AB* (13. 6.) let this be *FG*, and upon *FG* describe the pentagon *FGHKL* similar and similarly situated to *ABCDE* (18. 6.) then will the former be $\frac{1}{5}$ of the latter. Again, let it be required to find the side of a polygon 3 times as large as *ABCDE*, and similar to it?

Thus $\sqrt{AB \times \frac{1}{3}AB}$ = *the side required.*

233. *Prop.* 22. By means of this proposition, the reason of the algebraic rule for multiplying surd quantities together, may be readily shewn. Thus, let it be required to prove that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, First, since *unity* : the *multiplier* : : the *multiplicand* : the *product*; therefore, in the present case, $1 : \sqrt{a} :: \sqrt{b} : \sqrt{a} \times \sqrt{b}$ = the product, but by the proposition ($1^2 : \sqrt{a}^2 :: \sqrt{b}^2 : \sqrt{a}^2 \times \sqrt{b}^2$, that is) $1 : a :: b : ab$ = the square of the product, wherefore $\sqrt{ab}$ = the product.

234. *Prop.* 23. Hence, if two triangles have one angle of the one equal to one angle of the other, they will have to each other the ratio compounded of the ratios of the sides about their equal angles; this will appear by joining *DB* and *GE*; for the triangles *DBC*, *GEC* have the same ratio to one another, that the parallelograms *DB* and *GE* have (1. 6.). Also it appears from hence, that parallelograms and triangles have to one another respectively, the ratio compounded of the ratios of their bases and altitudes.

235. *Prop.* 30. This proposition has been introduced under a different form in another part of the Elements, (viz. 11. 2.) *there*, we have merely to divide a straight line, so that the rectangle contained by the whole and the less segment, may equal the square of the greater; we have to determine the properties of a figure, but the idea of ratio does not occur; *here* we are to divide a line, so that the whole may be to the greater segment, as the greater segment is to the less, and the idea of figure has no place; but our business is solely with the agreement of certain ratios. I do not recollect a single reference to this proposition in any subsequent part of the Elements, except in some of the books which are omitted.

236. *Prop.* 31. What was proved of squares in prop. 47. b. 1.

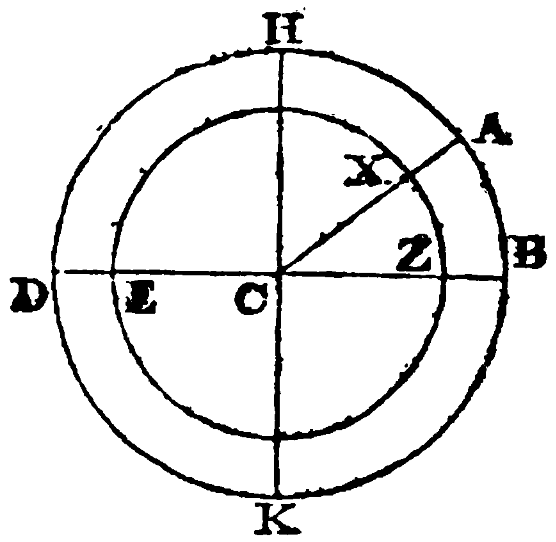
is here shewn to be true of rectilineal figures in general; and the same property belongs likewise to the circle, and to all similar curvilineal and similar mixed figures, with respect to their diameters or similar chords; but the six former books of Euclid's Elements do not furnish us with sufficient principles to extend the doctrine beyond what is proved in this proposition. We are here taught how to find the sum and difference of any two similar rectilineal figures, that is, to find a similar figure equal to the said sum or difference. *See the observations on* 47. 1.

237. *Prop.* 33. This useful proposition is the foundation of *Goniometry*, or the method of measuring angles. If about the angular point as a centre with any radius, a circle be described, it is here shewn, that the *arc* intercepted between the legs of the angle will vary as the angle it subtends varies; thus, if the angle be a right angle, the subtendiug *arc* will be a quadrant (or quarter of a circle); if it be half a right angle, the subtending *arc* will be half a quadrant; if it be equal to two right angles, the subtending *arc* will be a semi-circle; and if it equal four right angles, the subtending *arc* will be the whole circumference. Now if two things vary directly as each other, it is plain that the magnitude of one, will always indicate the contemporary magnitude of the other; that is, it will be a proper measure of the other. Such then is the intercepted *arc* described about an angle, to that angle; and therefore if the whole circumference be divided into any number of equal parts, the number of those parts intercepted between the legs of the angle, will be the measure of that angle. It is usual to divide the whole circumference into 360 equal parts called degrees, to subdivide each degree into 60 equal parts called minutes, and each minute into 60 equal parts called seconds, &c. wherefore, if an angle *at the centre* be subtended by an *arc* which consists of suppose 30 degrees, that angle is said to be *an angle of* 30 *degrees*, or *to measure* 30 *degrees;* if it be subtended by an *arc* of 45 deg. 54 min. the angle is said *to measure* 45 *deg.* 54 *min. &c.*

238. Hence the whole circumference which subtends *four* right angles *at the centre* (Cor. 1. 15. 1.) being divided into 360 degrees, a semicircle which subtends *two* right angles will contain 180 degrees, and a quadrant which subtends *one* right angle will contain 90 degrees, wherefore two right angles are said to measure 180 degrees, one right angle 90 degrees, &c. and note,

degrees, minutes, and seconds, are thus marked °, ′, ″, thus 12 degrees, 3 minutes, 4 seconds, are usually written 12°, 3′, 4″, &c.

238. B. Hence, if about any angular point *C* as a centre, several concentric circles be described, cutting *CA* and *CB* in the points *X, Z, A, B,* the *arc AB,* will be to the whole circumference of which it is an *arc,* as the *arc XZ* is to the whole circumference of which it is an *arc.* Produce *BC* to *D,* and through *C* draw *HK* at right angles to *DB* (11. 1.); then *BA* : : *BH* : : angle *BCA* : angle *BCH* (13. 6.) ∵ *BA* : 4 × *BH* : : angle *BCA* : 4 × angle *BCA,* (15. 5.); that is, *BA* is to the whole circumference *BHDK,* as the angle *BCA,* is to four right angles; in the same manner it is shewn, that *XZ* is to the whole circumference *ZXE* as the same angle *BCA* to four right angles; wherefore *AB* : the whole circumference *BHDK* : : *XZ* : the whole circumference *ZXE.* Q. E. D.



239. Hence also, if the circumferences of these two circles be each divided into 360 degrees, as above (Art. 236.) *AB* will contain as many degrees of the circumference *BHDK,* as *XZ* does of the circumference *ZXE.*

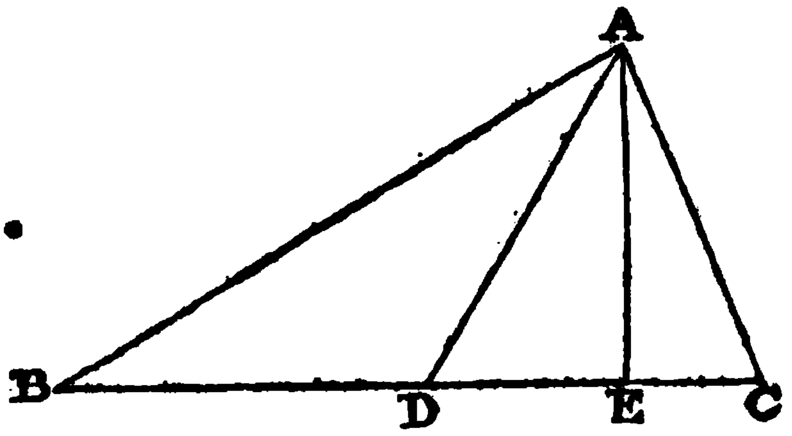
AN APPENDIX TO THE FIRST SIX BOOKS OF EUCLID.

Containing some useful propositions which are not in the Elements.

240. If one side of a triangle be bisected, the sum of the squares of the two remaining sides is double the square of half the side bisected, and of the square of the line drawn from the point of bisection to the opposite angle.

Let *ABC* be a triangle, having *BC* bisected in *D,* and *DA* drawn from *D* to the opposite angle *A;* then will $\overline{BA}|^2+\overline{AC}|^2=2.\overline{BD}|^2+\overline{DA}|^2$.

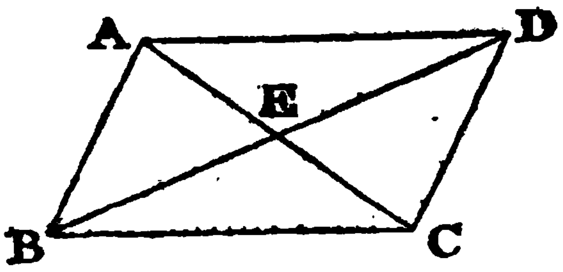
Let *AE* be perpendicular to *BC,* then because *BEA* is a right angle, $\overline{AB}|^2=\overline{BE}|^2+\overline{EA}|^2$; and $\overline{AC}|^2=\overline{CE}|^2+\overline{EA}|^2$, (47. 1.)



$\because \overline{BA}^2 + \overline{AC}^2 = \overline{BE}^2 + \overline{EC}^2 + 2.\overline{EA}^2$. But since BC is divided into two equal parts in D, and into two unequal parts in E, $\overline{BE}^2 + \overline{EC}^2 = 2.\overline{BD}^2 + \overline{DE}^2$ (9.2.) $\therefore \overline{BA}^2 + \overline{AC}^2 = 2.\overline{BD}^2 + \overline{DE}^2 + \overline{EA}^2$. But $\overline{DE}^2 + \overline{EA}^2 = \overline{DA}^2$, (47.1.) $\therefore 2.\overline{\overline{DE}^2 + \overline{EA}^2} = 2.\overline{DA}^2$, $\therefore \overline{AB}^2 + \overline{AC}^2 = (2.\overline{BD}^2 + 2.\overline{\overline{DE}^2 + \overline{EA}^2} = 2.\overline{BD}^2 + 2.\overline{DA}^2 =)$ $2.\overline{\overline{BD}^2 + \overline{DA}^2}$; and the same may be proved if the angle at C be obtuse, by using the 10th proposition of the second book instead of the 9th. Q. E. D.

241. In any parallelogram, the sum of the squares of the diameters, is equal to the sum of the squares of the sides.

Let $ABCD$ be a parallelogram, AC and BD its diameters, then will $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2$.



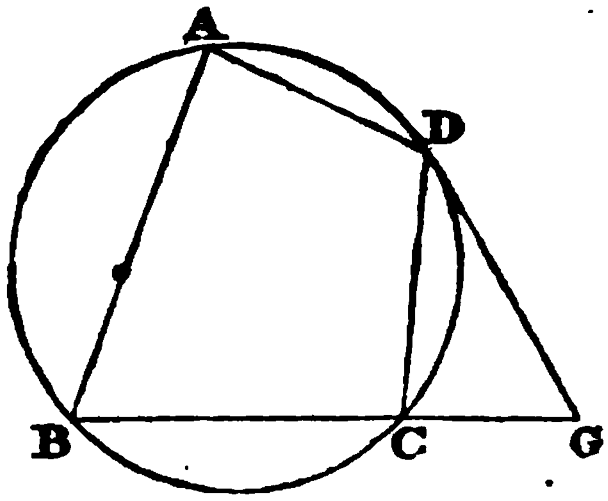
Because the angle $AED = CEB$ (15. 1.) and $EAD = ECB$ (29. 1.) the triangles AED, CEB have two angles of the one = two angles of the other each to each, and a side opposite to the equal angles in each, equal, viz. $AD = BC$ (34. 1.) $\therefore AE = EC$ and $DE = EB$ (26. 1.); and because BD is bisected in E, $\overline{BA}^2 + \overline{AD}^2 = 2.\overline{\overline{BE}^2 + \overline{EA}^2}$, and $\overline{BC}^2 + \overline{CD}^2 = (2.\overline{\overline{BE}^2 + \overline{EC}^2}$ (Art. 239.) $=)$ $2.\overline{\overline{BE}^2 + \overline{EA}^2}$; $\therefore \overline{BA}^2 + \overline{AD}^2 + \overline{DC}^2 + \overline{CB}^2 = 4.\overline{\overline{BE}^2 + \overline{EA}^2} =$ (since $4.\overline{BE}^2 = \overline{BD}^2$, and $4.\overline{EA}^2 = \overline{AC}^2$. by 4.2) $\overline{BD}^2 + \overline{AC}^2$. Q. E. D.

Cor. Hence the diameters of a parallelogram bisect each other.

242. If the sum of any two opposite angles of a quadrilateral figure be equal to two right angles, its four angles will be in the circumference of a circle.

Let $ABCD$ be a quadrilateral figure, having the sum of any two of its opposite angles equal to two right angles, and let a circle be described passing through the three points, A, B, D, (5. 4. and Art. 194.) I say the circumference shall likewise pass

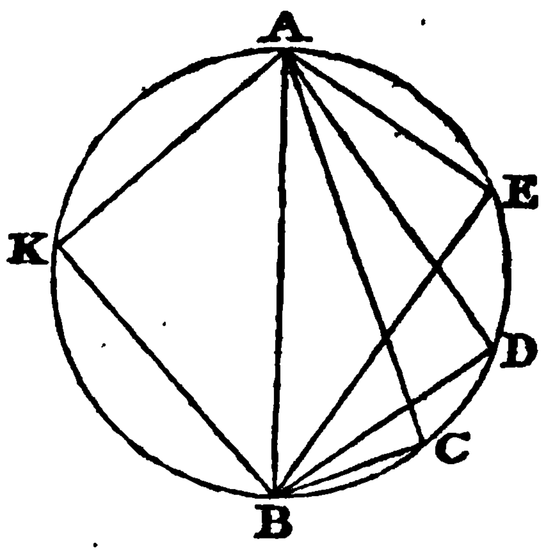
through the fourth point C; for if not, let the fourth point fall without the circumference at G, and join DC; then since by hypothesis the sum of any two opposite angles of the figure are equal to two right angles, $\because BAD+BGD=$ two right angles, but $BAD+BCD=$ two right angles (22. 3.) $\because BAD+BGD=BAD+BCD$, take away the common angle BAD, and $BGD=BCD$, the interior and opposite equal to the exterior which is impossible (16. 1.) $\because$ the fourth point cannot fall without the circle, in the same manner it may be shewn that it cannot fall within it, $\because$ it must fall on the circumference at. C. Q. E. D.



Cor. If one side BC of a quadrilateral figure inscribed in a circle be produced, the exterior angle $DCG=$ the interior and opposite BAD; for $DCG+DCB=$ two right angles (13. 1.) and $BAD+DCB=$ two right angles (22. 3.) $\because DCG+DCB=BAD+DCB$, take away DCB, and $DCG=BAD$.

243. If the vertical angles of several triangles described on the same base, be equal to each other, and the circumference of a circle pass through the extremities of the base, and one of the vertical angles, it shall likewise pass through all the others.

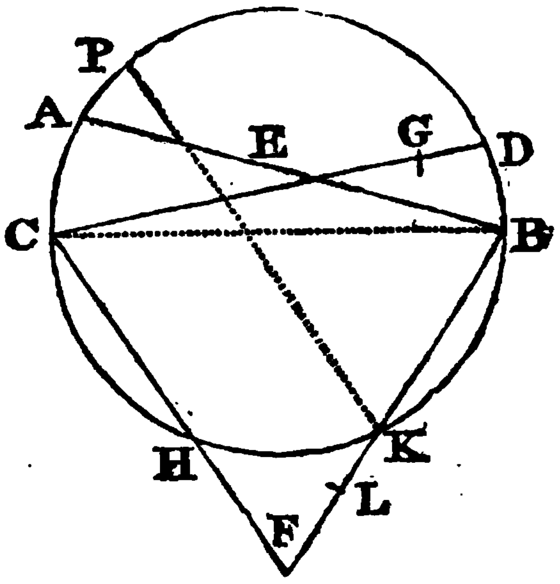
Let ACB, ADB, AEB, &c. be the several equal vertical angles of triangles described on the common base AB, if a circle pass through A, B, and C, it shall likewise pass through the remaining points D, E, &c. Take any point K in the circumference on the other side of AB, and join AK, KB, then will $ACB+AKB=2$ right angles, (22. 3.); but $ADB=AEB=ACB$ by hypothesis, $\because$ each of the angles AEB, ADB together with $AKB=2$ right angles, $\because$ (Art. 241.) the angles E and D are in the circumference. Q. E. D.



243. If two straight lines cut one another, and the rectangle

contained by the segments of one of them, be equal to the rectangle contained by the segments of the other, the circumference which passes through three of the extremities of the two given straight lines, shall likewise pass through the fourth.

Let AB and CD cut each other in E, so that $AE \times EB = CE \times ED$, the circumference ACB, which passes through the three points A, C, and B, shall likewise pass through the fourth D.



For if not, let the circumference, if possible, cut CD in some other point G; then since A, C, B, and G, are in the circumference, the rectangle $AE \times EB = CE \times EG$ (35.3.) but $AE \times EB = CE \times ED$ by hypothesis; $\because CE \times EG = CE \times ED$, $\because EG = ED$, the less = the greater, which is absurd; therefore G is not in the circumference; and in the same manner it may be shewn, that no other point in CD, except D, can be in the circumference. Q. E. D [*].

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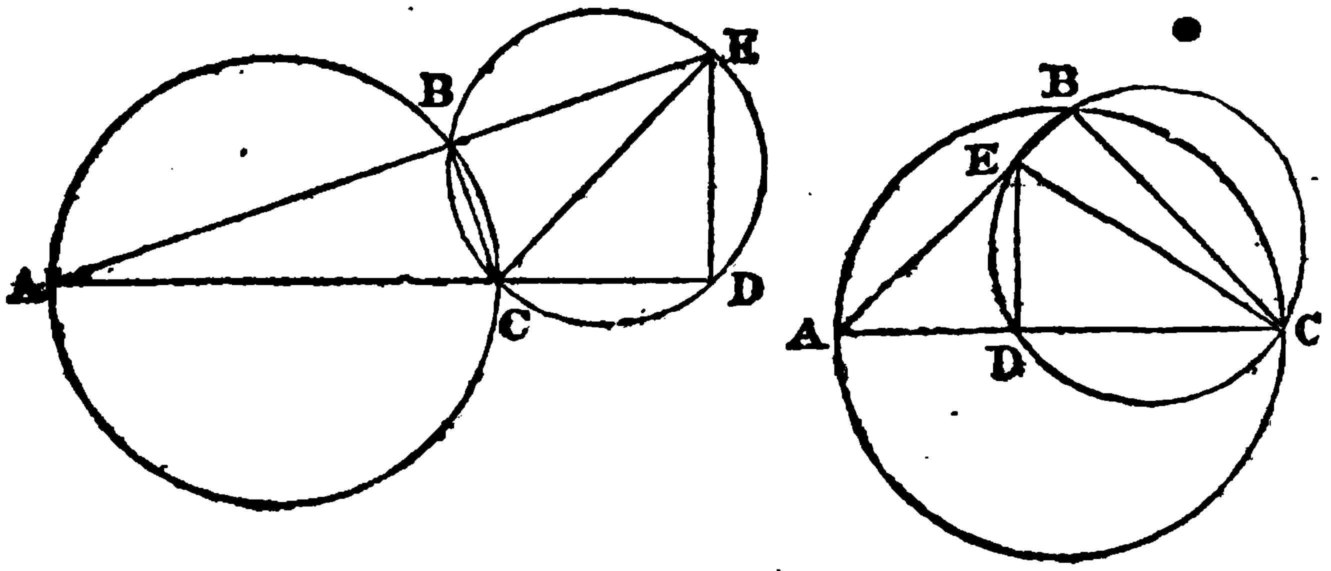
[*] Join CB, and through K draw KP parallel to FC; then since the angle $AEC = ABC + DCB$ (32. 1.) if the angular point E were in the circumference, it is plain that it would be subtended by an arc equal to $AC + DB$; and consequently, if E were at the centre, it would be subtended by an arc equal to to $\frac{AC+DB}{2}$ (20. 3.) Again, if KC be joined, it may be proved (29. 1. and 26. 3.) that CP and HK are equal, but the arc $BDP = (CPB - CP =)$ $CPB - HK$; and since the angle $BKP = BFC$, and BKP is subtended by the arc BDP, if BKP were in the circumference, it would be subtended by an arc equal to BDP; but if it were at the centre, BKP would be subtended by an arc $= (\frac{BDP}{2}$ (20. 3.) that is $=) \frac{CPB-HK}{2}$ by what has been shewn. And since an angle is measured by the subtending arc described about the angular point as a centre (Art. 262.) it follows, that *if two straight lines AB, CD cut one another within a circle, the angle AEC is measured by half the sum of the subtending arcs AC and BD, and (by similar reasoning) the angle AED is measured by half the sum of the arcs APD, CKB. But if two straight lines CF, FB cut one another without the circle*, the angle BFC is measured by half the *difference of the intercepted arcs CPB and HK*; this is connected with Art. 261, 262.

244. Let there be two straight lines CF and FB, cutting the circle in two points, and each other in a point F without the circle; and let CF cut the circumference in C and H, and FB cut it in B; then if a point K be taken in FB, so that $CF\times FH=BF\times FK$, I say the point K is in the circumference.

For if not, let the circumference HAB cut FB in L, then $CF\times FH=BF\times FL$ (36. 3. cor.) but $CF\times FH=BF\times FK$ by hypothesis, $\because$ $BF\times FL=BF\times FK$ and $FL=FK$, the less = the greater, which is absurd. $\because$ L is not in the circumference; and in like manner it may be shewn that no other point in BF, except B and K, can be in the circumference; K is therefore in the circumference. Q. E. D.

245. If a straight line AB be drawn from the extremity A of the diameter AC, meeting the perpendicular ED in E, then will the rectangle $BA\times AE=CA\times AD$.

Join BC, CE, then because the angle ABC in a semicircle is a right angle (31. 3.) CBE is also a right angle (13. 1.) and if a circle $CDEB$ be described on CE as a diameter, its circumfe-



rence shall pass through the points C, B, E, and D; and since BE and CD meet in the point A, $BA\times AE=CA\times AD$ by 35.3 or cor. 36.3 Q. E. D.

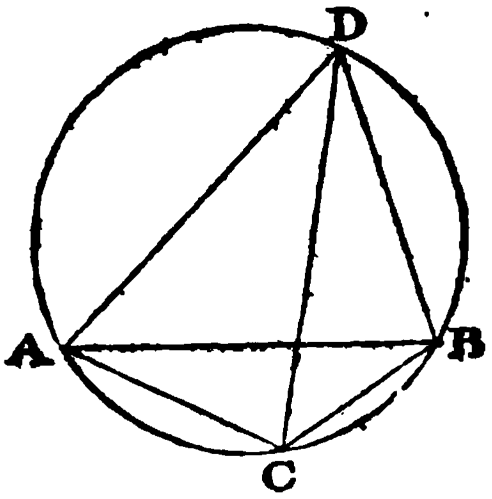
Hence $EA : AD :: CA : AB$ (by 16. 6.)

246. If an *arc* of a circle be bisected, and from the extremities of the *arc* and the point of bisection, straight lines be drawn to any point in the circumference, then will the sum of the two lines drawn from the extremities of the *arc*, have to the line drawn from the point of bisection, the same ratio which the chord of the *arc* has to the chord of half the *arc*.

Let AB be an *arc* bisected in C, and D any point in the

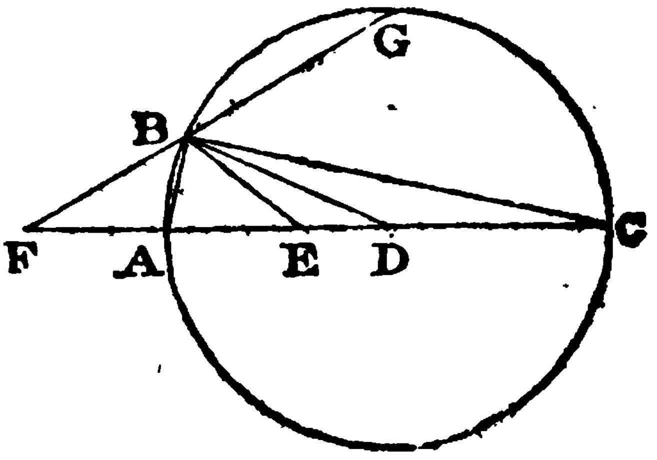
circumference, join AD, CD, BD, AC, and BC, then will $AD+DB : DC :: BA : AC$.

Because $ACBD$ is a quadrilateral figure inscribed in a circle, $AB.CD$ $(=AD.CB+DB.AC$ (D. 6.) which, because $CB=AC)$ $=AD.AC+BD.AC$, $=AC.\overline{AD+BD}$ (1. 2.) and because the sides of equal rectangles are reciprocally proportional (14. 6.) $AD+BD : CD :: AB : AC$. Q. E. D.



247. If two points be taken in the semidiameter of a circle, such, that the rectangle contained by the segments between them and the centre, is equal to the square of the semidiameter; the straight lines drawn from these points to any point in the circumference, shall have the same ratio, that the segments of the diameter between the two fore-mentioned points and the circumference, have to one another.

Let D be the centre of the circle, ABC and DF the semidiameter produced, in which let E and F be taken, such, that $ED.DF=\overline{AD}|^2$; then if EB and FB be drawn from E and F, to any point B in the circumference $EB : FB :: EA : AF$.



Join AB, BD; then since by hypothesis $ED.DF=(\overline{AD}|^2=)\ \overline{DB}|^2$; $DF : DB :: DB : DE$ (17. 6.); that is, the sides about the common angle D of the triangles FBD, EBD are proportionals, ∵ these triangles are equiangular (6. 6.), and the angle $FBD=BED=EAB+ABE$ (32. 1.); but $EAB=ABD$ (5. 1.) ∵ $(FBD=)$ $FBA+ABD=ABD+ABE$, take away the common angle ABD, and $FBA=ABE$, ∵ BA bisects the angle FBE, ∵ $EB : BF :: EA : AF$ (3. 6.) Q. E. D.

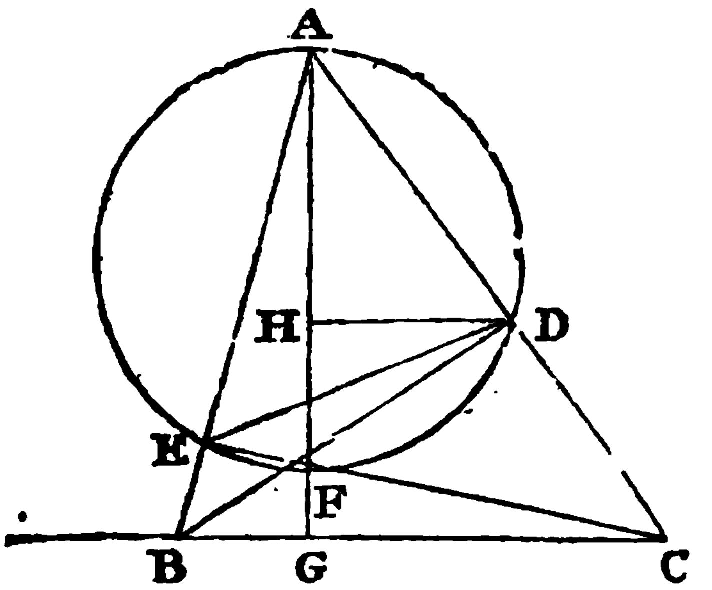
Cor. Hence, if FB be produced to G, and BC joined, the exterior angle ABG will be bisected by BC. For since ABC is a right angle (31. 3.) it is half the sum of the angles FBE and EBG (13. 1.): but $ABE=\frac{1}{2}\ FBE$, ∵ $EBC=\frac{1}{2}EBG$.

248. If from the three angles of any triangle, perpendiculars

be drawn to the opposite sides, these perpendiculars shall intersect one another in the same point.

First. In the acute angled triangle *ABC*, let the perpendiculars *BD* and *CE* intersect one another in *F*, join *AF*, and produce it to *G*, *AG* is perpendicular to *BC*.

Join *DE*, and let a circle be described about the triangle *AEF* (5.4.) then since by hypothesis *AEF* is a right angle, *AF* will be the diameter of the circle (31.3.); and because *ADF*= *AEF*, the circumference of the same circle shall pass through the point *D* (Art. 242.) and the points *A*, *E*, *F*, *D*, will be all in the circumference. But because the angle *EFB*=*DFC* (15.1.) and *BEF*=*CDF* (by hypothesis) ∴ the triangles *BEF* and *CDF* are equiangular (32.1.) ∴ *BF* : *EF* : : *CF* : *FD* (4.6.)

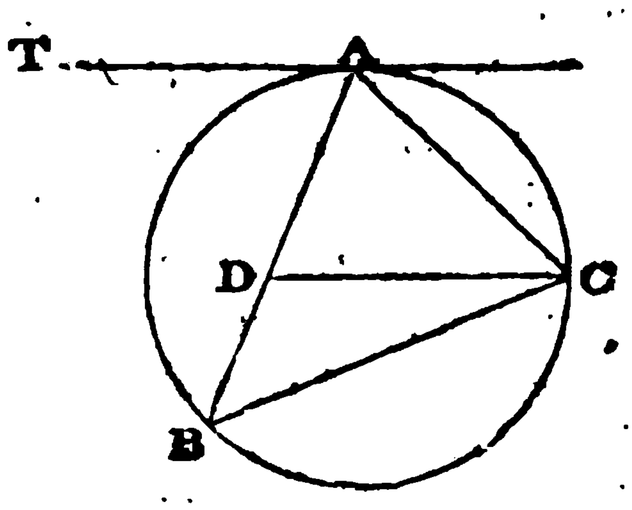


∴ *BF* : *CF* : : *EF* : *FD* (16.5.) and since the angle *BFC*=*EFD* (15.1.) and the sides about these equal angles are proportionals, the triangles *BFC* and *EFD* are equiangular (6.6.) ∴ the angle *FCB*=*EDF*=*EAF* (21. 3.) ∴ *EAF*=*FCG*; and *AFE*=*CFG* (15.1.) ∴ *AEF*=*FGC* (32.1.); but *AEF* is a right angle by hypothesis, ∴ *FGC* is also a right angle and *AG* is perpendicular to *BC*.

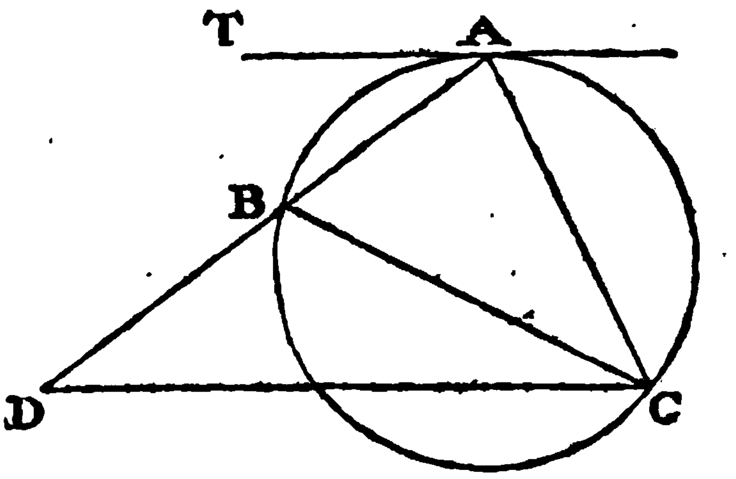
Secondly. In the right angled triangle *AFD*, draw *DH* perpendicular to *AF*, ∴ *AD*, *HD*, and *FD*, are the three perpendiculars, and it is plain that they all meet in *D*.

Thirdly. In the obtuse angled triangle *BFC*, *BE* is perpendicular to *CF* produced, *CD* perpendicular to *BF* produced, and *GF* perpendicular to *BC*; and it appears by the foregoing demonstration, that these three perpendiculars *BE*, *GF*, and *CD* intersect each other in the same point *A*. Q. E. D.

249. If a straight line touch a circle, and from the point of contact two chords be drawn, and if from the extremity of one of them, a straight line be drawn parallel to the tangent meeting, the other chord (produced, if necessary); then will the two chords and the segment intercepted between the parallels, be proportionals.



Let TA touch the circle in A, from whence let the chords AB and AC be drawn, and from C the extremity of one of them, let CD be drawn parallel to TA (31. 1.) meeting AB in D, then will $BA : AC :: AC : AD$. Join BC, then because the angle $ACB = TAD$ (32. 3.) $= ADC$ (29. 1.) and BAC common, the triangles ACB, ADC are equiangular, and $AB : AC :: AC : AD$ (4. 6.) Q. E. D.



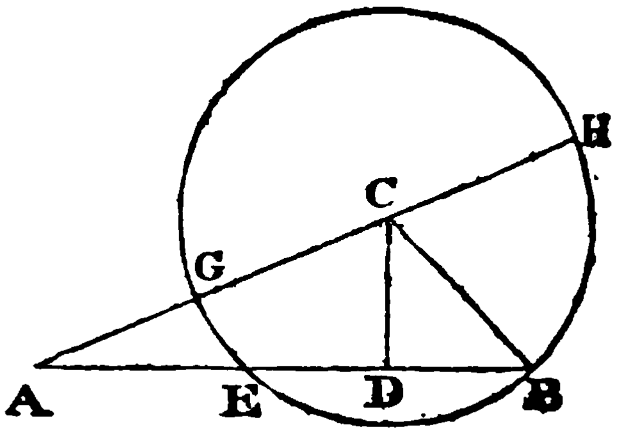
Cor. 1. Hence $BA.AD = \overline{AC}|^2$.

2. If AB pass through the centre, then will ACB be a right angle (31. 3.), and CD will be perpendicular to AB (18. 3. and 29. 1.); and since $AB : AC :: AC : AD$, the side AC of the triangle ACB is a mean proportional between the hypothenuse AB and the segment of it, AD adjacent to AC, as is shewn in cor. 8. 6.

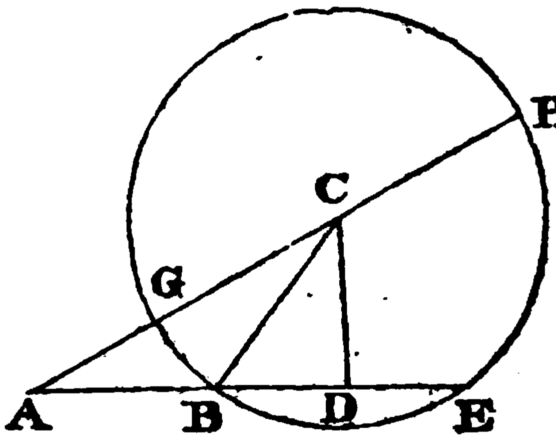
250. If a perpendicular be drawn from the vertical angle of any triangle to the base, (produced if necessary), then will the rectangle contained by the sum and difference of the sides of the triangle, be equal to the rectangle contained by the sum and difference of the segments of the base.

Let ABC be a triangle, and CD a perpendicular drawn from the vertical angle C to the base AB, meeting it (pro-

duced if necessary, as in fig. 2.) in D, then will $\overline{AC+CB}.\overline{AC-CB} = \overline{AD+DB}.\overline{AD-DB}$.



From C as a centre, with the shortest side CB for a distance, describe a circle, cutting AC produced in G and H, and AB (produced in fig. 2.) in E and B. Then since $CG=CH=CB$, $\because$ $AH=AC+CB=$ the sum of the sides, and $AG=$ ($AC-CG=$) $AC-CB=$ their difference; and because $DB=DE$ (3. 3.) ($AD+DB=$) AB is the sum of the segments, and ($AD-DB=AD-DE=$) AE their difference in fig. 1. also ($AD+DB=AD+DE=AE=$ the sum of the segments in fig. 2. and ($AD-DB=$) $AB=$ their difference. Wherefore, (cor. 36. 3.) $AH.AG=AB.AE$; that is, the rectangle contained by the sum and difference of the sides AC and CB, is equal to the rectangle contained by the sum and difference of the segments AD and BD, intercepted between the extremities A and B of the base, (or base produced,) and the perpendicular CD. Q. E. D.



Cor. 1. Hence $AB : AH :: AG : AE$ (16. 6.) that is, the base of a triangle : is to the sum of the sides :: as the difference of sides to the sum : (fig. 2.), or difference (fig. 1.), of the segments of the base, according as the perpendicular CD falls without, or within the triangle. This inference is particularly useful in trigonometry, when the three sides of a triangle are given to find the angles.

2. Because $DB=DE$, and $BE=2\,BD$, $\because$ $AB.AE=(AB.\overline{AB\mp BE}=)$ $AB.\overline{AB\mp 2\,BD}=\overline{AB}|^2\mp 2\,AB.BD$; $\because$ since $\overline{AC}|^2-\overline{CB}|^2 = \overline{AC+CB}.\overline{AC-CB}$ (cor. 5. 2.) $=\overline{AB}|^2\mp 2\,AB.BD$, the rectangle contained by the sum and difference of two sides of a triangle, is equal to the square of the base *minus* or *plus* twice the rectangle contained by the base, and its least segment.

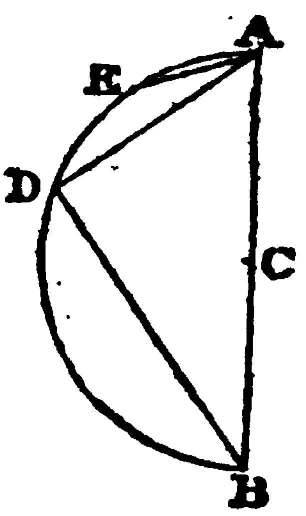
3. If ABC be a right angle, the point B coincides with D, and the circle described from C with the distance CB will touch the base AB in D, and (36. 6.) $HA.AG=(\overline{AD}|^2$; that is, since B coincides with $D=$) $\overline{AB}|^2$; $\because$ the rectangle contained

by the sum and difference of the hypothenuse, and one of the sides is equal to the square of the other side.

4. Since by cor. 2. $\overline{AC+CB}.\overline{AC-CB}=\overline{AB}|^2\mp 2.AB.BD$, and $\overline{AC+CB}.\overline{AC-CB}=\overline{AC}|^2-\overline{CB}|^2$ (5. 2.) $\therefore \overline{AC}|^2-\overline{CB}|^2=\overline{AB}|^2\mp 2.AB.BD$, and $\overline{AC}|^2=\overline{AB}|^2+\overline{CB}|^2\mp 2.AB.BD$. Or the square of the side AC is less or greater than the sum of the squares of AB and CB, by twice the rectangle contained by the base, and the segment CB; according as the angle ABC is acute or obtuse. This is the same as 12 and 13. 2 Euclid.

250. B. The chord of one sixth part of the circumference being given, to find the chord of half that arc, and thence to inscribe within the circle a polygon of a great number of sides.

Let ABD be a semicircle, C its centre, draw the chord $DA=AC$ (1. 4.), bisect the arc DA in E (30. 3.), and join EA; EA will be the side of a regular polygon of 12 sides. Bisect EA, and draw a straight line from A to the point of section, and it will be the side of a polygon of 24 equal sides; and by continually bisecting, we obtain the sides of polygons of 48, 96, 192, 384, &c. equal sides.



251. *To find the circumference and area of a circle, having the diameter given*[p].

RULE. *First.* Since there is no geometrical method for determining accurately, the length of the whole, or any part of the circumference, we must be content with an approximation; which however, may be obtained to such a degree of exactness, as to differ from the truth by a line less than any given line.

Secondly. If two similar polygons of a great number of sides, be one inscribed in, and the other circumscribed about a circle,

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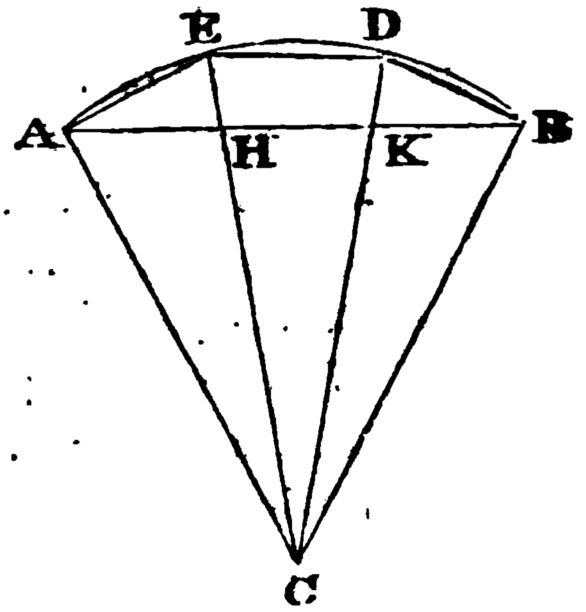
[p] This problem will serve to shew by what laborious methods Wallis, Romanus, Metius, Snellius, Van Ceulen, and others, obtained approximations to the circles periphery; the same thing may however be performed with much more expedition and ease, by the method of fluxions, infinite series, &c. See Simpson's *Doctrine and Application of Fluxions*, part. 1. sect. 8.

the circumference will be greater than the sum of the sides of the former, but less than the sum of the sides of the latter; and therefore, if the numbers expressing these sums agree in a certain number of figures, those figures may be considered as expressing (as far as they go) the length of the circumference which lies between the two polygons; and if half the difference of the remaining figures be added to the less number, or subtracted from the greater, the result will afford a still more accurate expression for the length of the circumference.



Draw any straight line AC, and on it describe the equilateral triangle ABC (1. 1.) from C as a centre, with the distance $CA=CB$ describe the *arc* $AEDB$; then because $AB=AC=$ the side of an equilateral and equiangular hexagon inscribed in the circle (15. 4.) ∴ $AEDB$ will be one sixth of the whole circumference.

Let $r=AC=1$, $c=AB=1$, the *arc* $AE=ED=DB$, and $x=AE=$ the chord of one third of the *arc* AB; then since the *arc* EB is double the *arc* AE, the angle $EAH=ACE$ (20. 3.) and AEC is common, ∴ the triangles AEC and AEH are equiangular (32. 1.) and $CA:AE::AE:EH$ (4. 6.); that is, $r:x::x:\frac{xx}{r}=EH$; also $CE:AE::AH:EH$ ∴ $AE=AH$; in like manner it is shewn that $BD=BK$, ∴ $AH=BK$, ∴ $AH+BK=2x$, and $HK=(AB-AH-BK=)\ c-2x$; but $CE:ED::CH:HK$ (4. 6.), or $r:x::r-\frac{xx}{r}:c-2x$; whence, multiplying extremes and means, $cr-2rx=rx-\frac{x^2}{r}$; which by transposition, &c. (since c and r each $=1$,) becomes $x^3-3x=-1$, the root of which is the chord of AE, or of $\frac{1}{18}$ part of the whole circumference.

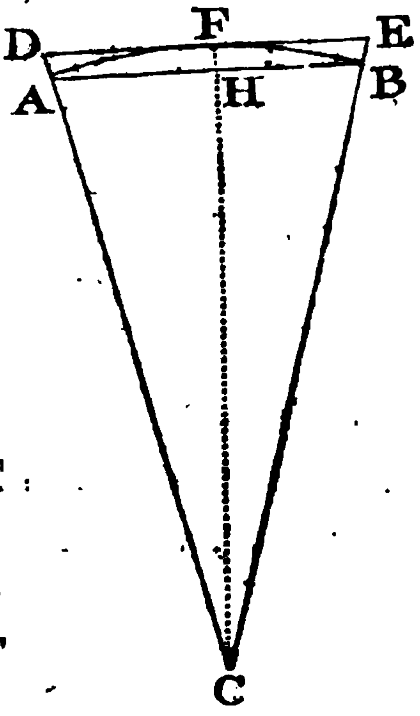
Next to trisect the *arc* AE, let $3y-y^3=x$, the chord of AE,

we shall have $x^3 = 27y^3 - 27y^5 + 9y^7 - y^9$
and $-3x = -9y + 3y^3$
and $+1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots +1$

Their sum $x^3 - 3x + 1 = -9y + 30y^3 - 27y^5 + 9y^7 - y^9 + 1 = o$, the root of which is the chord of $\frac{1}{54}$ part of the whole circumference.

Again, to trisect the *arc* of which y is the chord; let $3z - z^3 = y$, and if this value be substituted for y in the last equation, we shall obtain an expression in which the value of z will be the chord of the $\frac{1}{162}$ part of the whole circumference. Proceeding in this manner after sixteen trisections, the chord of $\frac{1}{258280326}$ part of the circumference (the radius being unity) will be found to be .000000024326999289832033, which number being multiplied by 258280326 (or the number of sides of the polygon, of which the above number expresses the length of one side) the product will be 6.283185307179585968482758 =the perimeter of the inscribed polygon.

252. Next, we are to find the length of the side of a circumscribed polygon of the same number of sides, in order to which, let AB = the side of the inscribed polygon as found above, DE the side of a similar circumscribed polygon; bisect AB in H, join CH and produce it to F. Then $\overline{CA}|^2 - \overline{AH}|^2 = \overline{CH}|^2$ or $1^2 - \overline{.000000012163499644916}|^2 = 1 - .00000000000000014795072361187165 8084647056 = .9999999999999998520492 76388128342$, &c. $= \overline{CH}|^2$, the square root of which number is .99999999999999999 &c. $= CH$; now $CH : HA :: CF : \frac{HA.CF}{CH}$



$= DF$ (4. 6.) that is $\frac{.000000012163499644916016 \times 1}{.99999999999999999} =$ .000000012163499644916, &c. $= DF$, which number multiplied by 2 gives .000000024326999298 32, &c. $= DE$
But .00000002432699928983, &c. $= AB$
and since these two numbers agree as far as the 16th place of decimals, and the *arc* AFB lies between DE and AB, it follows, that those 16 decimal places will express the length of the *arc*

AFB very nearly; that is, the above number will differ from the truth by a very small decimal, whose highest place is 17 places below unity. Whence .0000000243269992849=the length of the *arc AFB* or of the $\frac{1}{[illegible]}$ part of the whole circumference extremely near. Now if the length of the *arc AFB* as above determined be multiplied into the denominator of this fraction, the product will be 6.2831853061898472=the circumference of a circle whose diameter is 2, very nearly.

253. Having found the circumference of a circle, we can readily find the area, if not with strict accuracy, at least sufficiently near the truth for any practical purpose, in order to which, let us suppose an indefinite number of straight lines drawn from the centre to the circumference, these will divide the circle into as many sectors, the bases of which will be indefinitely small *arcs*, and their common altitude the radius of the circle; now since these small *arcs* coincide indefinitely near with the sides of a circumscribed or inscribed polygon of the same number of sides as there are sectors, these sectors may evidently be considered as triangles, the bases of which are the above small *arcs*, and their common altitude the radius; but half the base of a triangle, multiplied into the altitude, will give the area (42. 1.) wherefore, (half the sum of the bases, that is) half the circumference of the circle, multiplied into the radius, will give the area of the triangles, that is, the area of the circle; thus $\frac{6.2831853, \&c. \times 1}{2}$=3.1415926, &c.=the area of a circle, whose diameter is 2.

254. Having found the circumference of a circle, whose diameter is 2, we are by means of it enabled to find the circumference of any other circle, whatever its diameter may be; for let the inscribed polygon (whose sides coincide indefinitely near with the circumference) have n sides, the length of each being r; and let a similar polygon be inscribed in any other circle having the length of its side $=s$, then will nr=the periphery of the first polygon, and ns=that of the second. Let 1=the radius of the former circle, t=that of the latter; then if lines be drawn from each centre to the points of division in the respective circumferences, we shall have $1 : r :: t : s$, (4. 6.) whence (16. 5.) $1 : t :: r : s$, and consequently (15. 5.) $1 : t :: nr : ns$, that is, the peripheries of the similar polygons are to each

other as the radii of their circumscribed cricles; but these polygons coincide indefinitely near with their circumferences: wherefore the circumferences of circles are as their radii.

255. The area of one circle being known, that of another circle having a given diameter, may be found; let $D=$ the diameter of a circle, $A=$ its area, and $d=$ the diameter of another circle, whose area x is required; then (2. 12.) $D^2 : d^2 :: A : x$, whence $x=\frac{d^2A}{D^2}$, the area required.

PRACTICAL GEOMETRY.

255. Practical Geometry teaches the application of theoretical Geometry, as delivered by Euclid and other writers, to practical uses *.

256. *To draw a straight line from a given point A, to represent any length; in yards, feet, inches, &c.*

RULE. I. Let each of the divisions on any convenient scale of equal parts represent a yard, foot, inch, or other unit of the measure proposed.

II. Extend the compasses on that scale until the number of units proposed be included *exactly* between the points.

III. With this distance in the compasses, and one foot on A, describe a small *arc* at B; lay the edge of a straight scale or ruler from A to B, and draw the line AB with a pen or pencil, and it will be the line required.

EXAMPLES.—1. To draw a straight line from the point A to represent 12 inches.

* The following problems are intended as an introduction to the practical application of some of the principal propositions in the Elements of Euclid, and likewise to assist the student in acquiring a knowledge of the use of a case of mathematical instruments. From a great variety of problems usually given by writers on Practical Geometry, we have selected such as appear most necessary, and likewise such methods of solving them as appear most simple and obvious; to a learner well acquainted with Euclid, other methods will occur, and he should be encouraged to exercise his ingenuity in discovering and applying them. The best elementary treatises on Practical Geometry and Mensuration, are those of Mr. Bonnycastle and Dr. Hutton.

With one foot on 0 extend the other to the 12th division on the scale you choose to adopt, and apply that distance from A as above directed, and it will give the length proposed.

2. To draw a line that shall represent 35 yards.

Let each primary division be considered as 10 yards, then will each subdiv sion represent 1 yard; apply the compasses from 3 backwards (to the left) to the 5th subdivision, and 35 subdivisions will be included between the points; apply this from the given point and draw the line as before.

3. To draw a line equal to 263.

On the diagonal scale, let each primary division represent 100, then will each subdivision represent 10, and the distance which each diagonal slopes on the first parallel will be 1, on the second 2, on the third 3, and so on; therefore for 263 extend from the number 2 backwards to the sixth subdivision, on the third parallel, (viz. the 4th line downwards) and it will be the distance required.

257. *To measure any straight line* [r].

RULE. Extend the compasses from one extremity of the given line to the other, and apply this distance to any convenient scale of equal parts, the number of parts intercepted between the points, will be the length required.

Note. If the sides of a rectilineal figure are to be measured, *the same* scale must be used for them all; and one scale must be used for each of two or more lines, when their relative length is required to be ascertained [s].

258. *To bisect a given straight line AB.*

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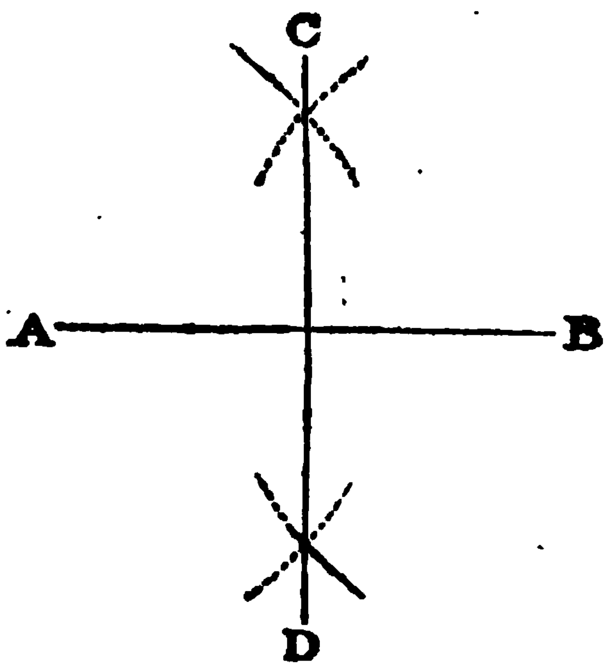
[r] By the word *measure* is meant the *relative* measure of a line, that is, the length of that line compared with the length of another line, both being measured from the same scale; if we call the subdivisions of the scale feet or yards, the line will *represent* a line of as many feet or yards as it contains such subdivisions; to find the *absolute* measure of a line in yards or feet, we must evidently apply a scale of actual yards or feet to it.

[s] Any scale of equal parts may be employed for this purpose, but it will be proper to choose one that will bring the proposed figure within the limits you intend it to occupy; every part (viz. every line) of the figure must be measured by one scale, and not one line of the figure by one scale, and another line by another.

Rule I. With any distance in the compasses greater than half the given line, let *arcs* be described from the centres *A* and *B*, cutting each other in *C* and *D*.

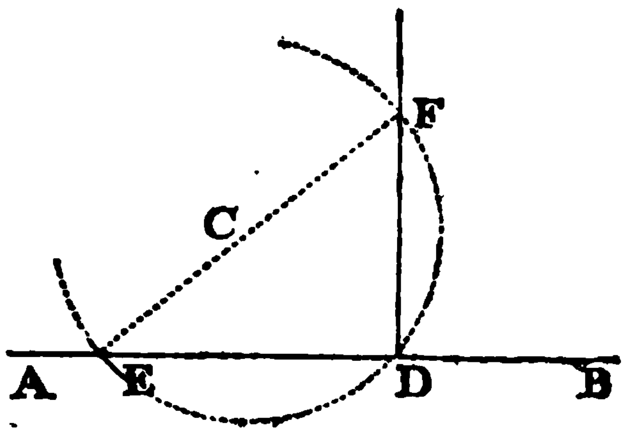
II. Draw a straight line from *C* to *D*, and it will bisect the given straight line, as was required [t].



259. *From a given point D, in a given straight line AB, to erect a perpendicular FD.*

Rule I. From any point *C* (without *AB*) as a centre, with the distance *CD*, describe the circle *EDF* cutting *AB* (produced if necessary) in *E* and *D*, and draw the diameter *ECF*.

II. Join *FD*, and it will be perpendicular to *AB*, as was required [u].



By the Protractor.

Lay the centre of the protractor on *D*, and let the 90 on its circumference exactly coincide with the given line; draw the line *FD* along the radius, and it will be the perpendicular required.

259.B. *From a given point F, to let fall a perpendicular to a given straight line AB.* See the preceding figure.

Rule I. In *AB* take any point *E*, join *FE*, and bisect it in *C*, (Art. 258.)

II. From *C* as a centre with the distance *CF* or *CE*, describe

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[t] If the points *AC* and *BC* be joined, this rule may be proved by Euclid 8. 1.

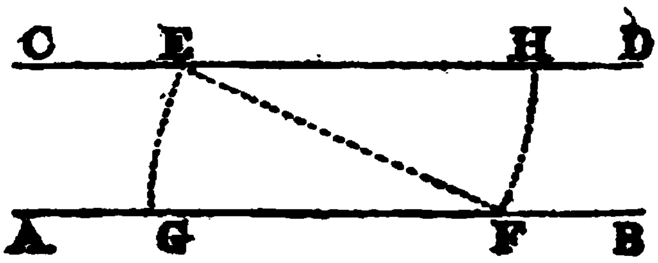
[u] The proof of this rule depends on Euclid 31. 3. Of the various methods for erecting a perpendicular, given by writers on Practical Geometry, this is the most simple and easy.

the circle *EDF*; join *FD*, and it will be the perpendicular required [x].

260. *Through a given point E to draw a straight line parallel to a given straight line AB.*

Rule I. Take any point *F* in *AB*, and from *E* and *F* as centres, with the distance *EF*, describe the *arcs EG*, *FH*.

II. Take the distance *EG* in the compasses, and apply it from *F* to *H* on the *arc FH*.



III. Through *E* and *H* draw the straight line *CD*, and it will be parallel to *AB* as was required [y].

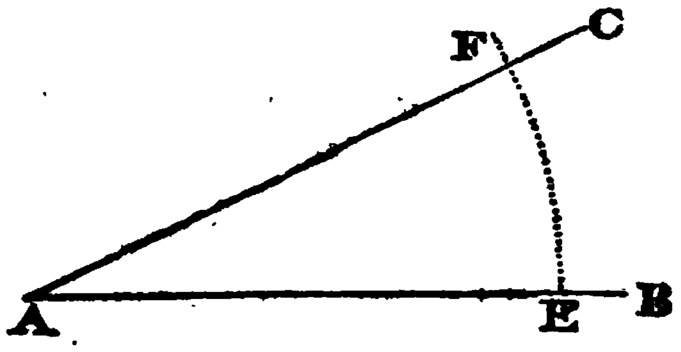
By the Parallel Ruler.

Lay the ruler so, that the edge of one of its parallels may exactly coincide with the line *AB*. Holding it steady in that position, move the other parallel up or down until it cut the point *E*, through which draw a line *CED*, and it will be parallel to *AB*.

If *E* be too near, or too distant for the extent of the ruler, first draw a line parallel to, and at any convenient distance from *AB*, to which draw a parallel through *E* as before, and it will be parallel to *AB*.

261. *At a given point A, in a given straight line AB, to make an angle BAC, which shall measure any given number of degrees.*

Rule I. Extend the compasses from the beginning of the scale of chords (marked *C*,) to the 60th degree, and from the given point *A*, with this distance, describe an *arc* cutting *AB* (produced if necessary) in *E*.



II. Extend the compasses from the beginning of the scale of

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[x] This depends on Euclid 31.3.

[y] Since the *arcs EG*, *HF* are equal, the angles *EFG*, *FEH* at the centres are equal, (Euclid 27.3.) and therefore *AB* is parallel to *CD*, Euclid 27.1.

chords, to the number denoting the measure of the proposed angle, and from *E* as a centre, with this distance, cut the above *arc* in the point *F*.

III. Through *F* draw the straight line *AB*, and the angle *BAC* will be the angle required [a].

EXAMPLES—1. Let the angle proposed measure 30 degrees.

Having described the arc EF with the radius 60, extend the compasses from the beginning of the scale to 30; lay off this extent from E, and draw a line through the point marked with the compasses, and the angle of 30° will be made.

2. At the point *A* in *AB* make an angle measuring 150 degrees [b].

Here the proposed angle being greater than 90, it will be convenient to take it at twice; lay off 80° first, on EF; then from F, lay off 70° more; draw a line through the extremity of the 70, and it will make with AB an angle of 150 degrees.

BY THE PROTRACTOR.

Lay the central point on *A*, and the fiducial edge of the radius along *AB*, so that they exactly coincide; then with the pointer, make a fine dot, opposite the proposed degree (reckoning from the line *AB*) on the circumference; through *A* and this dot, draw a straight line, and it will make with *AB* the angle required.

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[a] If the circumference of a circle be divided into 360 equal parts called degrees, one sixth part of the circumference will measure 60 degrees, and its chord will be equal to the radius of the circle (Euclid 15. 4.); wherefore, if the first 60 degrees on any scale of chords be taken in the compasses, and a circle be described with that distance as radius, the chords on the scale, will be the proper measure for the chord of every *arc* of that circumference, as well as for the circumference itself; and since the *arc* intercepted between the legs of the angle, (being described from the angular point as a centre,) is the measure of the angle it subtends, (Euclid 33. 6. Art. 236.) the rule is manifest. By this problem an angle may be made, equal to any given angle.

[b] To measure, or lay down, an angle greater than 90°, the *arc* must be taken in the compasses *at twice*; thus for 100°, take 60° first, and then 40°; or 50° first, and then the remaining 50°, &c. For an *arc* of 170° take 90° and 80°, or 50°, 50°, and 70°, viz. at three times, &c. &c. If two straight lines cut one another within a circle, their angle of inclination is measured by half the sum of the intercepted *arcs*; but if they cut without the circle, their angle of inclination is measured by half the difference of the intercepted *arcs*. See the note on Art. 243.

EXAMPLES. **Make at given points, in given straight lines, the following angles, viz. of 20°, 35°, 45°, 58°, 90°, 160°, and 171°½.**

262. *To measure a given angle BAC.* See the preceding figure.

RULE I. From the angular point *B* as a centre, with 60° from the scale of chords as a radius, describe the *arc EF,* cutting the legs of the given angle (produced if necessary) in *E* and *F.*

II. Extend the compasses from *E* to *F*, and apply the extent to the scale of chords, so that one point of the compasses be on the beginning of the scale; then the number to which the other point reaches will denote the measure of the given angle [b].

EXAMPLE. To measure the angle *BAC.*

Having with the radius 60° described the arc EF, extend the compasses from E to F; then if this extent reaches from the beginning of the scale to 35°, the angle BAC measures 35 *degrees.*

BY THE PROTRACTOR.

Lay the fiducial edge on *AB,* so that the central notch may

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[b] The reason of this rule will be evident from the preceding note. An ingenious method of measuring angles, by means of an undivided semicircle, and a pair of compasses, without the assistance of any scale whatever, was proposed by M. De Lagni, in the memoirs of the French Academy of Sciences; some account of his method may be found in Dr. Hutton's Mathematical Dictionary, under the word *Goniometry*. Thomas Fantet De Lagni was born at Lyons in the 17th century, and died in 1734; he was successively professor royal of Hydrography at Rochford, sub-director of the General Bank at Paris, and associate geometrician and pensioner in the Ancient Academy. De Lagni excelled in Arithmetic, Algebra, and Geometry, sciences which are indebted to him for improvements; he invented a binary Arithmetic, requiring only two figures for all its operations; likewise some convenient approximating theorems for the solution of higher equations, particularly the irreducible case in cubics. He gave a general theorem for the tangents of *multiple-arcs*, and determined the ratio of the circumference of a circle to its diameter to 120 places, which is the nearest approximation for the purpose, that has been made. Our author was particularly fond of calculating, and it may be truly said of him, that "He felt the ruling passion strong in death;" for on his death bed, when he was apparently insensible, one of his friends asked him, What is the square of 12? to which he immediately replied, 144; we regret, that the last moments of this ingenious man, were not employed on subjects of infinitely greater importance.

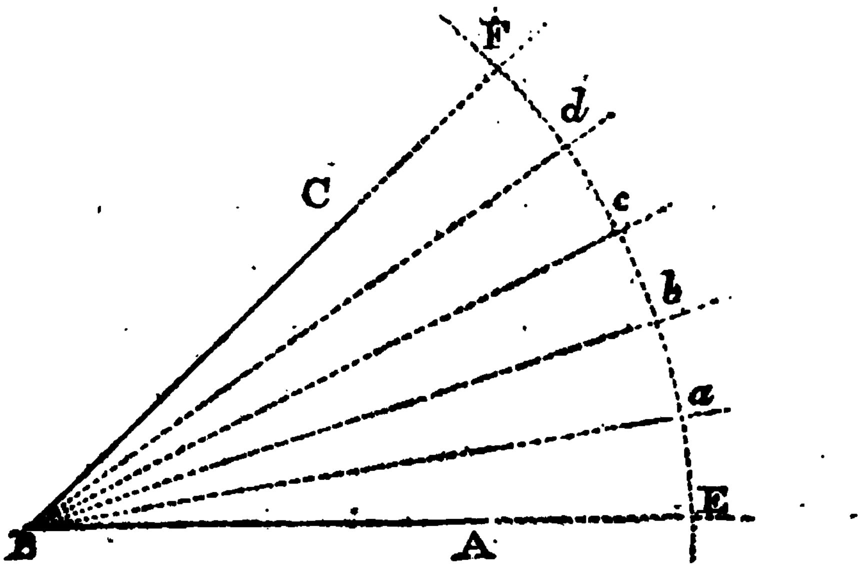
be on A, then will the degrees (on the circumference) intercepted between AB and AC, be the measure of the angle.

EXAMPLE. To measure the angle BAC by the protractor.

Let the centre coincide with A, and the fiducial edge with AB; count the degrees (on the circumference) from AB to AC, and the number will be the measure of BAC.

263. *To divide a given angle ABC into any number of equal parts.*

RULE I. From the angular point B as a centre, with the radius 60° (from the scale of chords,) describe the *arc EF* as before, and find the measure of the angle *ABC*.



II. Divide the number of degrees in this measure by the number denoting the number of parts into which the angle is to be divided, and the quotient will be the degrees each part will measure.

III. Extend the compasses, from the beginning of the scale of chords, to the degree denoted by the above quotient, and apply this extent successively along the *arc EF*.

IV. Through B and each of these divisions, draw straight lines *Ba, Bb, Bc, Bd,* &c. and the angle *ABC* will be divided, as was proposed *.

EXAMPLE. To divide the angle *ABC* into 5 equal parts.

Having described EF with the radius 60°, *let EF measure*

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* If either of the lines *BC, BA* be less than the proposed radius, (viz. the chord of 60°) it must be produced to the circumference *EF*; likewise *BC, BA* may be either, or both, so long, that *EF* cuts them; in either case the rule is the same as is plain. See the note on Art. 261. So to measure an angle with the protractor, it will sometimes be necessary to produce the lines containing the angle, until they meet the circumference of the instrument; this may be done with a lead pencil, and the produced parts may be rubbed out, after the angle is measured.

suppose 55 *degrees, then* $\frac{55}{5}=11°=$ *the number of degrees in each of the parts; take* 11° (*from the scale of chords*) *in the compasses, and apply it from E to a, from a to b, from b to c, and from c to d; and through the points a, b, c, and d, draw Ba, Bb, Bc, and Bd, and ABC will be divided into* 5 *equal parts.*

264. In like manner the whole circumference may be divided into any number of equal parts, and by joining the points of division, polygons of any number of sides may be inscribed in it; and if straight lines be drawn perpendicular to the several radii which pass through the points of division, at their extremities, polygons of the same number of sides will be circumscribed about the circle, as is evident.

By the Protractor.

Let the fiducial edge coincide with the diameter of the circle, and the central notch with the centre, and suppose a polygon of 36 equal sides be required to be inscribed in the circle, mark with the pointer opposite every 10th degree (on the protractor); draw straight lines from the centre to these points, and join the points where they cut the circumference; and a polygon of 36 sides will be inscribed: and if at the extremities of these radii, and perpendicular to them, lines be drawn meeting each other, a polygon will be circumscribed about the circle, similar to the former; and by a similar method, any other regular polygon may be inscribed, or circumscribed.

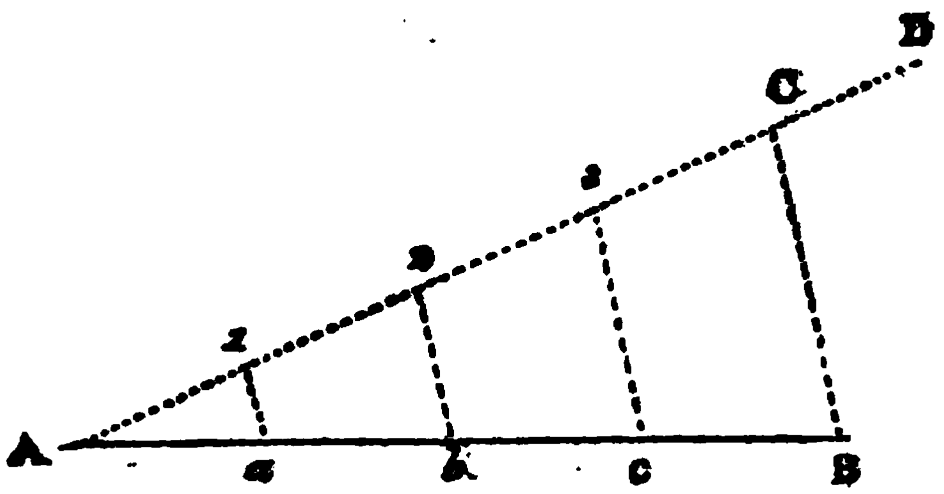
Examples—1. To inscribe in, and circumscribe about, a given circle, an equilateral triangle, and a square.

2. To inscribe in, and circumscribe about, a circle, regular polygons of 10, 15, 20, 24, and 30 sides, respectively.

265. *To divide a given straight line AB into any number of equal parts.*

Rule I. Draw the straight line *AD* making any angle with *AB*.

II. Beginning at *A*, with any extent in the compasses, take as many equal divisions (*a*1, 12, 23, 3*c*, &c.) in *AD*, as *AB* is to be divided into, let these terminate at *C*, and join *CB*.



III. Through these divisions draw straight lines parallel to *CB*, and cutting *AB* in the points *a*, *b*, *c*, &c. these will divide *AB* into the number of equal parts required [d].

EXAMPLES—1. It is required to divide a given line *AB* into 4 equal parts.

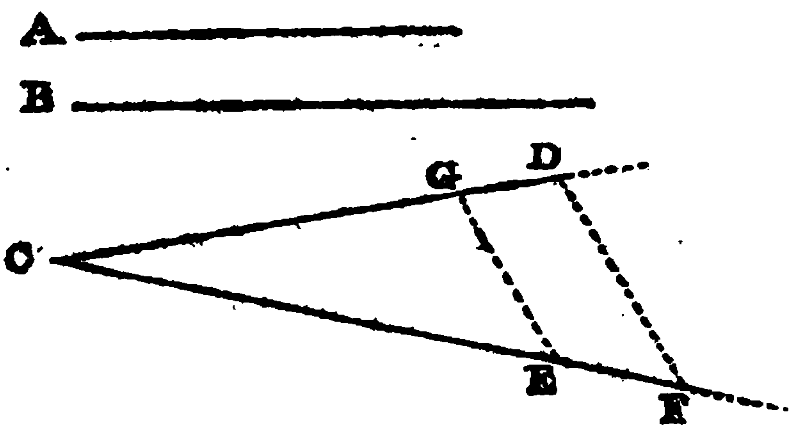
First, draw an indefinite line AD, making any angle (DAB) with AB. Secondly, open the compasses to any convenient extent, (as A1) and with it lay off the equal distances A, 1; 1, 2; 2, 3 and 3, C. Thirdly, join CB, and through 3, 2, and 1, draw 3 c, 2 b, 1 a each parallel to CB, (Art. 260.) then will AB be divided into 4 equal parts in a, b, and c.

2. To divide a line of $4\frac{1}{2}$ inches in length into 10 equal parts.

Note. By this problem any straight line may be divided into parts which are proportional to those of a given divided straight line [e].

266. *To find a third-proportional to two given straight lines A and B.*

RULE I. Draw two indefinite straight lines *CD*, *CF*, making any angle *DCF*.



II. In these, take *CG* equal to *A*, *CD* and *CE* each equal to *B*, and join *GE*.

III. Through *D* draw *DF* parallel to *GE* (Art. 260.) and *CF* will be the third proportional required; that is, (*CG* : *CE* : : *CD* : *CF*, or) *A* : *B* : : *B* : *CF* [f].

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[d] The reason of this rule will appear from Euclid 10. 6. it is preferable to the complex methods proposed by some of the modern writers.

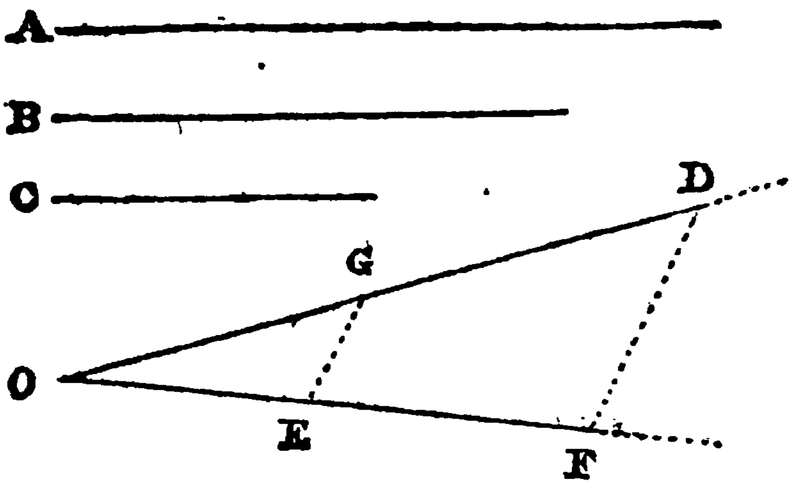
[e] See Euclid 10. 6.

[f] This is the same with Euclid 11. 6.

267. *To find a fourth proportional to three given straight lines A, B, and C.*

RULE I. Draw two indefinite lines *OD*, *OF*, as before.

II. Take *OD* equal to *A*, *OF* equal to *B*, and *OG* equal to *C*.

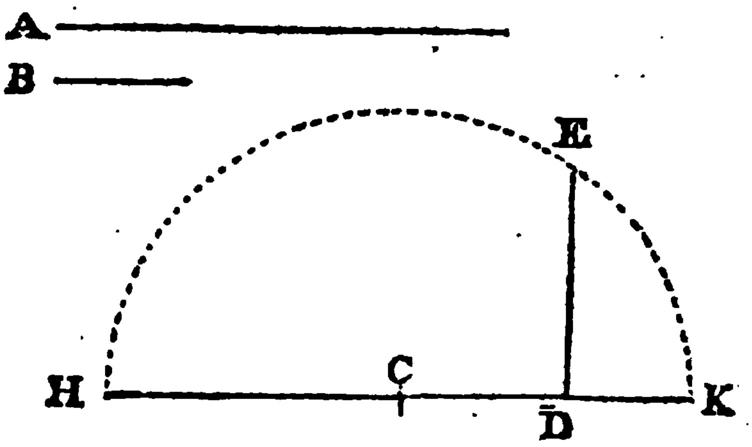


III. Join *DF*, and through *G* draw *GE* parallel to *DF* (Art. 260.), and *OE* will be the fourth proportional required; for (*DO* : *OF* : : *GO* : *OE*, that is) *A* : *B* : : *C* : *OE* [g].

268. *To find a mean proportional between two given straight lines A and B.*

RULE I. Draw the indefinite straight line *HK*, and in it take *HD* equal to *A*, and *DK* equal to *B*.

II. Bisect *HK* in *C* (Art. 258.), and from *C* as a centre with the distance *CH* (=*CK*) describe the semicircle *HEK*.



III. Through *D*, draw *DE* perpendicular to *HK*, (Art. 259.) and it will be the mean proportional required; for (*HD* : *DE* : : *DE* : *DK*, that is) *A* : *DE* : : *DE* : *B* [h].

269. *To find the centre of a given circle ABD.*

RULE I. Draw any straight line *BD* in the given circle, and bisect it in *H*, (Art. 258.)

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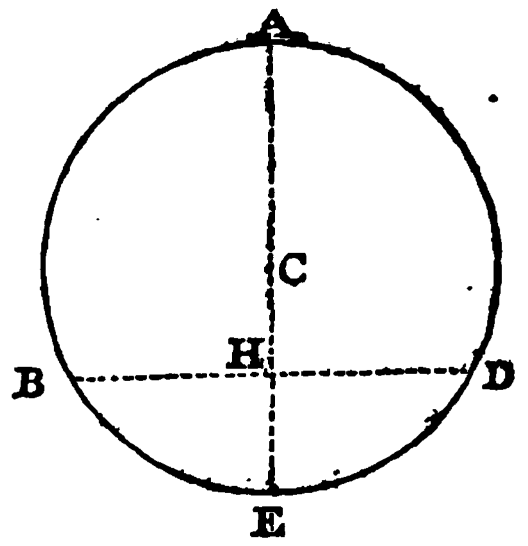
[g] This is the same with Euclid 12. 6.

[h] This is Euclid's 13. 6.

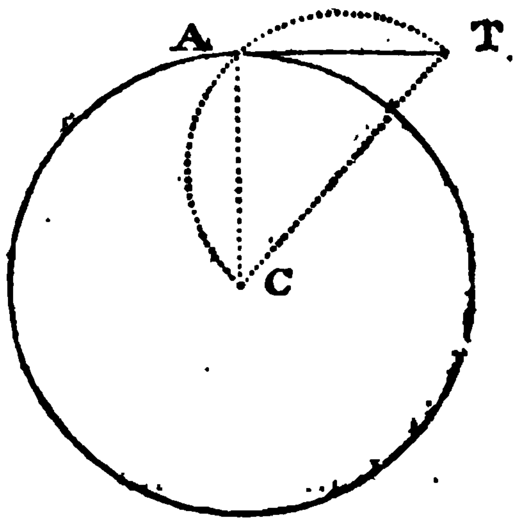
II. Through H draw AH perpendicular to BD, (Art. 259.) and produce it to E.

III. Bisect AE in C, (Art. 258.) the point C will be the centre of the given circle [i].



270. *To draw a tangent to a circle from any given point, either in the circumference, or without the circle.*

RULE I. Find the centre C, (Art. 269.) and let T be a given point without the circle, from which the tangent is required to be drawn.



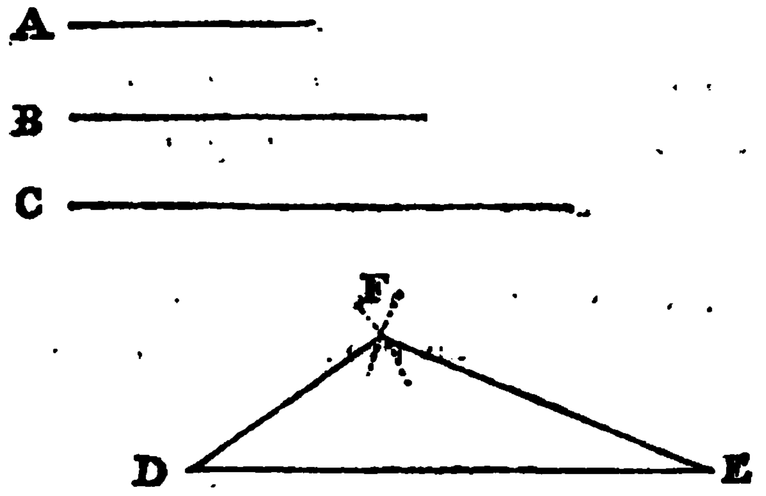
II. Join CT, and on it describe the semicircle CAT.

III. Join AT, and it will touch the circle as was required.

IV. If the tangent be required to be drawn from any point A in the circumference, join CA, and draw AT perpendicular to it (Art. 259.); AT will touch the circle [k].

271. *To describe a triangle, having its three sides given.*

RULE I. Let A, B, and C, be the three sides of the required triangle, draw a straight line DE equal to one of them, suppose A, (Art. 256.).



II. Take the length of the line B in the compasses, and from D as a centre, with this distance, describe an *arc*.

III. From E as a cen-

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[i] This rule depends on Euclid 1. 3. Other methods may be derived from Euclid 19, 3; 31, 3; 32, 3; and various other parts of the Elements.

[k] This depends on Euclid 31. 3. and 16. 3.

tre, with the length of the line C in the compasses, describe an *arc*, cutting the former *arc* in F.

IV. Join DF, EF; and DEF will be a triangle, having its sides respectively equal to A, B, and C [l].

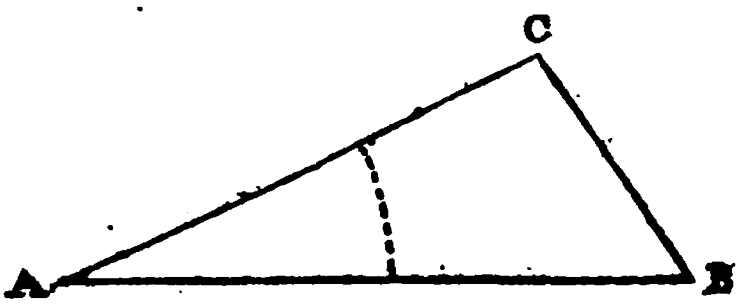
EXAMPLES.—1. Describe a triangle of which the sides are 4, 3, and 2, respectively, and measure the angles. *Ans.* $103^\circ\frac{1}{2}$, 47°, *and* $29^\circ\frac{1}{2}$.

2. Describe a triangle, the sides of which are 25, 36, and 47, and find the measure of its angles.

272. *To describe a triangle having two sides and the included angle given.*

RULE I. Draw a straight line AB equal to one of the given sides.

II. At the point A, make the angle BAC equal to the proposed angle, (Art. 261.); and make AC equal to the remaining given side.



III. Join BC, and ABC will be the triangle required [m].

EXAMPLES.—1. Given $AB=8$, $AC=6$, and the angle $BAC=30^\circ$; to describe the triangle, and measure the remaining side CB, and likewise each of the angles C and B. *Ans. side* $CB=4.25$, *ang.* $C=100^\circ$, *ang.* $B=50$.

2. Given 2 sides equal to 210 and 230 respectively, and the included angle 105° to find the rest.

273. *To describe a triangle having two sides AB, AC, and an angle ABC, opposite to one of them, given.*

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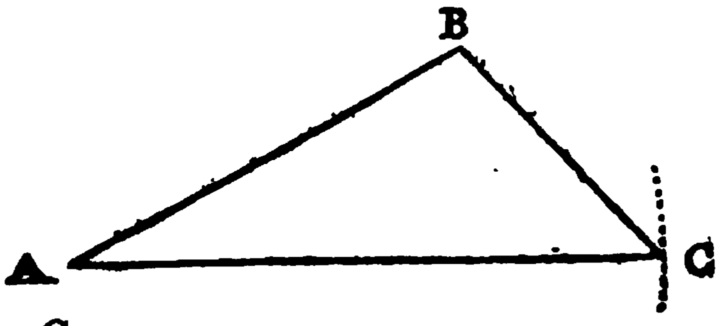
[l] The proof of this rule may be found in Euclid 22. I.

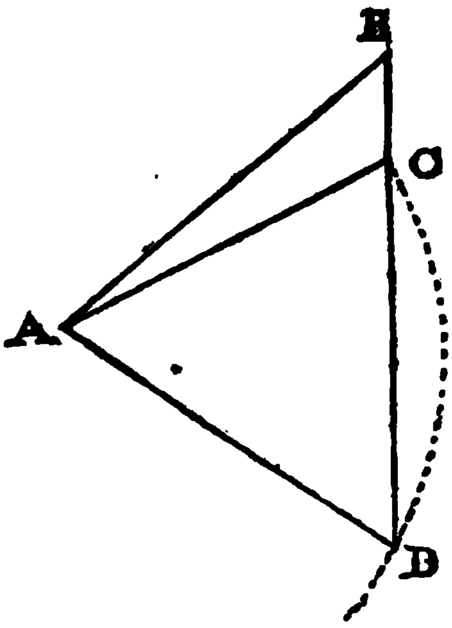
[m] This rule and the two next are sufficiently obvious.

RULE I. Draw the side AB, and at its extremity B make an angle ABC equal to the proposed angle (Art 261.); and produce the line BC.



II. From A as a centre, with the given length of AC in the compasses, describe an *arc*, cutting BC in C.

III. Join AC, and ABC will be the required triangle.



Note. If the given angle be (a right angle, or obtuse, viz.) opposite the *greater* given side (as in fig. 1.), the *arc* will cut BC (on the same side of B), in one point C only; but if the given angle be (acute, viz.) opposite the *less* side (as in fig. 2.), the *arc* will cut BC in two points C, D; and either of the triangles ABC or ABD will answer the proposed conditions; hence this case is ambiguous.

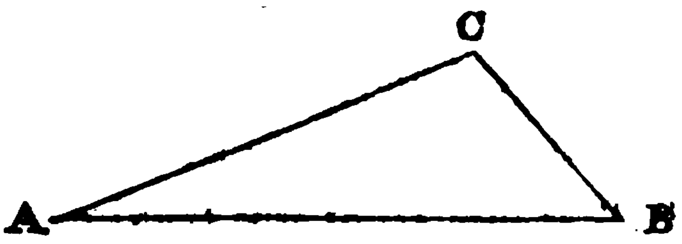
EXAMPLES.—1. Given $AB=195$, $AC=291$, and the angle $ABC=90^\circ$ (fig. 1.); to construct the triangle, and determine (instrumentally) the remaining side and angles. *Ans.* $BC=216$. *ang.* $A=48^\circ$, $C=42^\circ$.

2. Given $AB=136$, $AC=53$, and the angle $B=22^\circ\frac{1}{2}$ (fig. 2.) to find the rest. *Ans.* $BC=117$. *ang.* $BCA=99^\circ$, *ang.* $BAC=58^\circ\frac{1}{2}$, *or* $BD=183\frac{1}{2}$, *ang.* $D=81^\circ$, *ang.* $BAD=76^\circ\frac{1}{2}$.

274. *To describe a triangle, having two angles, and the adjacent side, given.*

RULE I. Draw a straight line AB, equal to the given side.

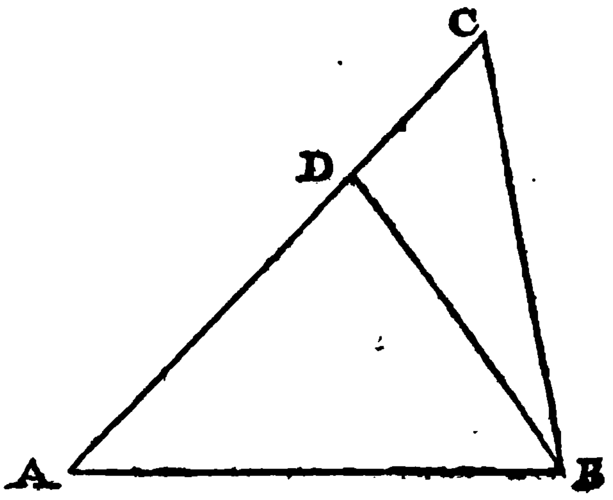
II. At A and B respectively, make angles CAB, CBA equal to the given angles (Art. 261.); and produce AC, BC, to meet in C; ABC will be the triangle required.



EXAMPLES.—1. Given $AB=72$, ang. $B=22^\circ\frac{1}{2}$, ang. $A=20^\circ$, to make the triangle, and find the rest. *Ans.* $AC=89\frac{1}{2}$, $CB=36\frac{1}{2}$, ang. $C=137^\circ\frac{1}{2}$.

2. Given $AB=10$, ang. $A=45^\circ$, ang. $B=50^\circ$, to construct the triangle, and find the rest.

275. *To describe a triangle, having two angles and a side opposite one of them, given.*



RULE I. Add the two given angles together, and subtract their sum from 180° (see Art. 236.B.).

II. Draw AB equal to the given side, and at the point A, make the angle BAC equal to the above remainder (Art. 261.).

III. At the point B, make the angle ABC equal to one of the given angles; then will ACB be the other, and the triangle will be described [n].

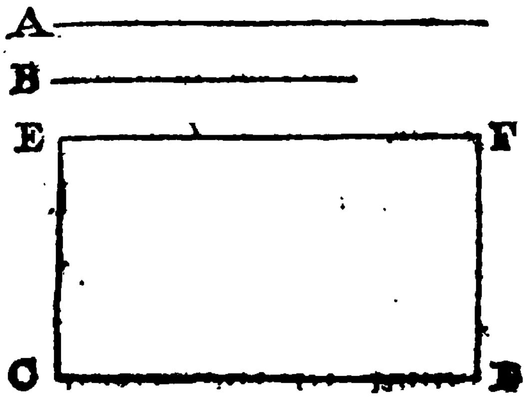
Note. If AB be opposite the *less* angle, then ABC is the triangle; but if AB be opposite the *greater*, then ABD will be the triangle required.

EXAMPLES.—1. Given $AB=40$, the angle $ABC=80^\circ$, and the angle $ACB=70^\circ$, to describe the triangle, and find the rest. *Ans.* $AC=86$, $BC=45$, *ang.* $A=30^\circ$.

2. Given $AB=40$, and two angles $=100^\circ$, and 40°, to make the triangle, and determine the rest.

276. *To describe a rectangle, the sides of which are given.*

RULE I. Let A be one side of the rectangle, and B the other; draw CD equal to A.



II. At the point C, draw CE perpendicular to CD (Art. 259.); and make it equal to B.

III. Through E draw EF parallel to CD (Art. 260.), through D draw DF parallel to CE, and $ECDF$ will be the rectangle contained by A and B, as was required [o].

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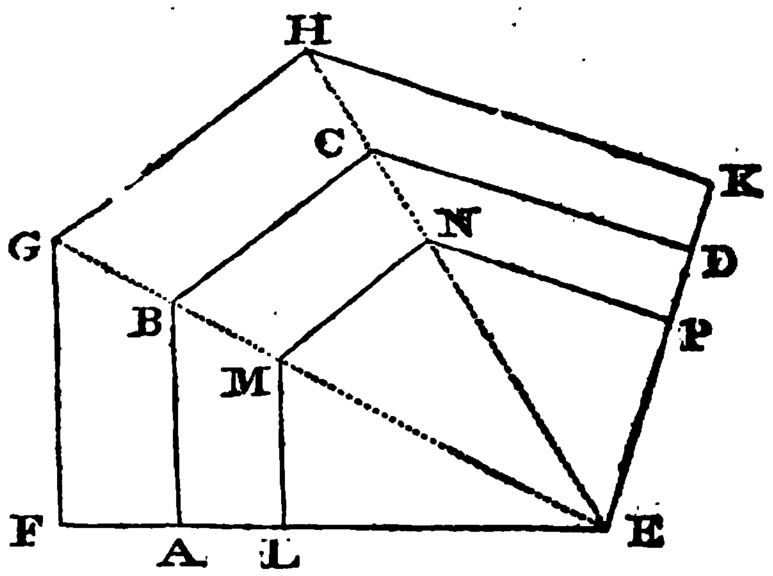
[n] The three angles of a triangle are together equal to two right angles (Euclid 32. I.) that is, to 180°; wherefore if the sum of two angles of a triangle be subtracted from 180°, the remainder will be the third angle.

[o] The proof of this problem may be inferred from Euclid 46. I.

In like manner a square may be described on a given line CD, by making CE equal to CD [p].

277. *To make a figure, similar to a given rectilineal figure; having the sides of the former greater, or less, in any ratio, than those of the given figure.*

RULE. I. Let $ABCDE$ be the given figure, draw the lines EB, EC, &c. from any one of the angles E, to the other angles B and C; and first, let it be required to increase the figure, to another whose side is EF.



II. Produce EA, EB, EC, and ED, to F, G, H, and K; and draw FG parallel to AB, GH to BC, and HK to CD (Art. 260.); $EFGHK$ will be similar to the given figure $ABCDE$.

III. In like manner, if it be required to lessen the figure, to another whose side is EL; through L draw LM, MN, and NP respectively parallel to AB, BC, and CD (Art. 260.); and $LMNPE$ will be similar to $ABCDE$ [q].

278. *To make a regular polygon of any number of sides, on a given straight line AB.*

RULE I. Let n=the number of sides of the polygon to be

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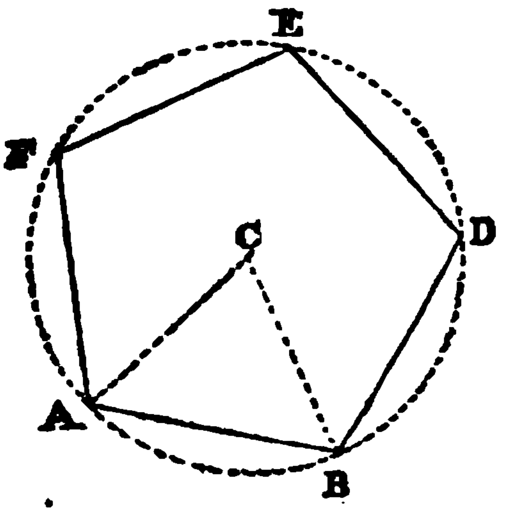
[p] See Euclid 46. 1.

[q] The truth of this construction is evident, for the triangles ELM, EAB, EFG, being equiangular, $EL : LM :: EA : AB :: EF : FG$ (Euclid 4. 6.) in like manner it may be shewn that the sides about the remaining equal angles of the figures are proportionals, wherefore (Euclid def. 1. 6.) the three figures are similar.



described, then will the sum of its interior angles be $=2n-4$ right angles, and each of its angles $=\frac{2n-4}{n}$ right angles [r].

II. At the points A and B make the angles BAC, ABC each equal to half the above angle, that is $=\frac{n-2}{n}$ (Art. 261 [s].).

III. From the point C where these lines intersect, with the distance $CA=CB$, describe a circle.

IV. Take the distance AB in the compasses, and apply it to the circumference (as AF, FE, ED, &c.), which will contain it as many times exactly, as the proposed polygon has sides; draw the straight lines AF, FE, ED, &c. and the polygon will be described.

EXAMPLES.—1. To make a regular pentagon on AB.

Here $n=5$, $\because \frac{n-2}{n}=(\frac{3}{5}$ *of a right angle* $=\frac{3}{5}$ *of* $90^{\circ}=)\ 54^{\circ}$.

Make BAC, ABC *each* $=54^{\circ}$; *from the centre* C *with the radius* CB *or* CA *describe the circle* EAB, *then* AB *taken in the compasses, and applied to the circumference, will meet it in the points* $ABDEF$ *and* A; *which points being joined, the pentagon will be described as proposed.*

2. To make a hexagon, and a heptagon on AB.

For the hexagon, $n=6$; $\because \frac{n-2}{n}=(\frac{2}{3}$ *of a right angle* $=)\ 60^{\circ}$ $=BAC$.

For the heptagon, $n=7$; $\because \frac{n-2}{n}=(\frac{5}{7}$ *of a right angle* $=) 64^{\circ}\frac{2}{7}$ $=BAC$; *and proceed for both figures as before.*

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[r] This depends on cor. 1. 32. 1. of Euclid.

[s] That the lines CA, CB drawn from the centre to the angular points A and B bisect the angles FAB, ABD, appears from Euclid book 4; viz. in the equilateral triangle, prop. 5; in the square, prop. 6; in the regular pentagon, prop. 14; and in the regular hexagon, prop. 15; and the same may be proved of any regular polygon whatever.

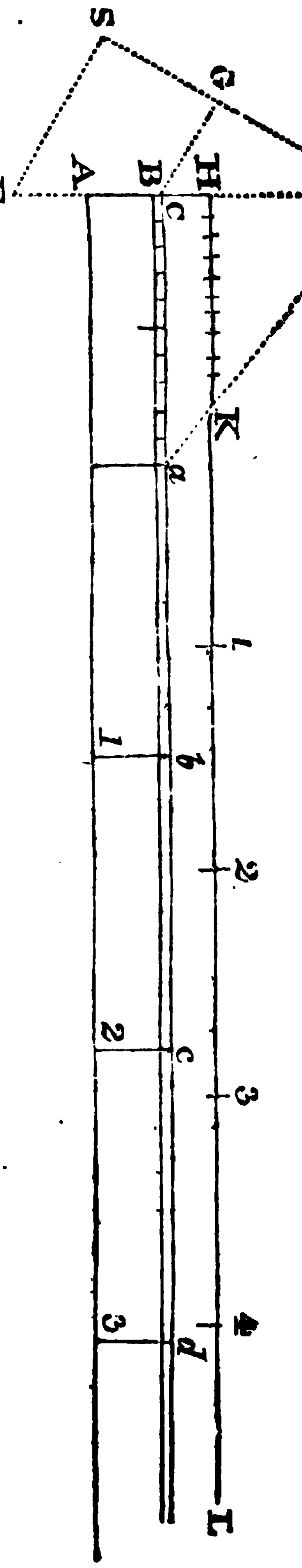
279. *To construct a scale of equal parts.*

RULE I. Draw three lines *A*, *B*, and *C*, at convenient distances, and parallel to one another (Art. 260.); and in *C*, take the parts *Ca*, *ab*, *bc*, *cd*, &c. equal to one another.

II. Through *C*, draw *DCE* perpendicular to *Ca* (Art. 259.); and through *a*, *c*, *d*, &c. draw lines parallel to *DCE*, cutting the parallels *A*, *B*, and *C*; the distances *Ca*, *ab*, *bc*, *cd*, &c. are called the *primary divisions* of the scale.

III. Divide the left hand primary divisions *Ca*, into 10 equal parts (Art. 265.); and draw lines through these points, parallel to *DCE*, across the parallels *B* and *C*; this primary division will be divided into 10 equal parts, called *subdivisions* of the scale.

IV. Number the primary divisions from left to right, viz. 1, 2, 3, &c. and the scale will be complete.



280. *To make a scale of which any number of its subdivisions will be equal to an inch.*

RULE I. Let one of the primary divisions *Ca*, of the scale *C*, be an inch; and let it be divided into 10 equal parts, as above.

II. From any point *D* in *AD*, draw *Da*; draw *DS* making any angle with *DA*, and make *DS*=*Ca*.

III. Take the number of subdivisions (which are proposed to make an inch) in the compasses from the scale *C*, and apply this distance from *D* to *E*.

IV. Draw *ES*, and through *C* draw *CG* parallel to *ES*, and make *DH*=*DG*.

V. Through H, draw HL parallel to Ca, cutting Da in K; then will HK be one of the primary divisions, containing 10 of the parts proposed [t].

VI. If lines be drawn through D to each of the subdivisions in Ca, it will divide the line HK into 10 equal parts (Art. 224.), which will be the subdivisions of the scale HL; and if the successive distances $K1$, 12, 23, 34, &c. be taken in KL, each equal to HK, these will form the primary divisions, and the scale HL will be constructed.

EXAMPLES.—1. To construct a plane scale, having 20 of its subdivisions equal to an inch.

Take the distance Cb (=2 inches =20 subdivisions of Ca) in the compasses, make $DE=Cb$, $DS=Ca$, and proceed as before.

2. To construct a scale of which 35 subdivisions make an inch.

Extend the compasses from d backwards to the 5th subdivision between C and a, this extent (=35 subdivisions of the scale Cd) being applied from D in the straight line DE, proceed as before.

3. To make scales of which 15, 25, 30, and 40 respective subdivisions will equal an inch.

281. *To construct scales of chords, sines, tangents, secants, &c.*

RULE I. With any convenient radius CA describe the circle $ABDE$, draw two diameters AD, BE, perpendicular to each other (Art. 259.), produce EB indefinitely towards F, draw DT parallel to EF (Art. 260.), and join AB, BD, DE, and EA.

II. Divide the quadrant BD into 9 equal parts (Art. 263.), and from the centre C, through each of the divisions, draw straight lines cutting DT in 10, 20, 30, 40, &c. this will be the scale of tangents.

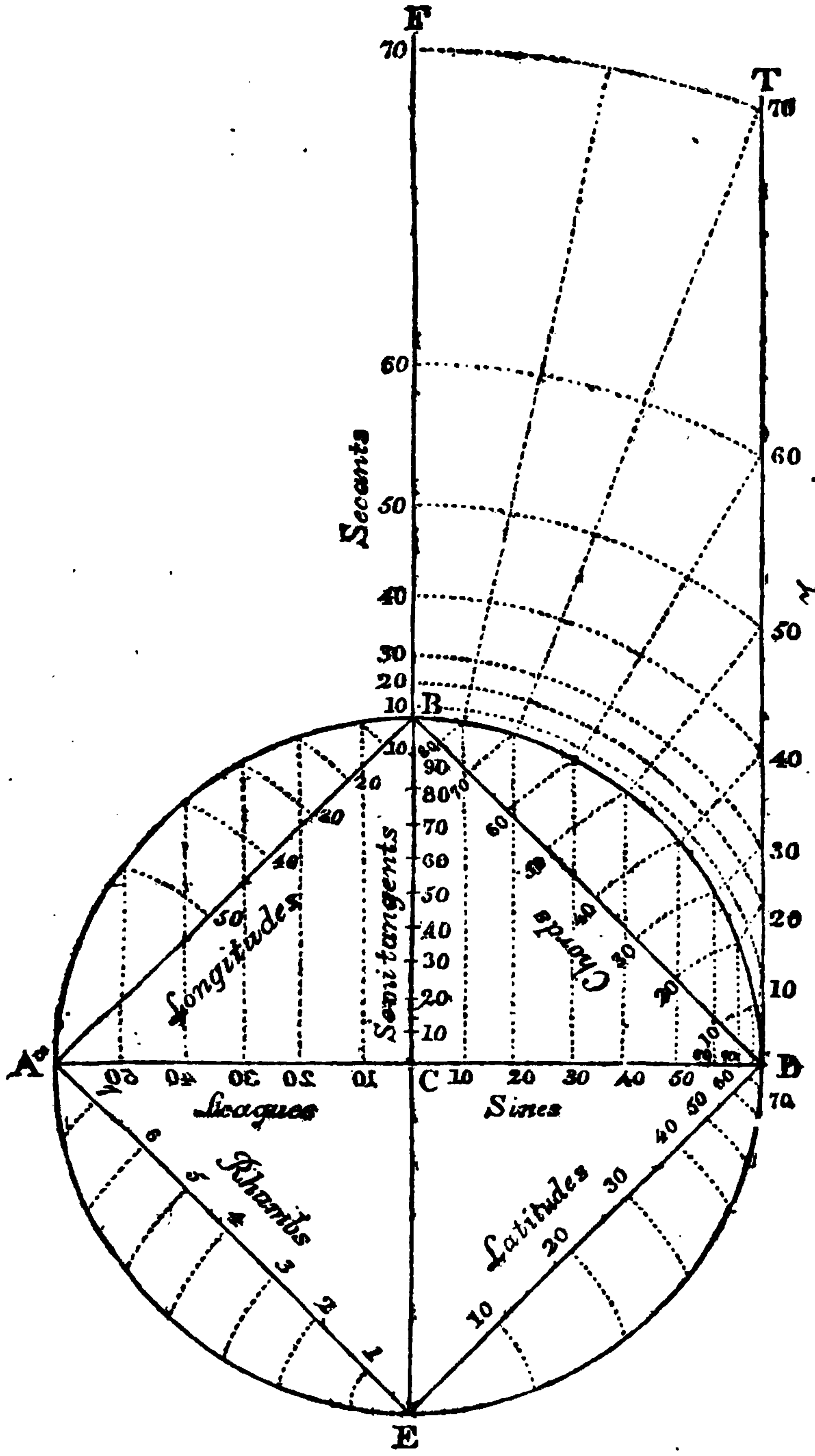
III. From D as a centre, through each of the divisions of the quadrant, describe *arcs* cutting BD in 10, 20, 30, 40, &c. this will be the scale of chords.

---

[t] To demonstrate the truth of this construction, let the number of subdivisions of HK contained in $Ca=Ba$ be called n, also by construction Ca contains 10 subdivisions of itself; $\because DE=n$, $DS=10$; but $DE:DS::DC:(DG=)\ DH$ (4.6.) and $DC:DH::Ca:HK$; $\because DE:DS::Ca:HK$, or $n:10::Ca:HK$, $\because HK=\frac{10Ca}{n}$; let $n=20$ (as in Ex. 1.) then $HK=\frac{Ca}{2}$; let $n=35$ (as in Ex. 2.) then $HK=\frac{2Ca}{7}$, &c. Q.E.D.

IV. Through the divisions of the quadrant, draw lines parallel to *BC*, cutting *CD* in 80, 70, 60, 50, &c. this will be the scale of sines and cosines.

V. If straight lines be drawn from *A* to the several divisions (10, 20, 30, &c.) of *DT*, cutting the radius in 10, 20, 30, 40, &c. *CB* will be a scale of semi-tangents.



VI. If from the centre *C*, through the several divisions of *DT*, arcs be described, cutting *BF* in 10, 20, 30, &c. *BF* will be a scale of secants.

VII. Divide the radius AC into 60 equal parts, draw straight lines through each of these divisions parallel to CB, cutting the *arc* AB; and from A as a centre, through the points where these parallels cut the quadrant AB, describe *arcs* cutting AB in 10, 20, 30, 40, &c. AB will be a scale of longitudes.

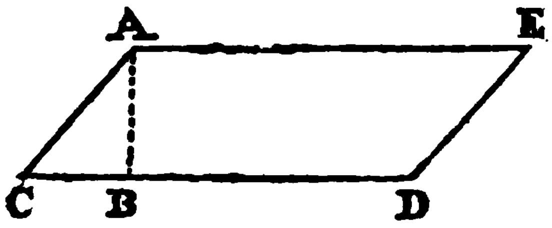
VIII. Divide the quadrant AE into 8 equal parts, and through these, from E as a centre, describe *arcs* cutting AE in 1, 2, 3, 4, &c. AE will be a scale of rhumbs.

IX. Draw straight lines from B, through the several divisions of the scale of sines (CD), these will cut the quadrant ED in as many points; from A as a centre, through each of these points, describe *arcs* cutting ED in 10, 20, 30, &c. ED will be a scale of latitudes.

X. If the above constructions be accurately made, with a circle the radius of which is 2 inches, the several lines will exactly correspond with those on the common scales; wherefore to construct a scale, we have only to take the several lines respectively in the compasses, and apply them (with their respective divisions) to a flat ruler; and what was required will be done.

282. *To find the area of a parallelogram $ACDE$.*

RULE. Let a=the altitude AB, b=the base CD; then will ab=the area required [a].



EXAMPLES.—1. To find the area of a square whose side is 12 inches.

Here $a=12$, $b=12$, *and* $ab=12\times 12=144$ *square inches = the area required.*

2. To find the area of a parallelogram, the base of which is 20 inches, and its altitude 25.109.

Here $a=25.109$, $b=20$, *and* $ab=25.109\times 20=502.18$ *square inches = the area required.*

3. To find the area of a rhombus, whose base is 42, and altitude 23.

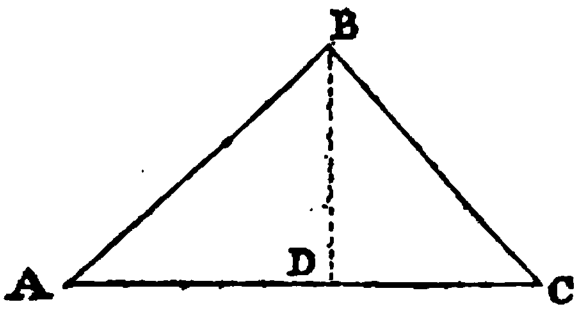
4. To find the area of a rhomboid, whose base is 10, and altitude $7\frac{1}{2}$.

---

[a] Every parallelogram, is equal to the rectangle contained by its base and perpendicular altitude (see Euclid 35. 1; 1, 2, &c.); whence the rule is plain.

283. *To find the area of a triangle ABC.*

RULE. Let fall a perpendicular BD from the vertical angle B to the base AC, and let $a=BD$, $b=AC$, then will $\frac{ab}{2}$=the area required [x].



EXAMPLES.—1. The perpendicular height of a triangle is 28 inches, and its base 16 inches; what is the area?

Here $a=28$, $b=16$, *and* $\frac{ab}{2}=\frac{28\times16}{2}=224$ *square inches, the area required.*

2. The base of a triangle is 1.03, and its perpendicular altitude 2.11, what is the area? *Ans.* 1.08665.

3. The altitude $7\frac{1}{3}$, and the base $8\frac{3}{4}$ being given, to find the area of the triangle.

284. *To find the area of a triangle, having its three sides given.*

RULE. Let a, b, and c, represent the three sides respectively, and let $\frac{a+b+c}{2}=p$; then will $\sqrt{p.\overline{p-a}.\overline{p-b}.\overline{p-c}}$=the area of the triangle [y].

---

[x] This depends on Euclid 41. 1.

[y] Let $AB=a$, $AC=b$, $BC=c$, $AD=x$, then $DC=b-x$, and (Euc. 47. 1.) $c^2-\overline{b-x}|^2=\overline{BD}|^2=a^2-x^2$, or $c^2-b^2+2bx-x^2=a^2-x^2$, whence $x=\frac{a^2+b^2-c^2}{2b}$. But $\overline{BD}|^2=\overline{AB}|^2-\overline{AD}|^2=\overline{AB+AD}.\overline{AB-AD}=(a+\frac{a^2+b^2-c^2}{2b})\times(a-\frac{a^2+b^2-c^2}{2b})=\frac{2ab+a^2+b^2-c^2}{2b}\times\frac{2ab-a^2-b^2+c^2}{2b}=\frac{\overline{a+b}|^2-c^2}{2b}\times\frac{c^2-\overline{a-b}|^2}{2b}$, $\because BD=\frac{1}{2b}\sqrt{(\overline{a+b}|^2-c^2)\times(c^2-\overline{a-b}|^2)}$; and the area $\frac{1}{2}AC\times BD=\frac{1}{4}\sqrt{(\overline{a+b}|^2-c^2)\times(c^2-\overline{a-b}|^2)}=\frac{1}{4}\sqrt{(\overline{a+b+c}.\overline{a+b-c}.\overline{c+a-b}.\overline{c-a+b})}=\sqrt{\frac{a+b+c}{2}.\frac{a+b-c}{2}.\frac{c+a-b}{2}.\frac{c-a+b}{2}}$; this expression, by putting $p=\frac{a+b+c}{2}$, becomes $\sqrt{p.\overline{p-c}.\overline{p-b}.\overline{p-a}}$, which is the rule. Q. E. D.

Cor. If $s=a+b$, and $d=b\backsim c$, then will $\sqrt{\overline{s^2-a^2}.\overline{a^2-d^2}}$ be the rule. *Bonnycastle's Mensuration, p.* 47.

EXAMPLES.—1. To find the area of a triangle, whose sides are 4, 5, and 6.

Here $a=4$, $b=5$, $c=6$, $p=(\frac{a+b+c}{2}=\frac{15}{2}=)\ 7.5$. *and*

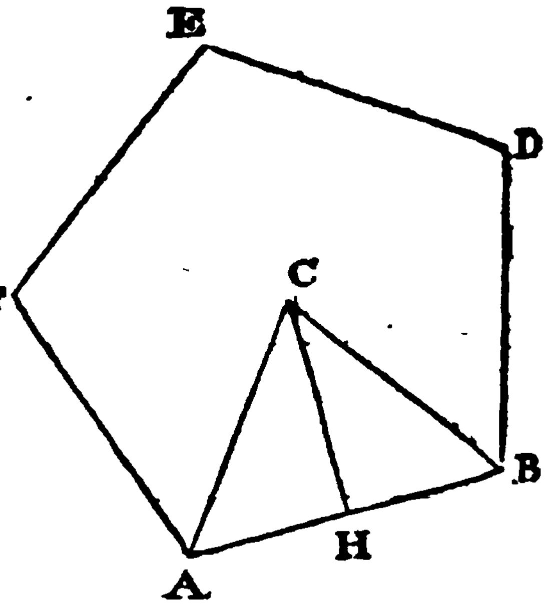
$\sqrt{p.\overline{p-a}.\overline{p-b}.\overline{p-c}}=\sqrt{7.5\times\overline{7.5-4}\times\overline{7.5-5}\times\overline{7.5-6}}=$

$\sqrt{7.5\times3.5\times2.5\times1.5}=\sqrt{98.4375}=9.9215=$ *the area required.*

2. Required the area of a triangle, of which the three sides are 20, 30, and 40, respectively? *Ans.* 290.4737, &c.

3. The sides are 12, 20, and 25, required the area of the triangle?

285. *To find the area of a regular polygon, having one side, and also the number of sides given.*



RULE I. Let *ABDEF* be any regular polygon, bisect the angles *FAB, ABD* by the lines *AC, BC,* and from the point of intersection *C* let fall the perpendicular *CH.*

II. Let $n=$the number of sides of the polygon, $a=CH$, and $b=AB$, then will $\frac{nba}{2}=$the area of the polygon [*].

This rule is given, without a demonstration, in the *Geodesia* of Hero the younger; but the invention is supposed to belong to some preceding, and more profound Geometer. Tartalea is the first among the moderns who introduces the rule, viz. in his *Trattato di Numeri et Misure,* fol. Venice, 1556.

[*] This rule is evident, for the area of each of the triangles will be $=\frac{ba}{2}$ (Art. 283.); but there are n triangles, wherefore the area of their sum, (viz. of the given polygon,) will be $n\times\frac{ba}{2}=\frac{nba}{2}$.

If the side of each of the following figures be *unity,* then will the radius of the inscribed and circumscribed circles be as below:

EXAMPLES.—1. The side of a pentagon is 4, and the perpendicular from the centre 2.01, required the area?

Here $n=5$, $b=4$, $a=2.01$, *and* $\frac{nba}{2}=\frac{5\times4\times2.01}{2}=20.1$, *the area required.*

2. The side of a hexagon is 7.3, and the perpendicular from the centre 6.32 required the area?

Here $n=6$, $b=7.3$, $a=6.32$, *and* $\frac{nba}{2}=\frac{6\times7.3\times6.32}{2}=$ 138.408, *the area required.*

3. To find the area of an octagon, whose side is 9.941, and perpendicular 12. *Ans.* 477.168.

4. To find the area of a heptagon, whose side is 4.845, and perpendicular 5.

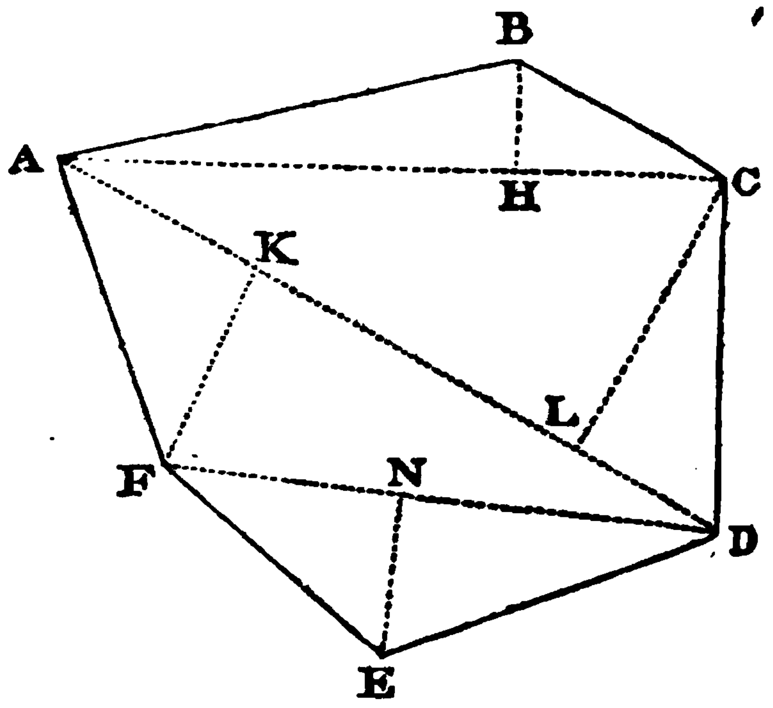
	Inscribed circle.	*Circum. cir.*	*Perp. height.*
Equilateral triangle	0.28867518	0.57735027	0.86602540
Square	0.50000000	0.70710678	....
Pentagon	0.68819096	0. 8506508	1.53884176
Hexagon	0.86602540	1.00000000	....
Octagon	1.20710678	1.30656296	....
Decagon	1.53884176	1.61803398	....
Dodecagon	1.86632012	1.93185165	....

Hence the areas of these figures may be readily found, and likewise those of similar figures, whatever be the length of the given side; since similar polygons are to one another as the squares of their homologous sides, (Euclid 20. 6.) or as the squares of the diameters of their circumscribing circles by 1. 12.

If the square of the side of any regular polygon in the following table, be multiplied into the number standing against its name, the product will be the area.

No. of sides.	*Names.*	*Multipliers.*
3	Trigon, or equilateral triangle	0.433013—
4	Tetragon, or square	1.000000
5	Pentagon	1.720477 +
6	Hexagon	2.598076 +
7	Heptagon	3.633912 +
8	Octagon	4.828427 +
9	Nonagon	6.181824 +
10	Decagon	7.694209—
11	Undecagon	9.365640—
12	Dodecagon	11.196152+

286. *To find the area of any given rectilineal figure ABCDEF.*



RULE I. Join the opposite angles of the figure, viz. *AC*, *AD*, *FD*, so that it may be divided into triangles *ABC*, *ACD*, *ADF*, *FDE*.

II. Find the area of each of the triangles *ABC*, *ACD*, *ADF*, *ADE*, (Art. 283.), and add these areas together, the sum will be the area of the figure *ABCDEF*.

EXAMPLES.—1. Let $AC=10$, $BH=4$, $CL=6$, $AD=12$, $CL=6$, $FD=8$, $EN=3$, and $FK=5$.

$$\text{Then } \frac{AC.BH}{2}=\frac{10\times 4}{2}=\frac{40}{2}=20=\text{area of } ABC.$$

$$\frac{AD.LC}{2}=\frac{12\times 6}{2}=\frac{72}{2}=36=\text{area of } ACD.$$

$$\frac{AD.FK}{2}=\frac{12\times 5}{2}=\frac{60}{2}=30=\text{area of } AFD.$$

$$\frac{FD.NE}{2}=\frac{8\times 3}{2}=\frac{24}{2}=12=\text{area of } FDE.$$

Their sum 98=*area of ABCDEF.*

2. Let $AC=45$, $BH=10$, $AD=50$, $CL=20$, $FD=100$, $EN=20$, and $FK=14$, to find the area. *Ans.* 2075.

287. *The diameter of a circle being given, to find the circumference; or the circumference being given, to find the diameter.*

RULES I. As 7 : 22 } :: the diameter : the circumference [a].
or, as 113 : 355
or, as 1 : 3.1415927

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[a] The first of these proportions is that of Archimedes, which is the easiest, although the least exact, of any of the rules which have been proposed for this purpose; the second proportion is that of Metius; the third is Van Ceulen's rule, and depends on Art. 252, where it is shewn, that if the diameter be 2, the circumference will be 6.2831853, &c. wherefore, if the diameter be 1, the circumference will be 3.1415927 nearly, which is the same as the rule.

II. As 22 : 7
or, as 355 : 113
or, as 3.1415927 : 1
} :: the circumference : the diameter [b].

EXAMPLES.—1. The diameter of a circle is 12, required the circumference?

Thus, as $7 : 22 :: 12 : \frac{22 \times 12}{7} = \frac{264}{7} = 37.714285$ *the circumference nearly.*

Or, as $113 : 355 :: 12 : \frac{355 \times 12}{113} = \frac{4260}{113} = 37.699115$ *the circumference more nearly.*

Or, as $1 : 3.1415927 :: 12 : 3.1415927 \times 12 = 37.6991124$ *the circumference very nearly.*

2. The circumference is 30, required the diameter?

Thus, as $22 : 7 :: 30 : \frac{30 \times 7}{22} = \frac{105}{11} = 9.54545$, &c. *the diameter.*

Or, as $355 : 113 :: 30 : \frac{113 \times 6}{71} = \frac{678}{71} = 9.549295$, &c. *the diameter.*

Or, as $3.1415927 : 1 :: 30 : \frac{30}{3.1415927} = 9.549296$, &c. *the diameter.*

3. The diameter of a circle is 6, required the circumference? *Ans.* 18.8495562, &c.

4. The circumference is 5, required the diameter? *Ans.* 1.5915493, &c.

5. If the diameter be 100, what is the circumference? And if the circumference be 100, what is the diameter?

288. *To find the area of a circle.*

RULE I. Let c = the circumference, d = the diameter, then will $\frac{cd}{4}$ = the area of the circle.

Or, 2nd. $.7854\, d^2$ = the area. Or, 3rd. $.07958\, c^2$ = the area.

EXAMPLES.—1. The diameter of a circle is 4, required the circumference and area?

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[b] These proportions are the converse of the former.

Thus (*Art.* 252.) $3.1415927 \times 4 = 12.5663708 =$ *the circumference.*

Then $\frac{cd}{4} = \frac{12.5663708 \times 4}{4} = 12.5663708 =$ *the area, by rule* 1. (*Art.* 253.)

Or, $.7854\, d^2 = .7854 \times 16 = 12.5664 =$ *the area, by rule* 2.

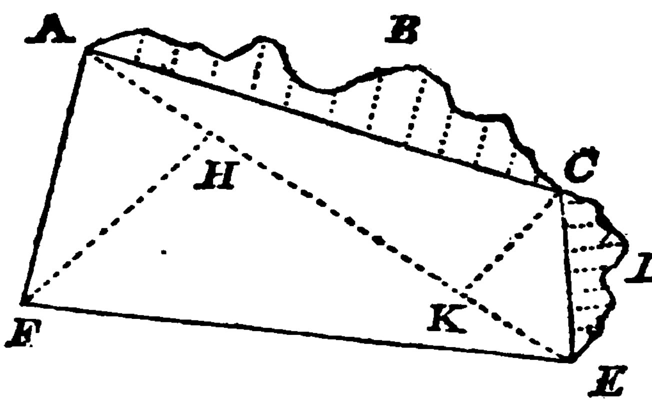
Or, $.07958\, c^2 = (.07958 \times \overline{12.5663708}|^2 = .07958 \times 157.913675$, &c. $=)$ $12.566769 =$ *the area, by rule* 3.

2. Required the area of a circle, whose diameter is 7, and its circumference 22? *Ans.* $38\frac{1}{2}$.

3. What is the area of a circle, whose diameter is 1, and circumference 3.1415927?

289. *To find the area of any irregular mixed figure ABCDEF.*

RULE I. Inscribe the greatest possible rectilineal figure *ACEF* in the proposed figure, and let *ABC*, *CDE* be the remaining irregularly curved boundaries.



II. From as many points as possible in the curve *ABC*, let fall perpendiculars (Art. 259.), to *AC*; and find their sum.

III. Divide this sum by the number of perpendiculars taken, and multiply the quotient by the base *AC*, the product will be the area of the curved space *ABC*.

IV. Proceed in like manner, to find the area of the space *CDE*.

V. Find the area of the rectilineal figure *ACEF* by Art. 286. then lastly, add the three areas together, and the sum will be the area of the figure *ABCDEF* [c].

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[c] This method of approximation is used for measuring fields and other enclosures, which have very crooked and irregular boundaries; the greater the number of perpendiculars be, the nearer truth will the approximation be, as is evident.

To find the area of a regularly tapering board, measure across the two ends, add both measures together, and half the sum multiplied into the length of the board, will give the area.

EXAMPLES.—1. Let $AE=20$, the perpendicular $FH=10$, the perpendicular $CK=9$, $AC=14$, $CE=11$, the sum of 9 perpendiculars let fall on $AC,=37$; and the sum of 7 perpendiculars let fall on CE, $=25$, to find the area of the figure $ABCDEF$.

First, $\frac{AE\times\overline{FH+KC}}{2}=(\frac{20\times\overline{10+9}}{2}=\frac{20\times 19}{2}=\frac{380}{2}=)\ 190$ *=the area of the rectilineal space* $ACEF$.

Secondly, $\frac{37}{9}=4.1111$, &c. *then* $AC\times 4.1111$, &c. $=(14\times 4.1111$, &c. $=)$ 57.5555, &c. $=$*the area of the curved space* ABC.

Thirdly, $\frac{25}{7}=3.571428$, &c. *then* $CE\times 3.571428$, &c. $=(11\times 3.571428$, &c. $=)$ 39.285708, &c. $=$*the area of the curved space* CDE.

Lastly, these added together, viz.

190 . . . . =*the area* $ACEF$
57.555555= . . . . ABC
39.285708= . . . . CDE
The sum 286.841263= . . . . $ABCDEF$, *as was required.*

2. Let $AE=101$, $FH=25$, $CK=21$, $AC=87$, $CE=79$, the sum of 20 perpendiculars on $AC=103$, and the sum of 17 on $CE=72$; to find the area of the figure $ABCDEF$.

290. *To find the solid content of a prism.*

RULE. Find the area of its base by some of the preceding rules, and multiply this area into the perpendicular height of the prism, the product will be the solid content [d].

EXAMPLES.—1. The side of a cube is 13 inches, required its solidity?

Thus $13\times 13=169=$*area of the base* (*Art.* 282.)

Then $169\times 13=2197$ *cubic inches =the solidity of the cube.*

Or thus, $13\times 13\times 13=(\overline{13}|^3=)$ $2197=$ *the solidity, as before.*

If the board do not taper regularly, measure the breadth in several places, add all the measures together, divide the sum by the number of breadths taken, and multiply the quotient by the length of the board, and it will give the area.

[d] This rule depends on Euclid 2 cor. 7. 12.

2. The sides about one of the angles of the base of a rectangular prism are 7 and 5 respectively, and the altitude of the prism 20; required the solidity?

Thus $7\times5=35=$ *area of the base; then* $35\times20=700$ *the solidity.*

3. The sides of the base of a triangular prism are 2, 3, and 4, respectively, and the perpendicular altitude 30; required the solidity?

Thus (*Art.* 284.) $p=\dfrac{2+3+4}{2}=4.5$, *and*

$\sqrt{4.5\times\overline{4.5-2}\times\overline{4.5-3}\times\overline{4.5-4}}=\sqrt{8.4375}=2.9047375=$ *area of the base.*

Then $2.9047375\times30=87.1421250=$ *the solidity.*

4. The base of a prism is a regular hexagon, the side of which is 8 inches, and the altitude of the prism is 4 feet; required the solidity?

Here (*Art.* 285.) $b=8$, $a=\sqrt{8^2-4^2}=(\sqrt{48}=)\ 6.9282$, $n=6$, *and* $\dfrac{nba}{2}=\dfrac{6\times8\times6.9282}{2}=166.2768$ *square inches = the area of the base: wherefore by the rule* 166.2768×48 (*inches*) $=7981.2864$ *cubic inches* $=4$ *cubic feet* 1069.2864 *cubic inches.*

5. The length of a parallelopiped is 16 feet, its breadth $4\frac{1}{2}$ feet, and thickness $6\frac{3}{4}$ feet; required the solidity? *Ans.* 486 *cubic feet.*

6. The length of a prism is 5 feet, and its base an equilateral triangle, the side of which is $2\frac{1}{2}$ feet; required the solidity? *Ans.* 13.5315 *cubic feet.*

7. The base is a regular pentagon, the side of which is 12 inches, and the length 9 feet; required the solidity of the prism?

291. *To find the solid content of a pyramid.*

RULE. Find the solid content of a prism, having the same base and altitude as the pyramid, by the last rule; one third part of this prism will be the solid content of the pyramid *.

EXAMPLES.—1. The altitude of a pyramid is 20 feet, and its base is a square, the side of which is 12 feet; required the solidity?

* This depends on cor. 1, 7. 12. Euclid.

Here (*Art.* 282.) $12\times12=144=$*area of the base*, $144\times20=2880=$*solidity of the circumscribing prism, and* $\frac{2880}{3}=960$ *cubic feet* $=$ *the solid content of the pyramid.*

2. The altitude of a pyramid is 11 feet, and its base a regular hexagon, the side of which is 4 feet; what is the solidity?

Here (*Art.* 285.) $b=4$, $a=\sqrt{4^2-2^2}=\sqrt{12}=3.4641016$, $n=6$, *and* $\frac{nba}{2}=\frac{6\times4\times3.4641016}{2}=41.5692192=$*area of the base; also* $41.5692192\times11=457.2614112=$*solidity of the circumscribing prism* (*Art.* 290.), $\therefore\ \frac{457.2614112}{3}=152.4204704$ *cubic feet* $=$ *the solidity of the pyramid.*

3. What is the solid content of a triangular pyramid, the height of which is 10, and each side of the base 3? *Answer*, 12.99039.

4. What is the solidity of a square pyramid, each side of its base being 13, and the altitude 25?

292. *To find the solid content of a cylinder.*

RULE. Multiply the area of the base by the perpendicular altitude, and the product will be the solidity [f].

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[f] This rule depends on Euclid 11 and 14 of book 12. The convex superficies of a cylinder is found by multiplying the circumference of the base by the altitude of the cylinder; to which, if the areas of the two ends be added, the sum will be the whole external superficies.

To find the solidity of squared timber. 1. If the stick be equally broad and thick throughout, find the area of a section *any where taken*, and multiply it into the length, the product will be the solidity. 2. If the stick tapers regularly from one end to the other, find half the sum of the areas of the two ends, and multiply it into the length. 3. If the stick does not taper regularly, find the areas of several different sections, add them together, and divide the sum by the number of sections taken, this quotient multiplied into the length, will give the solidity.

To find the solidity of rough or unsquared timber. Multiply the square of one fifth of the mean girt by twice the length, and the product will be the solidity. Or, multiply the square of the circumference by the length, take $\frac{1}{16}$ of the product, and from this last number subtract $\frac{1}{5}$ of itself, the remainder will be the solidity. See on this subject *Hutton's* and *Bonnycastle's Treatises on Mensuration.*

EXAMPLES.—1. The altitude of a cylinder is 12 feet, and the diameter of its base 3 feet; required the solidity?

First, $3 \times 3.1415927 = 9.4247781$ = *circumference of the base. Art.* 287.

Then, $\frac{3 \times 9.4247781}{4} = 7.0685836$ = *area of the base. Art.* 288. $\therefore$ $7.0685836 \times 12 = 84.8230032$ *cubic feet = the solidity required.*

2. The altitude is 20 feet, and the circumference of the base 20 feet; required the solid content of the cylinder? *Ans.* 636.64 *feet.*

3. The diameter of the base is 4 feet, and the altitude 9 feet; required the solidity of the cylinder?

293. *To find the solid content of a cone.*

RULE. Find the solidity of a cylinder of the same base and altitude with the given cone, by the last rule; one third of this will be the solid content of the cone [s].

EXAMPLES 1. The circumference of the base of a cone is 12 feet, and its altitude 10 feet; required the solid content?

First, $\frac{12}{3.1415927} = 3.819718$ = *diam. of the base. Art.* 287.

Then, $\frac{12}{2} \times \frac{3.819718}{2} = 6 \times 1.909859 = 11.459154$ = *area of the base. Art.* 288.

Whence $11.459154 \times 10 = 114.59154$ = *solidity of the circumscribing cylinder. Art.* 292.

Lastly, $\frac{114.59154}{3} = 38.19718$ *cubic feet = the solidity of the cone.*

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[s] For the foundation of the rule see Euclid 10. 12. Let a = the axis of a cone, d = the semidiameter of its base, then (Euclid 47. 1.) $\sqrt{a^2 + d^2}$ = the slant height of the cone; and if the slant height be multiplied into the circumference of the base, the product will be the convex superficies of the cone, to which, adding the area of the base, the sum will be the whole external superficies. Rules for finding the superficies and solidities of the several sections of a prism, pyramid, cone, cylinder, sphere, &c. may be found in Mr. Bonnycastle's excellent *Introduction to Mensuration*, a work which cannot be too highly commended.

2. The altitude is 12, and the diameter of the base 3; required the solidity of the cone? *Ans.* 28.2743344.

3. The area of the base is 20, and the altitude 14; required the solid content of the cone?

294. *To find the solid content of a sphere.*

RULE. Find the solidity of a cylinder, of which the altitude, and the diameter of its base, are each equal to the diameter of the given sphere; two thirds of this will be the solidity of the sphere [b].

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[b] Euclid has proved that "spheres are to each other in the triplicate ratio of their diameters" (18. 12.); but this is the only property of the sphere to be found in the Elements. We are beholden to Archimedes for the most part of our original information on this subject; the above rule, which was taken from his treatise "on the sphere and cylinder," may be easily demonstrated by "indivisibles," "the method of increments," "fluxions," and some other modern methods of computation; but I believe it cannot be effected by elementary Geometry.

The superficies of a sphere is equal to the convex surface of its circumscribing cylinder; it is likewise equal to four times the area of a great circle of the sphere.

If the diameter of a sphere be 2, then will the circumference of a great circle be 6.28318

The superficies of a great circle		3.14159
The superficies of a sphere		12.56637
The solidity of the sphere		4.18790
And of the inscribed tetraëdron	its side	1.62209
	superficies	4.6188
	solidity	0.15132
The inscribed hexaëdron	its side	1.1547
	superficies	8.0000
	solidity	1.5396
The inscribed octaëdron	its side	1.41421
	superficies	6.9282
	solidity	1.33333
The inscribed dodecaëdron	its side	0.71364
	superficies	10.51462
	solidity	2.78516
The inscribed icosaëdron	its side	1.05146
	superficies	9.57454
	solidity	2.53615

Hence the superficial and solid content of a solid, similar to any of the above, may be readily obtained, its side being given; the superficies being as the squares (Euclid 20. 6.), and the solidities as the cubes (cor. 8. 12.) of the homologous sides.

EXAMPLES.—1. The diameter of a sphere is 3 feet; required its solidity?

First, $3 \times 3.1415927 = 9.4247781$ = *circumf. of the cylinder's base. Art.* 287.

Secondly, $\frac{3 \times 9.4247781}{4} = 7.0685836$ = *the cylinder's base. Art.* 288.

Thirdly, $7.0685836 \times 3 = 21.2057508$ = *the solidity of the cylinder. Art.* 292.

Lastly, $\frac{2}{3}$ of $21.2057508 = 14.1371672$ *cubic feet* = *the solidity of the sphere.*

2. The diameter of a sphere is 17 inches; required its solidity? *Ans.* 1.48868 *cubic feet.*

3. If the earth be a perfect sphere of 8000 miles diameter, how many cubic miles of matter does it contain?

PART IX.

TRIGONOMETRY.

HISTORICAL INTRODUCTION.

TRIGONOMETRY [a] is a science which teaches how to determine the sides and angles of triangles, by means of the relations and properties of certain right lines drawn in and about the circle; it is divided into two kinds, plane and spherical, the former of which applies to the computation of plane rectilineal triangles, and the latter to triangles formed by the intersections of great circles, on the surface of a sphere.

This science is justly considered as an important link connecting theoretical Geometry with practical utility, and making the former conducive, and subservient to the latter. Geography, Astronomy, Dialling, Navigation, Surveying, Mensuration, Fortification, &c. are indebted to it, if not for their existence, at least for their distinguishing perfections; and there is scarcely any branch of Natural Philosophy, which can be successfully cultivated without, the assistance of Trigonometry.

We are in possession of no documents that will warrant us even to guess at the period when Trigonometry took its rise; but there can be do doubt that it must have been invented not very long after the flood. The earliest inhabitants of Chaldæa and Egypt were acquainted with Astronomy, which

[a] The name is derived from *τρεις three*, *γωνος a corner*, and *μετρεω to measure*. The objects of Trigonometry are the sides and angles only, whatever respects the areas of triangles belongs to Geometry.

(admitting it to have been at that time merely an art, and in its rudest state) would still require the aid of some method similar to Trigonometry to make it of any benefit to mankind.

We may reasonably suppose that the ancient Greeks cultivated Trigonometry, in common with Geometry and Astronomy; but none of their writings on the subject have been preserved. Theon [b], in his Commentary on Ptolemy's Almagest, mentions a work consisting of twelve books on the chords of circular *arcs*, written by Hipparchus, an Astronomer of Rhodes, A.C. 130 [c]. This work is believed by the learned to have been a treatise on the ancient Trigo-

[b] Theon, a respectable mathematician and philosopher, and president of the Alexandrian school, flourished A.D. 370. He was not more famous for his acquirements in science, than for his veneration of the DEITY, and his firm belief in the constant superintendence of divine providence; he recommends meditation on the presence of God, as the most delightful and useful employment, and proposed, that in order to deter the profligate from committing crimes, there should be written at the corner of every street, REMEMBER GOD SEES THEE, O SINNER. Dr. Simson, in his notes on the Elements of Euclid, has ascribed most of the faults in that book to Theon, without mentioning on what authority he has done so.

[c] Hipparchus was born at Nice, in Bithynia: here, and afterwards at Rhodes and Alexandria, his astronomical observations were made. He discovered that the interval between the vernal and autumnal equinox is longer by 7 days than that between the autumnal and vernal; he was the first who arranged the stars into 49 constellations, and determined their longitudes and apparent magnitudes; and his labours in this respect were considered so valuable, that Ptolemy has inserted his catalogue of the fixed stars in his Almagest, where it is still preserved. He also discovered the precession of the equinoxes, and the parallax of the planets; and, after the example of Thales, and Sulpicius Gallus, foretold the exact time of eclipses, of which he made a calculation for 600 years. He determined the latitude and longitude, and fixed the first meridian at the *Fortunatæ Insulæ*, or *Canary Islands*; in which particular he has been followed by most succeeding geographers. Astronomy is particularly indebted to him for collecting the detached and scattered principles and observations of his predecessors, arranging them in a system; thereby laying that rational and solid foundation, upon which succeeding astronomers have built a most sublime and magnificent superstructure. Of the several works said to have been written by him, his Commentary on the Phœnomena of Aratus is the only one that remains.

nometry, and is the most ancient on that subject of which we have any account.

The Spherics of Theodosius [d] is the earliest work on Trigonometry at present known. It was written about 80 years before Christ, and consists of three books, "containing a variety of the most necessary and useful propositions relating to the sphere, arranged and demonstrated with great perspicuity and elegance, after the manner of Euclid's Elements."

We are in possession of three books on spherical triangles by Menelaus [e]. He is considered as the next Greek writer who treated expressly on the subject, and lived about a hundred years after Christ. This work of Menelaus was greatly

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[d] Theodosius was a native of Tripoli, in Bithynia; and, according to Strabo, excelled in mathematical knowledge. The work above-mentioned consists of three books; the first of which contains 22 propositions, the second 23, and the third 14. It was translated into Arabic, and afterwards from the Arabic into Latin, and published at Venice; but the Arabic edition being very defective, a complete edition was obtained by Jean Pena, Regius Professor of Astronomy at Paris, and published there in Greek and Latin, A. D. 1558. Long before this time, a good Latin translation of the work had been made by Vitellio, a respectable Polish mathematician of the 13th century, and the first of the moderns who wrote to good purpose on optics. The Spherics of Theodosius have been enriched with notes, commentaries, and illustrations, by Clavius, Heleganius, Guarinus, and De Chales; but the best editions are those of Dr. Barrow, 8vo. London, 1675; and Hunt, 8vo. Oxon, 1707.

There are still in existence in the National Library at Paris, two other pieces by Theodosius, one on *The Cœlestial Houses*, and the other on *Days and Nights*: a Latin translation of which was published by Peter Dasypody, A.D. 1572.

[e] Menelaus was a respectable mathematician and astronomer, probably of the Alexandrian school, but we have no particulars of his life or writings, except that he is said to have written six books on the chords of circular *arcs*, which is supposed to have been a treatise on the ancient method of constructing trigonometrical tables, but the work is lost. A Latin translation of the three books on spherical triangles was undertaken by Regiomontanus, but was first published by Maurolycus, together with the Spherics of Theodosius, and his own, (Messanæ, 1558, fol.) An edition of this work, corrected from a Hebrew manuscript, was prepared for the press by Dr. Halley, and published by Costard, the author of the History of Astronomy, in 8vo. 1758.

improved by Ptolemy, who, in the first book of his Almagest, has introduced a table of *arcs* and their chords, to every half degree of the semicircle; he divides the radius, and also the *arc* equal to one sixth of the whole circumference (whose chord is the radius) each into 60 equal parts, and estimates all other *arcs* by sixtieths of that *arc*, and their chords by sixtieths of that chord (or radius); which method he is supposed to have derived from the writings of Hipparchus, and other authors of antiquity.

No farther progress seems to have been made in the science, until some time after the revival of learning among the Arabians, namely, about the latter part of the eighth century; when the ancient method of computing by the chords of *arcs* was laid aside by that people, and the more convenient method of computing by the sines, substituted in its stead. This improvement has been ascribed by some to Mahomed Ebn Musa, and by others to Arzachel, a Moor, who had settled in Spain, about the year 1100: Arzachel is the first we read of who constructed a table of sines, which he employed in his numerous astronomical calculations instead of the chords, dividing the diameter into 300 equal parts, and computing the magnitude of the sines in those parts. We are indebted to the Arabs for the introduction of those axioms and theorems into the science, which are considered as the foundation of modern Trigonometry, and likewise for other improvements.

The sexagesimal division of the radius, according to the method of the Greeks, was still employed by the Arabians, although they had long been in possession of the Indian, or decimal scale of notation. But shortly after the diffusion of science in the west, an alteration was made by George Purbach, Professor of Mathematics at Vienna, who wrote about the middle of the 15th century; he divided the radius into 600000 equal parts, and computed a table of sines in

those parts, for every ten minutes of the quadrant, by the decimal notation. This work was further prosecuted by Regiomontanus, the disciple and friend of Purbach; but as the plan of his master was evidently defective, he afterwards changed it altogether, by computing anew the table of sines for every minute of the quadrant, to the radius 1000000. Regiomontanus also introduced the use of tangents into Trigonometry, the table of which he named *Canon Fœcundus*, on account of the numerous advantages arising from its use. He likewise enriched the science with many valuable theorems and precepts; so that, excepting the use of logarithms, the Trigonometry of Regiomontanus was little inferior to that of our own times.

About this period the mathematical sciences began to be studied with ardour in several parts of Italy and Germany, and it can hardly be supposed that a science so obviously useful as Trigonometry, would be without its share of admirers and cultivators, although scarcely any of their writings on the subject have been committed to the press. John Werner of Nuremburg, (who was born in 1468, and died in 1588,) is said to have written five books on triangles; but whether the work exists at present, or is lost, we are not informed. A brief treatise on plane and spherical Trigonometry was written about the year 1500, by Nicholas Copernicus, the celebrated restorer of the true solar system. This tract contains the description and construction of the canon of chords, nearly in the manner of Ptolemy; to which is subjoined a table of sines to the radius 100000 with their differences, for every ten minutes of the quadrant, the whole forming a part of the first book of his *Revolutiones Orbium Cœlestium*, first published at Nuremburg, fol. 1543. Ten years after, Erasmus Reinhold, Professor of Mathematics at Wirtemburg, published his *Canon Fœcundus*, or table of tangents; and about the same

time Franciscus Maurolycus, Abbot of Messina, in Sicily, and one of the best Geometers of the age, published his *Tabula Benefica*, or canon of secants.

But a more complete work on the subject than any that had hitherto appeared, was a treatise in two parts by Vieta, one of the ablest mathematicians in Europe, published at Paris, in 1579. The first part, entitled *Canon Mathematicus seu ad triangula, cum appendicibus*, contains a table of sines, tangents, and secants, with their differences for every minute of the quadrant, to the radius 100000. The tangents and secants towards the end of the quadrant are carried to 8 or 9 figures, and the arrangement is similar to that at present in use, each number and its compliment standing in the same line, opposite one another. The second part of this volume, entitled *Universalium Inspectionum ad Canonem Mathematicum liber singularis*, contains the construction of the foregoing table, a complete treatise on plain and spherical Trigonometry, with their application to various parts of the Mathematics; particulars relating to the quadrature of the circle, the duplication of the cube; with a variety of other curious and interesting problems and observations of a miscellaneous nature [f]. Besides the above masterly performance, Vieta was the author of several tracts on plane and spherical Trigonometry, which may be found in the collection of his works, published by Schooten, at Leyden, in 1646.

The triangular canon was next undertaken by George Joachim Rheticus, a pupil of the great Copernicus, and Professor of Mathematics at Wirtemburg; "he computed the

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[f] For further particulars of this interesting volume, see *The History of Trigonometrical Tables*, p. 4, 5, 6, 7, by Dr. Hutton. It appears that scarcely any copies of this excellent work are now to be found; for the Doctor says, in concluding his account of it, "I never saw one (copy) besides that which is in my own possession, nor ever met with any other person at all acquainted with such a book," p. 7.

canon of sines and co-sines for every ten seconds of the quadrant, and for every single second of the first and last degree;" he had proposed, in obedience to the desire of his master, to complete the trigonometrical canon, and extend it further than had hitherto been done; but, dying in 1576, the completion of this vast design was at his request consigned to his pupil and friend Valentine Otho, mathematician to the Electoral Prince Palatine; who, after several years of indefetigable labour and intense application, accomplished the work, and it was printed at Heidelberg, in 1596, under the title of *Opus Palatinum de Triangulis.* We have here an entire table of sines, tangents, and secants, for every ten seconds of the quadrant to ten places of figures, with their differences, being the first complete canon of these numbers that was ever published.

But notwithstanding the pains that had been taken in the calculation, the tables in this valuable performance were afterwards found to contain a considerable number of errors, particularly in the co-tangents and co-secants; the correction of these was undertaken by Bartholomew Pitiscus, a skilful mathematician of that time, who, having procured the original manuscript of Rheticus, added to it an auxiliary table of sines to 21 places, for the purpose of supplying the defect of the former, and published both in folio, at Frankfort, in 1613, under the title of *Thesaurus Mathematicus*, &c. Pitiscus then re-calculated the co-tangents and co-secants to the end of the first six degrees in Otho's work, which rendered it sufficiently exact for astronomical purposes, and published his corrections in separate sheets, making in the whole 86 pages in folio.

The *Geometrica Triangulorum* of Philip Lansbergius, in four books, was published in 1591; a brief, but very elegant work, containing the canon of sines, tangents, and secants, with their construction and application in the solution of

plane and spherical triangles; the whole being fully and clearly explained. This is the first work in which the tangents and secants are carried to 7 places of decimals to the last degree of the quadrant.

A complete and masterly work on Trigonometry by Pitiscus, was published at Frankfort, in 1599; the triangular canon is here given, and its construction and use clearly described, together with the application of Trigonometry to problems of surveying, altimetry, architecture, geography, dialling, and astronomy; forming the most commodious and useful treatise on the subject at that time extant.

Several other writers on Trigonometry appeared towards the close of the 16th, and at the beginning of the 17th century, of whom Christopher Clavius, a Jesuit of Bamberg, may be considered as one of the chief. In the first volume of his works, (which were printed at Mentz, in 5 volumes, folio, 1612,) he has given an ample and circumstantial treatise on Trigonometry. In this work the canon of sines, tangents, and secants, is computed for every minute to 7 places of decimals, and carried forward to the end of the quadrant, the sines having their differences computed to every second, and construction of the tables being accompanied with clear and satisfactory explanations, chiefly derived from the methods of Ptolemy, Purbach, and Regiomontanus.

Van Ceulen, in his celebrated treatise *De Circulo et adscriptis*, first published about the year 1600, treats of the chords, sines, and other lines connected with the circle; which work, with some other of Van Ceulen's pieces, was afterwards translated into Latin, and published at Leyden, in 1619, by Willebrord Snellius, who has also himself given in his *Doctrina Triangulorum Canonicæ*, the construction of sines, tangents, and secants, together with a very useful synopsis of the calculation of plane and spherical triangles.

A canon of sines, tangents, and secants, for every minute

of the quadrant, was published in 1627, at Amsterdam, by Francis Van Schooten, the ingenious commentator on the Geometry of Des Cartes. His assertion, that his table was without a single error, has been since found to be incorrect; some of his numbers have been discovered to err in the last figure, being not always calculated to the nearest unit [*].

[*] In the early ages of Geometry the circumference of the circle was divided into 360 degrees, each degree into 60 minutes, each minute into 60 seconds, &c.; this method was adopted by the moderns, and still prevails among the English, and most other nations in Europe; but the French mathematicians have introduced an improvement, which, when it is generally understood and adopted, will be of the greatest advantage to Trigonometry. Towards the latter part of the eighteenth century, a new system of weights and measures was instituted in France, in which they were decimally divided and subdivided; this was followed by another of equal importance, a new division of the quadrant. By this new method, the whole circumference is divided into 400 equal parts called degrees, each degree into 100 minutes, each minute into 100 seconds, &c. consequently the quadrant will contain 100 degrees. One advantage in this method is its convenient identity with the common decimal scale of numbers, for 1°, 23′, 45″, in the new French scale will be expressed by *the very same figures* in common decimals, viz. by 1.2345°; in like manner 21°, 3′, 4″, French, is expressed by 21.0304° common decimals; 170°, 1′, 2″, 34‴ by 170.010234°; 5′, 0″, 11‴ by .050011°; 12′, 13″, 14‴ by .121314°, &c. Among the works on this plan at present in use, are *Les Tables Portatives* de Callet, 2 Edit. Paris, 1795; the *Trigonometrical Tables* of Borda, improved by Delambre, 4to. an IX.; and the tables lately published by Hobert and Ideler, at Berlin. Likewise tables on the above plan, to an extent hitherto unknown, have been for many years under the hands of M. Prony, assisted by a number of able mathematicians, a work which, besides its great usefulness, will be the most ample monument existing, of human industry in the province of calculation.

To reduce degrees, minutes, &c. of the French scale, into degrees, minutes, &c. of the common scale, and vice versâ.

Since the quadrant is divided by the *French* method into 100°, and by the *common* method into 90°, ∵ 100° *French* = 90° *common*; ∴ *To reduce French degrees, minutes, &c. into common.*

RULE. Express the French measure *decimally*, subtract from this $\frac{1}{10}$ of itself; mark off the proper decimals in the remainder, multiply these by 60, mark off the decimals; multiply these again by 60, and mark off the decimals as before, &c.; the resulting whole numbers will be the degrees, minutes, seconds, &c. required, according to the common scale.

EXAMPLES.—1. In 34°, 56′, 32″ French, how many degrees, minutes, seconds, &c. common?

The invention of logarithms by Lord Napier, in 1614, and their subsequent improvement by Mr. Henry Briggs, greatly facilitated the practical operations of Trigonometry. Besides the invention of logarithms, we are indebted to Napier for the method of computing spherical triangles by means of the five circular parts, and other valuable improvements in spherical Trigonometry.

The doctrine of infinite series, introduced about the year 1668, by Nicholas Mercator, and improved by Newton, Leibnitz, the Bernoullis, and others, soon found its application to Trigonometry, by furnishing expressions for the sines, tangents, &c. for which purpose the exponential formulæ of Mr. Demoivre are extremely convenient.

But the greatest and most useful improvement of modern times in the analysis of sines, co-sines, tangents, &c. which

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First, *from* 34°, 56′, 32″ = 34.5632°
Subtract $\frac{1}{10}$ *of the same* = 3.45632
The remainder = 31.10688
Multiply the decimals by 60
6.41280
Multiply the decimals by 60
24.76800
Multiply the decimals by 60
46.08000

Therefore 34°, 56′, 32″ *French* = 31°, 6′, 24″, 46‴, 08 *common.*

2. In 8°, 12′, 8″ French, how many degrees, minutes, &c. common? *Ans.* 7°, 18′, 38″, 31‴.

3. In 12°, 1′, 2″ French, how many degrees, &c. common?

4. In 9°, 8′, 7″ French, how many degrees, &c. common?

To reduce common degrees into French.

Rule. Turn the minutes, seconds, &c. into decimals, to the whole add $\frac{1}{9}$ of itself; then the integers of the sum will be *degrees*, the two left hand decimals *minutes*, the two next decimals *seconds*, &c.

Examples.—1. To reduce 34°, 56′, 32″ *common*, to *French* measure.

First, *to* 34°, 56′, 32″ = 34.942222°, &c.
Add $\frac{1}{9}$ *of the same* = 3.882469
The sum is 38.824691 = 38°, 82′, 46″, 91‴ *French.*

2. In 24°, 44′, 6″ *common*, how many degrees *French*? *Ans.* 24°, 15′.

3. Turn 23°, 27′, 58″ *common* into *French*. *Ans.* 26°, 17′, 35″.

4. Turn 1°, 2′, 34″ *common* into *French*.

we owe to the penetrating, comprehensive, and indefatigable mind of the venerable Euler; by substituting the analytical mode of notation, in the room of the geometrical, which had hitherto been chiefly used, he simplified the methods of preceding writers, investigated a great variety of formulæ, applicable to the most difficult cases, and made the trigonometrical analysis assume the form of a new and interesting science.

Admitting that the Continental mathematicians are our superiors in the theory of Trigonometry, as well as in their writings on the science[a], still we have some very good and useful treatises on the subject; the chief of which are those of Thomas Simpson, Emerson, Maseres, Horsley, Keith, Vince, and Woodhouse; but Mr. Bonnycastle's *Treatise on Plane and Spherical Trigonometry*, is the most complete work on the subject of any that have hitherto appeared in this country.

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[a] See the *Quarterly Review* for November, 1810, page 401.

PLANE TRIGONOMETRY*.

DEFINITIONS AND PRINCIPLES.

1. PLANE Trigonometry teaches how to determine, from proper data, the sides and angles of plane rectilineal triangles, by means of the analogies of certain right lines, described in, and about a circle.

2. Every triangle contains six parts, viz. three sides, and three angles; any three of these, whereof one (at least) is a side, being given, the remaining three may be found.

3. The sides of plane rectilineal triangles are estimated in feet, yards, fathoms, chains, &c. or by abstract numbers: and each of the angles, by the *arc* of a circle included between the two legs; the angular point being the centre.

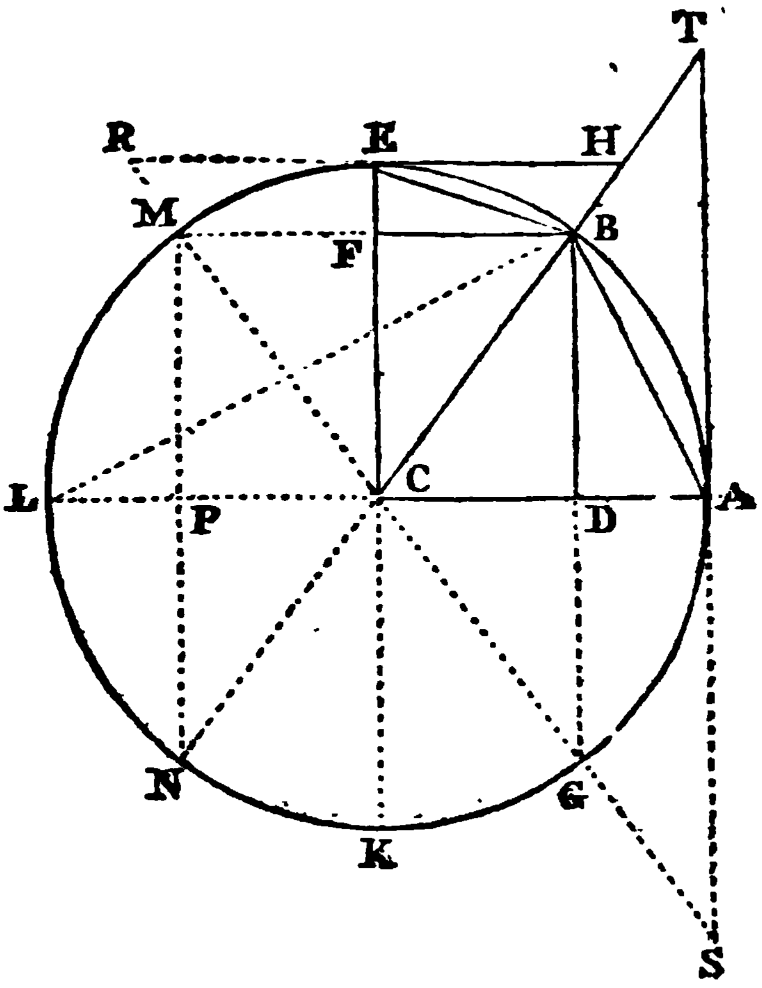
4. It has already been observed (Art. 237. part 8.), that the whole circumference is supposed to be divided into 360 degrees, each degree into 60 minutes, each minute into 60 seconds, &c.; as many degrees, minutes, and seconds therefore, as are contained in the *arc* intercepted between the legs of an angle, so many degrees, minutes, and seconds, that angle is said to measure; and, note, in the following definitions, whatever is affirmed of an *arc*, is likewise affirmed of the angle (at the centre,) which stands on that *arc*.

5. Draw any straight line AC; from C as a centre with the distance CA, describe the circle AEN; produce AC to L, and through the centre C draw ECK perpendicular to AL; in the *arc* EA take any point B, join BA, BE, and BC, and produce the latter to N; through A and B draw AT, BD each parallel to CE, and produce them to S and G; join CG, and produce it to R and S, produce CB to T, through E and B draw REH, MFB, each parallel to CA, and join BL, MN; then since TA, BD are both parallel to EC, they are parallel to one another (30. 1.), and both perpendicular to CA (29. 1.); for a like reason EH and FB

* An easy tract on Plane Trigonometry may be found in Ludlam's *Rudiments of Mathematics*. Mr. Bridge's lectures on the same subject, published in 1810, is likewise a neat and useful work.

are parallel, and both perpendicular to *EC*, and *BD*=*FC*, and *FB* =*CD* (34. 1.)

6. Because the four right angles *ACE*, *ECL*, *LCK*, *KCA* are subtended by the whole circumference, each of these angles will be subtended by one fourth part of the whole circumference, which is called a QUADRANT; the *arc ABE* is therefore a quadrant.



7. The difference of any *arc* from a quadrant, or 90°, or of any angle from a right angle, is called THE COMPLEMENT of that *arc* or angle.

Thus, the arc BE is the complement of the arc AB; and the angle BCE is the complement of the angle ACB [b].

8. The difference of any *arc* from a semicircle, or 180°, or of any angle from two right angles, is called THE SUPPLEMENT of that *arc* or angle.

Thus, the arc BL is the supplement of the arc AB, and the angle BCL of the angle ACB [c].

9. THE CHORD of an *arc* is a straight line drawn from one end of the *arc* to the other.

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[b] In like manner *AB* is the complement of *BE*, and the angle *ACB* of the angle *BCE*. The name *complement* likewise applies to the *excess* of an *arc* above a quadrant, or of an angle above a right angle; thus *EB* is the complement of the *arc BML*, and of the angle *BCL*; but in most practical questions it is usually restrained to what an *arc* or acute angle wants of 90°.

[c] The *arc AB* is likewise the supplement of the *arc BML*, and the angle *ACB* of the angle *BCL*. The term *supplement* means also the *excess* of an *arc* above a semicircle, thus the *arc AB* is the supplement of the *arc AMN*. The difference of an *arc* from the whole circumference is termed its *supplement to a circle*.

Thus, the straight line AB is the chord of the arc AB, or of the angle ACB.

Cor. The chord of 60° is equal to the radius (cor. 15. 4.); and the chord of 180° is the diameter.

10. THE CO-CHORD of an *arc*, is the chord of the complement of that *arc*.

Thus, the straight line BE (or the chord of the arc BE) is the co-chord of the arc AB, or of the angle ACB.

11. THE SUPPLEMENTAL CHORD of an *arc*, is the chord of its supplement.

Thus, BL (or the chord of the arc BML) is the supplemental chord of the arc AB, or of the angle ACB.

Cor. Hence it appears that the chord of any *arc*, is likewise the chord of its supplement to a whole circle; also that the chord can never exceed the diameter (15. 3.)

Thus, BL is not only the chord of the arc BML, but also of the arc BKL.

12. THE SINE of an *arc*, is a straight line drawn from one end of the *arc*, perpendicular to the diameter which passes through the other end of the *arc*.

Thus, BD is the sine of the arc AB, and of the angle ACB.

Cor. Hence the sine of an *arc*, is the same as the sine of its supplement, for *BD* is not only the sine of the *arc AB*, but also of the *arc BML*; for it is drawn from one extremity *B*, (of the *arc BML*,) perpendicular to the diameter *AL*, passing through the other extremity *L*.

13. THE CO-SINE of an *arc*, is that part of the diameter (passing through the beginning of the *arc*,) which is intercepted between the sine and the centre, and is equal to the *sine of the complement* of that *arc*.

Thus, CD is the co-sine of the arc AB, and of the angle ACB; and it is equal to BF (34. 1) *the sine of BE, which is the complement of AB.*

Cor. Hence the sine of a quadrant, or of a right angle (is not only equal to, but) is the radius; and the co-sine of a quadrant or right angle is nothing.

Thus, if the point B be supposed to move to E, the arc AB will become AE, the sine of which is EC; and the point D coinciding with C, the co-sine CD will vanish.

Hence also the sine or co-sine can never exceed the radius.

14. THE VERSED SINE of an *arc*, is that part of the diameter which is intercepted between the beginning of the *arc* and its sine.

Thus, DA is the versed sine of the arc AB, and of the angle ACB; and AP is the versed sine of the arc ABM, and of the angle ACM.

Cor. Hence the versed sine of an *arc less* than a quadrant, is equal to the *difference*; and of an *arc greater* than a quadrant, to the *sum* of the co-sine and radius.

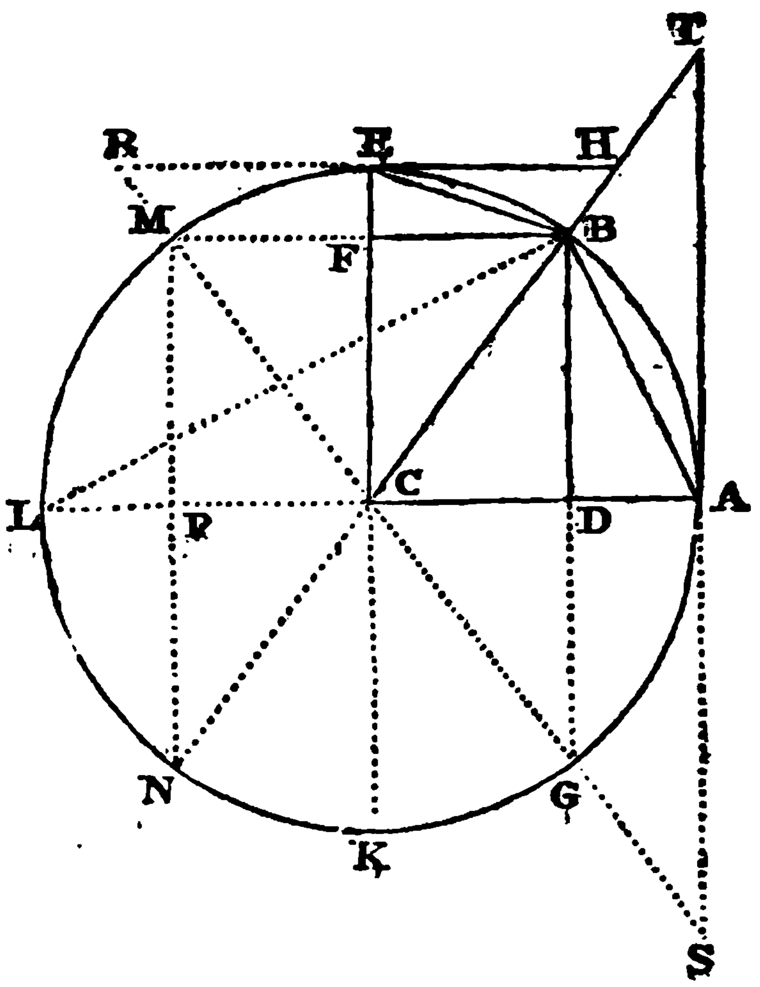
Thus, AD (the versed sine of AB) = CA—CD, and AP (the versed sine of ABM) = CA + CP.

Hence also the versed sine (being always within the circle,) can never exceed the diameter, (15. 3.)

15. THE CO-VERSED SINE of an *arc*, is the versed sine of its complement.

Thus, EF is the co-versed sine of the arc AB, and of the angle ACB.

Cor. Hence the co-versed sine is equal to the excess of the radius, above the sine.



16. THE TANGENT of an *arc*, is a straight line at right angles to the diameter, passing through one end of the *arc*, and meeting a diameter produced through the other end of the *arc*.

Thus, AT is the tangent of the arc AB, and of the angle ACB.

Cor. Hence a tangent may be of any magnitude (according to the magnitude of its *arc*) from *nothing* to *infinity*. Hence also the tangent of 45° is equal to the radius (6. 1.)

17. THE CO-TANGENT of an *arc*, is the tangent of the complement of that *arc*.

Thus, EH (the tangent of EB) is the co-tangent of the arc AB, and of the angle ACB.

18. THE SECANT of an *arc*, is a straight line drawn from the centre, through the end of the *arc*, and produced till it meet the tangent.

Thus, CT is the secant of the arc AB, and of the angle ACB.

Cor. Hence a secant can never be less than the radius, but it increases (as the *arc* increases) from the *radius* to *infinity.*

19. THE CO-SECANT of an *arc* is the secant of its complement.

Thus, CH (the secant of EB,) is the co-secant of the arc AB, and of the angle ACB [d].

THE VARIATIONS, AND ALGEBRAIC SIGNS, OF THE TRIGONOMETRICAL LINES IN THE FOUR QUADRANTS.

20. If the sine, co-sine, tangent, co-tangent, secant, co-secant, versed sine, and co-versed sine for every *arc* in the first quadrant *AE* be drawn, they will serve for the three remaining quadrants *EL, LK, KA,* that is, for the whole circle, as will be shewn further on; but previous to this, it will be necessary to suppose the point *B* to coincide with *A,* and to move from thence round the whole circumference, and this will lead us to explain the manner of applying the algebraic signs + and − to the lines peculiar to Trigonometry.

21. When the point *B* coincides with *A,* the *arc AB* will $=o$, and the points *D* and *T* will coincide with *A;* wherefore $AT=o$, $BD=o$, $DA=o$, *CB* and *CD* each = radius; that is, the tangent, sine, and versed sine, (of *o* degrees, or) at the beginning of the quadrant will be *nothing,* and the secant and cosine will be *radius.*

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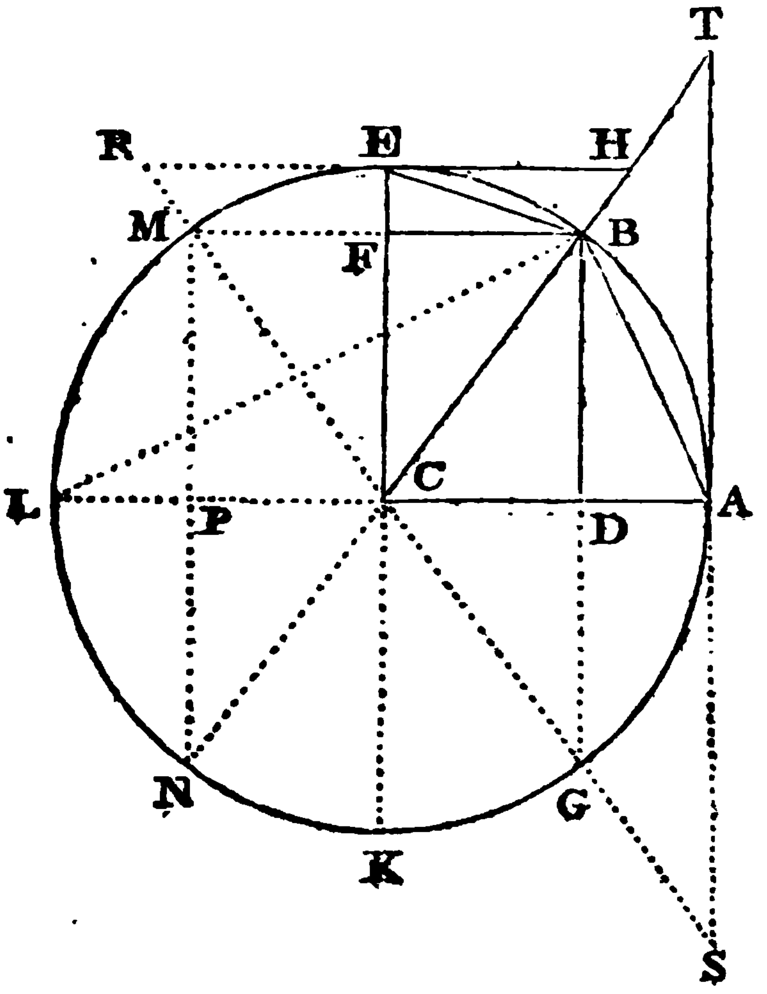
[d] Some of the trigonometrical lines received their names from the parts of an archer's bow, to which they bear a similitude; thus, ARC comes from *arcus,* a bow; CHORD from *chorda,* the string of a bow; SAGITTA (now generally called the versed sine) from *sagitta,* an arrow; SINE from *sinus,* the bosom, alluding to that part of the *chorda* or string, which is held near the breast in the act of shooting, the sine being half the chord of double the *arc.* The prefix co is an abbreviation of the word *complement;* thus *co-sine, co-tangent,* &c. imply *complement sine, complement tangent,* &c. or the sine, tangent, &c. of the complement of a given *arc.*

22. The sine *BD* increases (with the motion of *B*) from *o*, during the first quadrant *AE*; when the point *B* coincides with *E*, the sine *BD* will evidently coincide with *EC*, and become *radius*; it then decreases during the second quadrant, at the end of which, (when *B* is supposed to arrive at *L*,) it is again $=o$. During the progress of *B*, through the third quadrant *LK*, the sine again increases from *o*, and on the arrival of *B* at the point *K*, it again becomes radius; after which it gradually decreases through the fourth quadrant *KA*, at the end of which (where the *arc* is 360 degrees,) it is $=o$, after which it again increases as before.

23. The sines are considered as affirmative or negative with respect to their direction from the diameter *LA*, to which they are referred; those on one side that diameter being considered as affirmative, those on the other side, and in a contrary direction, will be negative; for instance, the sines of the first and second quadrants which are on one side the diameter being reckoned +, those of the third and fourth quadrants, being on the other side will be —.



24. The co-sine at the beginning of the first quadrant is *radius*, and decreases with the motion of the point *B* through the *arc AE* to *o*; when *B* arrives at *E*, *D* coincides with *C*; that is, the co-sine of a quadrant (or 90°) is $=o$. It afterwards increases from *o* to the end *L* of the second quadrant, where it is again radius; in the third, it continually decreases, at the end (*K*) of which it is again *nothing*; afterwards, during the fourth quadrant *KA*, it again increases, at the end of which (viz. at the point *A*) it is again *radius*.

25. The co-sines originate at the centre *C*; consequently if

those in the direction *CA* be considered as affirmative, those in the opposite direction *CL* will be negative. The co-sines then of the first and fourth quadrants will be alike, viz. +; those of the second and third will also be alike, but contrary to the former, viz. —.

26. At the beginning of the first quadrant (at *A*) the tangent is *nothing;* from *o* it increases continually, until the point *B* coincides with *E*, when it becomes parallel to the secant, (which will then coincide with *CE*) and is therefore *infinite.* When the point *B* has passed *E*, the tangent will change its direction, and (with the motion of *B*) will continually decrease, until *B* arrives at *L*, or the end of the second quadrant, when the tangent will again become *nothing*; from *o* it changes its direction to *AT*, and increases until *B* arrives at *K*, the end of the third quadrant; when it is again infinite, it decreases from infinite during the fourth quadrant, at the end of which it is again *nothing.*

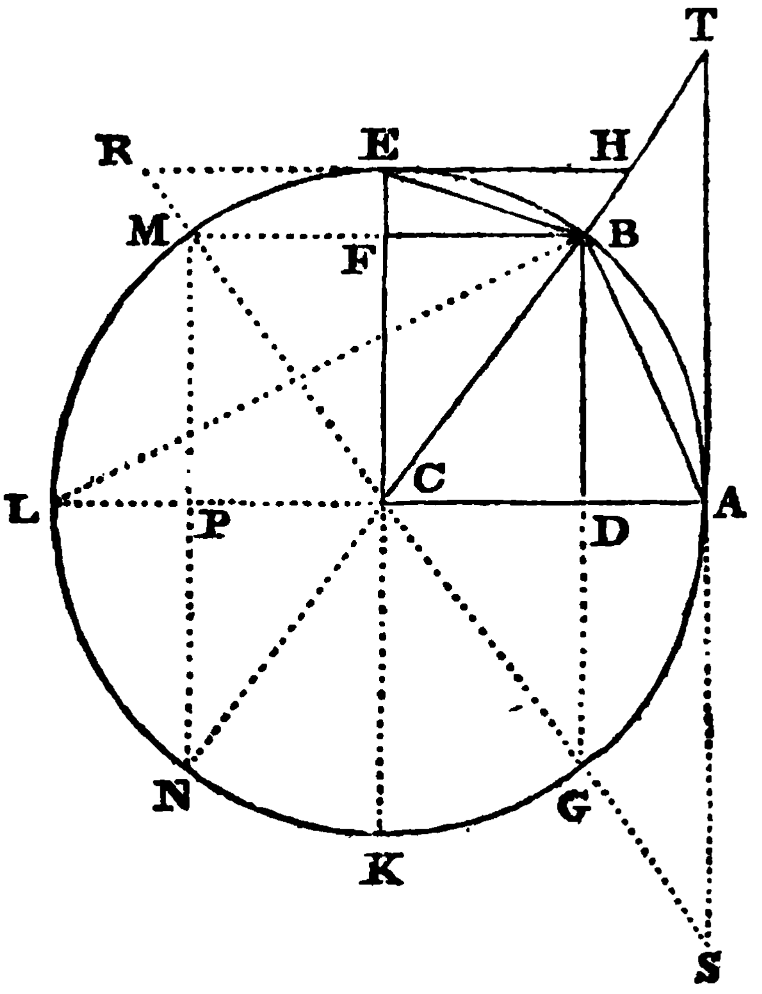
27. The tangent originates at the point *A;* consequently, if the tangent in the direction of *AT* be called affirmative, that in the direction of *AS* will be negative; but we have shewn that the tangents of the first and third quadrants are in the direction of *AT*, wherefore they are both +; whence the tangents of the second and fourth quadrants being in the direction of *AS* will, for the reason given above, be both —.

28. The secant at the point *A* is equal to *radius*, and it increases (by the motion of *B*) with the tangent, and with it becomes *infinite* at *E*, the end of the first quadrant. In the second quadrant *EL*, the secant changes its direction from *CT* to *CS*, and decreases from infinity to *radius;* in the third quadrant *LK*, it increases again in the direction *CT*, from *radius* to *infinity:* in the fourth quadrant *KA*, the secant once more changes its direction to *CS*, and decreases from *infinity* to *radius.*

29. The secant has its origin at the centre *C*, from whence its length is computed, and it will change its sign as often as the revolving radius *CB* passes the diameter *EK;* having the same algebraic sign as the co-sine; whence it appears that the secants of the first and fourth quadrants will be +, those of the second and third —.

30. The changes which take place in the magnitudes and directions of the co-tangent *EH*, and the co-secant *CH*, may be

explained in the same manner; the co-tangent being computed from the point E, will change its direction, and consequently its algebraic sign every quadrant, the first and third being $+$, the second and fourth will be $-$. The co-secant at the point A is infinite, at the point E it is *radius*, at the point L it is *infinite*, and at K it is again *radius*. In the first and second quadrants its sign will be $+$, in the third and fourth $-$, being the same as the sine.



31. The versed sine at A is $=o$, at E it is *radius*; at L it is *the diameter*; at K it has decreased to *radius*, and continues its decrease to A, where it is *nothing*. This line being computed from A, will be always affirmative.

32. It may be remarked, in general, of the above lines, that as oft as they become *infinite* or *nothing*, they change their direction, and consequently change their algebraic sign; these changes may be exhibited in one point of view, as follows [*]:

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[*] It is sometimes necessary in analytical computations to employ *arcs* greater than the whole circumference, which *arcs* will fall in the 5th, 6th, 7th, &c. quadrant (counting the quadrants again round the circle); in these cases, the proper sign of the *arc* in question must be particularly attended to; it may be readily found from the above table.

Let a = any *arc*, its sine, tangent, &c. may be found in terms of the rest from the foregoing figure, by means of similar triangles: thus,

1. Sine of $a = \sqrt{r^2 - \cos^2 a} = \frac{r.\cos a}{\text{co-tan } a} = \frac{\text{co-sec } a.\tan a}{r} = \frac{r.\tan a}{\sqrt{r^2 + \tan^2 a}} = \frac{r^2}{\sqrt{r^2 + \text{co-tan}^2 a}} = \frac{r^2}{\text{co-sec } a} = \frac{r.\tan a}{\sec a} = \frac{\cos a.\sec a}{\text{co-sec } a} = \frac{\tan a.\text{co-tan } a}{\text{co-sec } a} = \frac{r\sqrt{\sec^2 a - r^2}}{\sec a}$.

	1st quad.	2nd quad.	3rd quad.	4th quad.
Sine and co-secant	+	+	−	−
Co-sine and secant	+	−	−	+
Tangent and co-tan.	+	−	+	−
Versed sine	+	+	+	+

2. Co-sine of $a = \sqrt{r^2 - \sin^2 a} = \frac{r.\sin a}{\tan a} = \frac{\sin a.\text{co-tan } a}{r} =$
$\frac{r.\text{co-tan } a}{\sqrt{r^2 + \text{co-tan}^2 a}} = \frac{r^2}{\sqrt{r^2 + \tan^2 a}} = \frac{r^2}{\sec a} = \frac{r.\text{co-tan } a}{\text{co-sec } a} = \frac{\sin a.\text{co-sec } a}{\sec a} =$
$\frac{\tan a.\text{co-tan } a}{\sec a.} = \frac{r\sqrt{\text{co-sec}^2 a - r^2}}{\text{co-sec } a}.$

3. Tangent of $a = \frac{r^2}{\text{co-tan } a} = \frac{r.\sin a}{\cos a} = \frac{r^3 \cos a}{\sin a.\text{co-tan}^2 a} = \frac{r.\sin a}{\sqrt{r^2 - \sin^2 a}} =$
$\frac{r\sqrt{r^2 - \cos^2 a}}{\cos a} = \sqrt{\sec^2 a - r^2} = \frac{r.\sec a}{\text{co-sec } a} = \frac{\cos a.\sec a}{\text{co-tan } a} =$
$\frac{\sin a.\text{co-sec } a}{\text{co-tan } a} = \frac{r^2}{\sqrt{\text{co-sec}^2 a - r^2}}.$

4. Co-tangent of $a = \frac{r^2}{\tan a} = \frac{r.\cos a}{\sin a} = \frac{r^3.\sin a}{\cos a.\tan^2 a} = \frac{r.\cos a}{\sqrt{r^2 - \cos^2 a}}$
$= \frac{r\sqrt{r^2 - \sin^2 a}}{\sin a} = \sqrt{\text{co-sec}^2 a - r^2} = \frac{r.\text{co-sec } a}{\sec a} = \frac{\cos a.\sec a}{\tan a} =$
$\frac{\sin a.\text{co-sec } a}{\tan a} = \frac{r^2}{\sqrt{\sec^2 a - r^2}}.$

5. Secant of $a = \sqrt{r + {}^2\tan^2 a} = \frac{r^2}{\cos a} = \frac{r.\tan a}{\sin a} = \frac{\text{co-tan } a.\tan a}{\cos a} =$
$\frac{r\sqrt{r^2 + \text{co-tan}^2 a}}{\text{co-tan } a} = \frac{r^3}{\sin a.\text{co-tan } a} = \frac{r.\text{co-sec } a}{\text{co-tan } a} = \frac{\tan a.\text{co-sec } a}{r} =$
$\frac{\sin a.\text{co-sec } a}{\cos a} = \frac{r.\text{co-sec } a}{\sqrt{\text{co-sec}^2 a - r^2}}.$

6. Co-secant of $a = \sqrt{r^2 + \text{co-tan}^2 a} = \frac{r^2}{\sin a} = \frac{r.\text{co-tan } a}{\cos a} =$
$\frac{\tan a.\text{co-tan } a}{\sin a} = \frac{r\sqrt{r^2 + \tan^2 a}}{\tan a} = \frac{r^3}{\cos a.\tan a} = \frac{r.\sec a}{\tan a} = \frac{\cos a.\sec a}{\sin. a} =$
$\frac{\text{co-tan } a.\sec a}{r} = \frac{r.\sec a}{\sqrt{\sec^2 a - r^2}}.$

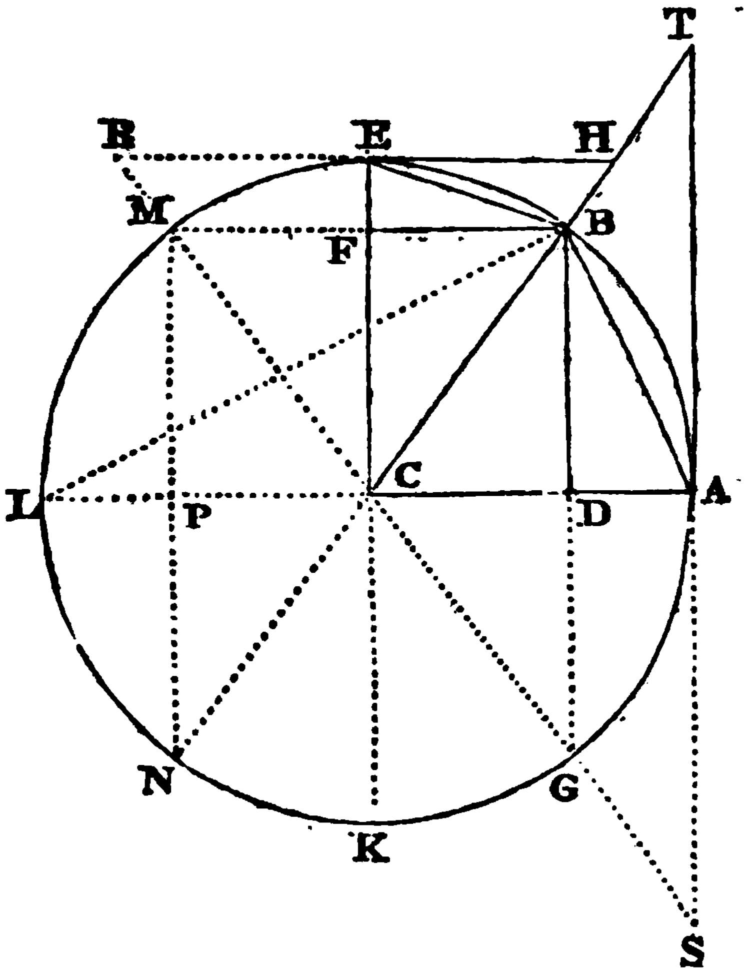
And since the versed sine of $a = r - \cos a$; the co-versed sine $= r - \sin a$; the supplemental versed sine $= r + \cos a$; the chord $= \sqrt{2r.\overline{r - \cos a}}$; the co-chord $= \sqrt{2r.\overline{r - \sin a}}$; and the supplemental chord =

At the beginning and end of each quadrant, the values of these lines will be as follow:

	0°	90°	180°	270°	360°
Sine	0	+ *rad.*	0	− *rad.*	0
Co-sine	+ *rad.*	0	− *rad.*	0	+ *rad.*
Tangent	0	*inf.*	0	*inf.*	0
Co-tangent	*inf.*	0	*inf.*	0	*inf.*
Secant	+ *rad.*	*inf.*	− *rad.*	*inf.*	+ *rad.*
Co-secant	*inf.*	+ *rad.*	*inf.*	− *rad.*	*inf.*
Versed sine	0	+ *rad.*	+ *diam.*	+ *rad.*	0

INTRODUCTORY PROPOSITIONS.

33. The sine, co-sine, tangent, and secant of any *arc*, are respectively equal to the sine, co-sine, tangent, and secant of the supplement of that *arc*.



Let the *arcs* AB and AM be supplements of each other, viz. AB less than a quadrant, and AM greater; then will the sine BD of the *arc* AB, be equal to the sine MP of the *arc* AM, and also the co-sine CD to the co-sine CP.

For since $AM+AB$

$\sqrt{2r.r+\cos a}$; either of these latter may be found in terms of any of the above by proper substitution, regard being had in every case to the change of signs, when the *arc* a is greater than a quadrant. From these expressions for the trigonometrical lines belonging to a single *arc*, others may be derived which are applicable to a great variety of cases, viz. for the sums, differences, multiples, sub-multiples, &c. of given *arcs*; but the prosecution of this useful part of Trigonometry further than is necessary for constructing the sines, tangents, &c. would require more room than can conveniently be spared; we must therefore refer the inquisitive student for the gratification of his wishes, to the writings of Euler, Cagnoli, Vince, Woodhouse, Bonnycastle, and some others, who have treated expressly on the subject.

$=180^\circ=AM+ML$; taking AM from both, the *arc* $AB=ML$, $\because$ the angle $BCA=MCL$ (27.3.); also the angles MPC, BDC are right angles, and the side $MC=BC$, $\because$ (26.1.) $MP=BD$, and $CP=CD$; that is, the sine and co-sine of any *arc* or angle, are respectively equal to the sine and co-sine of the supplement of that *arc* or angle, observing that the sines MP and BD will be both $+$, but the co-sines will have different signs, viz. CD will be $+$, and CP $-$.

Likewise AS the tangent, and CS the secant of the *arc* AM are respectively equal to AT the tangent, and CT the secant of the *arc* AB.

For the angle $TCA=MCL$ (as shewn above) $=ACS$ (15.1.), the angles at A right angles, and the side CA common, $\because$ (26.1.) $AS=AT$, and $CS=CT$; that is, the tangent and secant of any *arc* or angle, are respectively equal to the tangent and secant of the supplement of that *arc* or angle.

In like manner the sine, co-sine, tangent, and secant of an *arc* terminating in the third quadrant LK, will be those of an *arc* which is the excess of the proposed *arc* above a semicircle.

Thus the sine of the *arc* AMN is $PN=PM$ (3.3.) $=BD$, the sine of the *arc* AB; and the co-sine $PC=CD$, the co-sine of AB; only this sine and co-sine (PN and PC) will be negative. AT will likewise be the tangent, and CT the secant of the *arc* AMN, (as appears from Art. 16 and 18); the former of which will be $+$, and the latter $-$.

The sine, co-sine, tangent, and secant of an *arc* terminating in the fourth quadrant KA will be respectively the same with those of an *arc* which is the supplement of the proposed *arc* to the whole circle.

Thus the sine of the *arc* $AMNG$ is GD, which is $=BD$ (3.3.) the sine of the *arc* AB, only GD is negative; the co-sine CD is the *very same* as the co-sine of the *arc* AB.

AS is the tangent of $AMNG$, which is $=AT$; and CS the secant, which is $=CT$; AS will be $-$, $CS+$; see Art. 32.

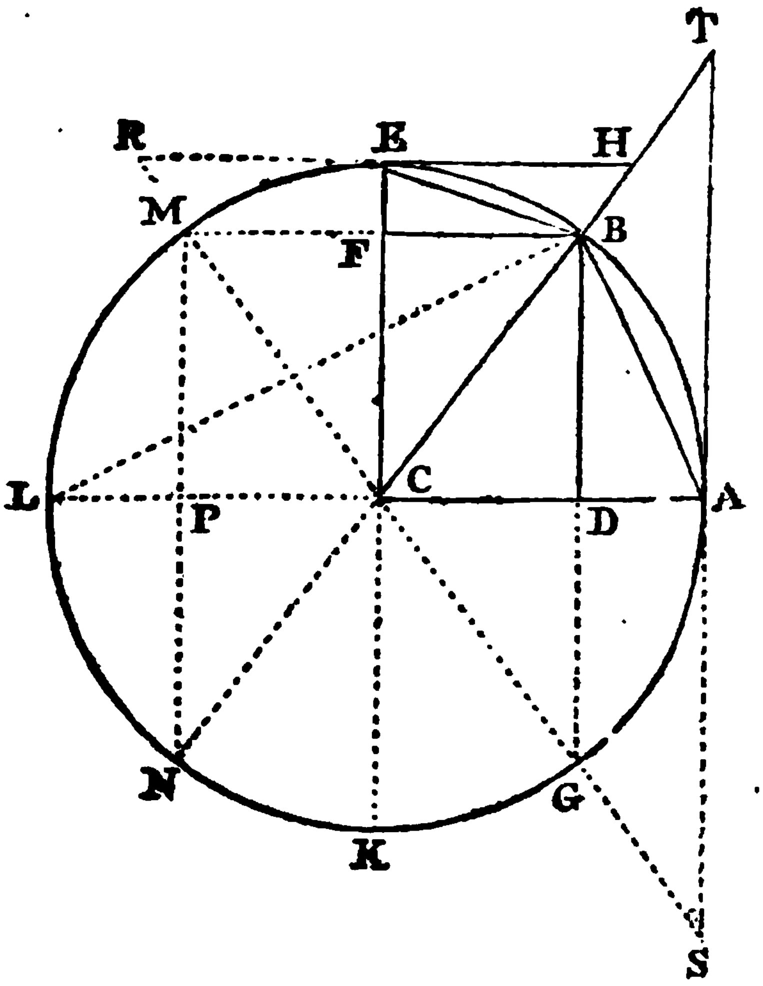
The versed sine AP of any *arc* AM, terminating in the second quadrant, is equal to the difference of the versed sine of its supplement and the diameter, or to the sum of the co-sine and radius.

Thus, (since $AD=LP$) $AP=(AL-LP=)$ $AL-AD=PC+CA$. The versed sine of any *arc*, terminating in the third

and fourth quadrants, is the same with the versed sine of its supplement to the whole circle: thus *AP* is the versed sine of the *arc AMN*, and also of the *arc NGA*; also *AD* is the versed sine of the *arc AMNG*, and likewise of *AG* its supplement to the whole circle. It has been already observed that all the versed sines are affirmative or +.

Thus we have shewn that the sine, co-sine, tangent, and secant of any *arc AB*, will be respectively equal in magnitude to the sine, co-sine, tangent, and secant of its supplement to either a semicircle, or to a whole circle, differing only in the algebraic signs; and therefore if the sine, co-sine, tangent, and secant for every degree and minute of the first quadrant be computed, and the whole arranged in a table, this table will serve for the whole circle.

34. The sine of any *arc* is equal to half the chord of double that *arc*; and conversely, the chord is double the sine of half the *arc*.



Because *CA* cuts *BG* at right angles $BD = DG$ (3. 3.) ∵ $BD = \frac{1}{2}BG$; also the *arc* $BA =$ the *arc* AG (30. 3.) ∵ the *arc* $BA = \frac{1}{2}$ the *arc* BG; that is, *BD* the sine of the *arc BA* is half the chord *BG* of (the *arc BAG*, which is) double the *arc BA*. Q. E. D. The converse is sufficiently evident from the preceding demonstration.

Cor. Hence, because the chord of $60° =$ the radius (Art. 9. cor.) ∵ the sine of $30° = (\frac{1}{2}$ the chord of $60° =) \frac{1}{2}$ radius. Hence also the co-sine of $60° =$ (sine of $30° =$) $\frac{1}{2}$ radius; and the versed sine of $60° =$ (radius — co-sine $=$) $\frac{1}{2}$ radius.

35. The sine or co-sine of any *arc*, together with the radius being given, we may thence determine the rest of the trigonometrical lines belonging to that *arc*, as follows:

First, Let CB the radius, and BD the sine of the *arc* BA, be given, to find the co-sine CD; then (47. 1.) $\overline{CB}|^2=\overline{BD}|^2+\overline{CD}|^2$, and $\overline{CB}|^2-\overline{BD}|^2=\overline{CD}|^2$, $\because CD=\sqrt{\overline{CB}|^2-\overline{BD}|^2}$; that is, *the co-sine of an arc is equal to the square root of the difference of the squares of the radius and sine.*

Secondly. Let CB the radius, and CD the co-sine be given, to find BD the sine; thus, (as above) $BD=\sqrt{\overline{CB}|^2-\overline{CD}|^2}$; that is, *the sine of an arc is equal to the square root of the difference of the squares of the radius and co-sine.*

Thirdly. Since $AD=CA-CD$, and $AP=AC+CP$; therefore *the versed sine of any arc* less than a quadrant, *is equal to the difference of the radius and co-sine; but of any arc* greater than a quadrant, *it is equal to the sum of the radius and co-sine.*

Fourthly. Because $\overline{BA}|^2=\overline{BD}|^2+\overline{DA}|^2$ (47. 1.) $\because BA=\sqrt{\overline{BD}|^2+\overline{DA}|^2}$; that is, *the chord of any arc is equal to the square root of the sum of the squares of the sine and versed sine* [f].

Fifthly. Because $\overline{EB}|^2=(\overline{BF}|^2+\overline{FE}|^2$ (47. 1.) $=\overline{DC}|^2+\overline{CE-CF}|^2=)$ $\overline{DC}|^2+\overline{CE-BD}|^2$ $\because EB=\sqrt{\overline{DC}|^2+\overline{CE-BD}|^2}$; that is, *the co-chord of an arc is equal to the square root of the sum of the squares of the co-sine and the excess of the radius above the sine.*

Sixthly. Because the right angled triangles BCD, TCA, BCF, and HCE have the acute angle TCA which is common to the two former, equal to each of the acute angles CBF, CHE in the two latter (by 29. 1.); these four triangles are equiangular (32. 1.), and have the sides about their equal angles proportionals (4. 6.); whence we have the following analogies.

1. $\left\{\begin{matrix} CD & : DB & :: CA & : AT \\ \text{co-sine} & : \text{sine} & :: \text{radius} & : \text{tangent}\end{matrix}\right\} \because AT=\dfrac{DB.CA}{CD}$,

or TANGENT $=\dfrac{\text{sine}\times\text{radius}}{\text{co-sine}}=(\text{if radius}=1)\dfrac{\text{sine}}{\text{co-sine}}$ [g].

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[f] In like manner it is shewn that $LB=(\sqrt{\overline{BD}|^2+\overline{DL}|^2}=)\sqrt{\overline{BD}|^2+\overline{AP}|^2}$ or, *The supplemental chord is equal to the square root of the sum of the squares of the sine and supplemental versed sine.*

[g] Hence it appears, that when the sine and co-sine have like algebraic signs, the tangent will be $+$, but when they have unlike signs, the tangent will be $-$.

2. $\left\{\begin{matrix} CD & : CB & :: CA & : CT \\ \text{co-sine} & : \text{radius} & :: \text{radius} & : \text{secant} \end{matrix}\right\} \therefore CT = \frac{CB.CA}{CD} = \frac{\overline{CB}|^2}{CD}$, or SECANT $= \frac{\overline{\text{radius}}|^2}{\text{co-sine}} =$ (if rad. $=1$) $\frac{1}{\text{co-sine}}$ [h].

3. $\left\{\begin{matrix} DB : CB & :: EC & : CH \\ \text{sine} : \text{radius} & :: \text{radius} & : \text{co-secant} \end{matrix}\right\} \therefore CH = \frac{CB.EC}{DB} = \frac{\overline{CB}|^2}{DB}$, or CO-SECANT $= \frac{\overline{\text{radius}}|^2}{\text{sine}} =$ (if rad. $=1$) $\frac{1}{\text{sine}}$ [i].

4. $\left\{\begin{matrix} DB : DC & :: EC & : EH \\ \text{sine} : \text{co-sine} & :: \text{radius} & : \text{co-tangent} \end{matrix}\right\} \therefore EH = \frac{DC.EC}{DB}$, or CO-TANGENT $= \frac{\text{co-sine} \times \text{radius}}{\text{sine}} =$ (if rad. $=1$) $\frac{\text{co-sine}}{\text{sine}}$ [k].

5. $\left\{\begin{matrix} TA & : AC & :: CE & : EH \\ \text{tangent} & : \text{radius} & :: \text{radius} & : \text{co-tangent} \end{matrix}\right\} \therefore EH = \frac{AC.CE}{TA} = \frac{\overline{AC}|^2}{TA}$, or CO-TANGENT $= \frac{\overline{\text{radius}}|^2}{\text{tangent}} =$ (if rad. $=1$) $\frac{1}{\text{tangent}}$.

6. $\left\{\begin{matrix} TA & : TC & :: CE & : CH \\ \text{tangent} & : \text{secant} & :: \text{radius} & : \text{co-secant} \end{matrix}\right\} \therefore CH = \frac{TC.CE}{TA}$, or CO-SECANT $= \frac{\text{secant} \times \text{radius}}{\text{tangent}} =$ (if rad. $=1$.) $\frac{\text{secant}}{\text{tangent}}$ [l].

Cor. Hence the radius is a mean proportional between the co-sine and secant; between the sine and co-secant, and between the tangent and co-tangent.

36. The secant of 60° is equal to the diameter.

For since the co-sine of $60° = \frac{1}{2}$ radius (cor. Art. 34.) $= \frac{1}{2}$ *CB*, if this value be substituted for *CD* in the second analogy (given above), we shall have $CT = (\frac{\overline{CB}|^2}{CD} =) \frac{\overline{CB}|^2}{\frac{1}{2}CB} = \frac{CB}{\frac{1}{2}} = 2\,CB$; that is, the secant of 60° is equal to the diameter. Q. E. D.

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[h] Hence the secant will always have the same algebraic sign with the co-sine.

[i] Hence the co-secant will have the same algebraic sign with the sine.

[k] Hence the co-tangent will be + when the sine and co-sine have like signs, and − when they have unlike, viz. it will always have the same sign as the tangent (see the 1st analogy.)

[l] Hence, when the tangent and secant have like signs, the co-secant will be +, but when they have unlike, −.

Cor. Hence the tangent of 60° = twice the sine; for since $CB : CT :: BD : TA$ (4.6. and 16.5.) and $CT = 2\,CB \therefore TA = 2\,BD$ (cor. 4.5.)

37. From what has been delivered, we can readily determine the arithmetical values of the chord, co-chord, supplemental chord, sine, co-sine, tangent, co-tangent, secant, co-secant, versed sine, co-versed sine, and supplemental versed sine of the *arcs* of 30°, 45°, 60°, and 90° to any given radius; thus, let the radius $=1$, then

Art. 36.	secant of 60°	= the diameter = 2.0000000.
Art. 19.	co-secant of 30°	
Art. 31.	versed sine of 180°	
Art. 9. *cor.*	chord of 180°	
Art. 9. *cor.*	chord of 60°	= the radius = 1.0000000.
Art. 10.	co-chord of 30°	
Art. 16. *cor.*	tangent of 45°	
Art. 17.	co-tang. of 45°	
Art. 13. *cor.*	sine of 90°	
Art. 31.	versed sine of 90°	
Art. 24.	co-sine of 180°	
Art. 34. *cor.*	sine of 30°; co-sine of 60°; versed sine of 60°	$= \frac{1}{2}$ the radius = 0.5000000.
Art. 15. *cor.*	co-versed sine of 30°	
Art. 34. *cor.* 35.	co-sine of 30°	$= \sqrt{\overline{\text{rad.}}\vert^2 - \frac{1}{4}\,\overline{\text{rad.}}\vert^2} = \sqrt{.75} =$ . . . . 0.8660254.
Art. 13.	sine of 60°	
Art. 35.	tangent of 30°	$= \textit{rad.} \times \frac{\text{sine } 30^\circ}{\text{co-sine } 30^\circ} = \frac{.5}{.8660254} =$ . . . . . 0.5773503.
Art. 17.	co-tang. of 60°	
Art. 35.	versed sine of 30°	= rad. − co-sine 30° = 1 − .8660254 = 0.1339746.
Art. 15.	co-versed sine of 60°	
Art. 35.	chord of 30°	$= \sqrt{\overline{\text{sine}}\vert^2 \text{of } 30^\circ + \overline{\text{v. sine}}\vert^2 \text{of } 30^\circ} = \sqrt{.25 + .0179492} =$ . . . . 0.5176380.
Art. 10.	co-chord of 60°	
Art. 35.	secant of 30°	$= \frac{\overline{\text{rad.}}\vert^2}{\text{co-sine } 30^\circ} = \frac{1}{.8660254} =$ . . . . 1.1547005.
Art. 19.	co-secant of 60°	

Art. 34. sine of 45°
Art. 13. co-sine of 45. $\Big\} = \frac{1}{2}$ chord of $90° = \frac{1}{2}\sqrt{2 \cdot \overline{\text{rad.}}|^2}$
$= \frac{1}{2}\sqrt{2} =$ .... 0.7071068.

Art. 35. versed sine of 45°
Art. 15. co-versed sine of 45° $\Big\} =$ rad. — co-sine $= 1 - .7071068$
$=$ .... 0.2928932.

Art. 36. secant of 45°
Art. 19. co-secant of 45° $\Big\} = \dfrac{\overline{\text{rad.}}|^2}{\text{co-sine}} = \dfrac{1}{.7071068} =$
.... 1.4142136.

Art. 35. chord of 45°
Art. 10. co-chord of 45° $\Big\} = \sqrt{\overline{\text{sine}}|^2 + \overline{\text{v. sine}}|^2} =$
$\sqrt{\overline{.7071068}|^2 + \overline{.2928932}|^2} =$
.... 0.7653668.

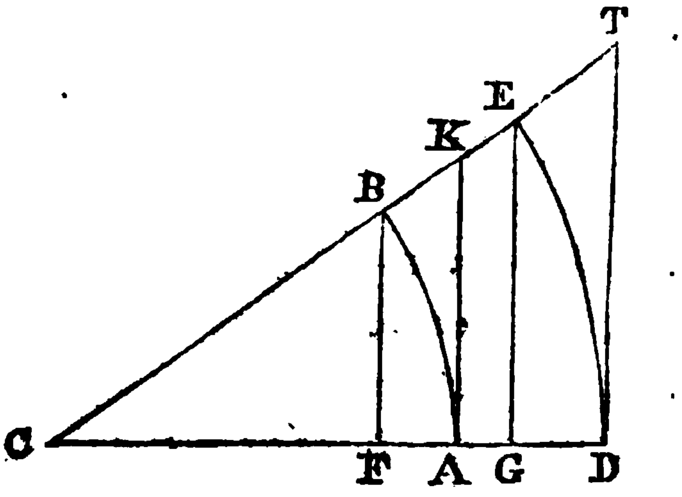
Art. 35. tangent of 60°
Art. 17. co-tangent of 30° $\Big\} = \text{rad.} \times \dfrac{\text{sine of } 60°}{\text{co-sine of } 60°} =$
$\dfrac{.8660254}{.5} =$ ...... 1.7320508.

In like manner (Art. 35.) the chord of the supplement of

$$\left.\begin{matrix}90°\\60°\\45°\\30°\end{matrix}\right\} = \text{chord of} \left\{\begin{matrix}90°\\120°\\135°\\150°\end{matrix}\right\} = \sqrt{\overline{\text{sine}}|^2 + \overline{\text{sup. v. sine}}|^2} = \left\{\begin{matrix}1.4142136\\1.7653668\\1.8477591\\1.9318516\end{matrix}\right.$$

38. The sine, co-sine, tangent, secant, &c. of any *arc AB* of a circle, whose radius is *CA*, is to the sine, co-sine, tangent, secant, &c. of a similar *arc DE*, whose radius is *CD*, as *CA* to *CD*.

From the point *B* let fall *BF* perpendicular to *CD* (12. 1.), and through *A*, *E*, and *D*, draw *AK*, *EG*, and *DT*, parallel to *BF* (31. 1.), then will *BF* be the sine of the *arc BA*, *CF* its co-sine, *AK* its tangent; *EG* the sine of *ED*, *CG* its co-sine, and *DT* its tangent (Art. 12. 16.); and since *AB* and *DE* each subtend the common angle at the centre *C*, they are similar, that is, they contain each the same number of degrees (part 8. Art. 239.); now since the angles at *F*, *A*, *G*, and *D*, are right angles, and the angle at *C* common, the triangles *BCF*, *KCA*, *ECG*, and *TCD*, are similar



(32. 1.), and have the sides about their equal angles proportionals (4.6.); that is,

First, $FB : BC :: GE : EC$, and alternately (16.5.) $FB : GE :: BC : EC$; that is, *sine of arc BA : sine of arc ED :: rad. (BC) of the former arc : rad. (EC) of the latter.*

Secondly, $FC : CB :: GC : CE$, and alternately $FC : GC :: CB : CE$; that is, *co-sine of arc BA : cosine of arc ED :: rad. (CB) of the former : rad. (CE) of the latter.*

Thirdly, $KA : AC :: TD : DC$, and alternately $KA : TD :: AC : DC$; that is, *tang. arc BA : tang. arc ED :: rad. of BA : rad. of ED.*

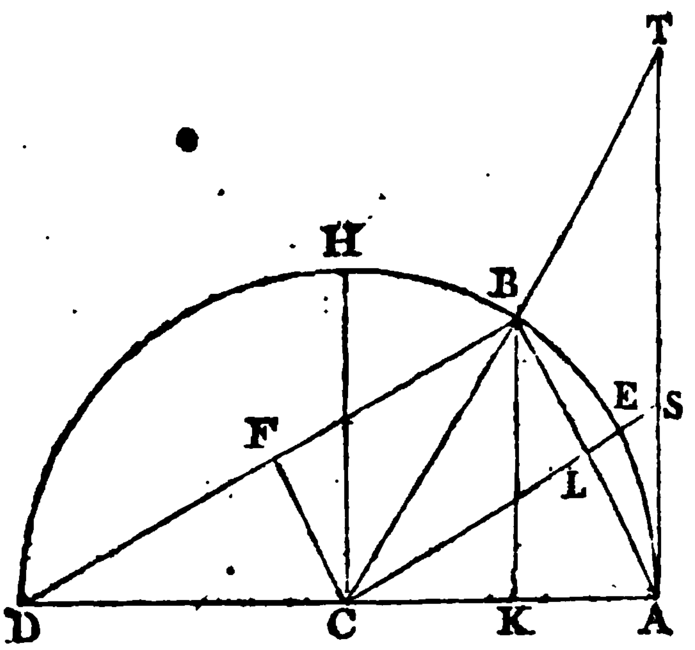
Fourthly, $KC : CA :: TC : CD$ ∵ alternately $KC : TC :: CA : CD$; that is, *secant of arc BA : secant arc ED :: rad. of BA : rad. of ED.*

Fifthly, Because $BC : CF :: EC : CG$ ∵ by conversion (prop. E. 5.) $BC : FA :: EC : GD$, ∵ inversely (prop. B. 5.), $FA : (BC=) AC :: GD : (EC=) DC$ ∵ alternately $FA : GD :: AC : DC$; that is, *versed sine of arc BA : versed sine of arc ED :: rad. of BA : rad. of ED.* Wherefore the sines, co-sines, tangents, secants, and versed sines of a given angle in different circles, are respectively as the radii of those circles. Q. E. D.

Hence, if sines, co-sines, tangents, &c. be computed to a given radius, they may be found to any other radius, by the above proportions.

39. The co-sine of any *arc*, is equal to half the chord of the supplement of double that *arc*.

Let AE be an *arc*, C the centre, join CE, and from A draw AL perpendicular to CE (12. 1.), and produce it to B, join BD, and from the centre C draw CF perpendicular to BD, ∵ $DF=$



FB (3.3.); also *CL* is the co-sine of *AE* (Art. 13.), *BD* the supplemental chord of (*AEB*=) double of *AE* (Art. 11.), and *FB*=half the said supplemental chord.

Because *DBA* is a right angle (13. 3.), and *CLB*, *CFB* right angles (by construction), ∵ *FB* is parallel to *CL*, and *BL* to *FC* (28. 1.), ∵ *FBLC* is a parallelogram, and *CL*=*FB* (34. 1.); that is, the cosine of the *arc* *AE* is equal to half the supplemental chord of (*AB*) double of *AE*. Q. E. D.

40. The chord of an *arc* is a mean proportional between its versed sine and the diameter.

Draw *BK* at right angles to *DA* (12. 1), then because *DBA* is a right angle (31. 3.), *DA* : *AB* :: *AB* : *AK* (cor. 8. 6.); that is, the diameter is to the chord of the *arc* *AEB*, as the same chord is to the versed sine of *AEB*. Q. E. D.

41. The sum of the tangent and secant of any *arc*, is equal to the co-tangent of half the complement of that *arc*.

Draw *CH* at right angles to *DA* (12. 1.), and let *AE* be any *arc*, *AS* its tangent, *CS* its secant, and the *arc* *EH* its complement. Bisect *EH* in *B* (30. 3.), and draw *CBT* meeting *AS* produced in *T*.

Then *AT* is the tangent of the *arc* *AEB* (Art. 16.) that is, the co-tangent of *HB* (Art. 17.) which is half the complement of *AE*.

Because *AT* and *CH* are parallel, the angle *HCB*=*CTA* (29. 1.); but *HCB*=*BCE* ∵ *BCE*=*CTA* ∵ *SC*=*ST* (6. 1.) *AS*+*SC*=*AT*; that is, the sum of the tangent and secant of the *arc* *AE* is equal to (*AT*) the co-tangent of (*HB*) half the complement of *AE*. Q. E. D.

42. The radius is to the co-sine of an *arc*, as twice the sine to the sine of double that *arc*.

Because the right angled triangles *ALC*. *AKB* have the angle at *A* common, they are equiangular (32. 1.), ∵ *AC*:

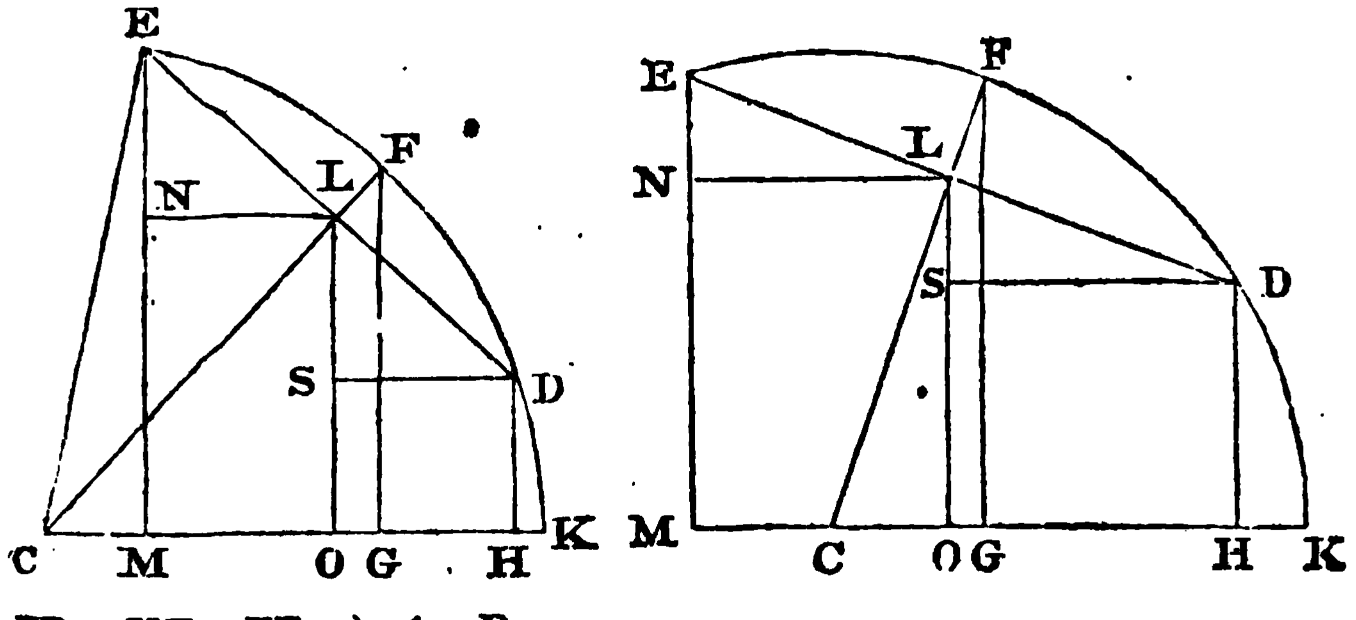
$CL :: AB : BK$, that is radius : co-sine of AE :: twice the sine of AE : sine of double of AE. Q. E. D.

THE INVESTIGATION OF FORMULÆ, NECESSARY FOR THE CONSTRUCTION OF THE TRIGONOMETRICAL CANON.

43. The sines and co-sines of two unequal *arcs* being given to determine the sine and co-sine of their sum and difference.

Let KF, FE be two unequal *arcs* of which the sines and co-sines are given, and let KF be the greater, from which cut off $FD=FE$ the less (34.6.), join ED, and from the centre C draw CF perpendicular to ED (12. 1.) $\therefore EL=LD$ (3. 3.); draw DH, FG, LO, EM, each perpendicular to the diameter CK, and DS, LN each parallel to it (31. 1.) meeting LO, EM in the points S and N.

Because $EL=LD$ $EF=FD$, $\therefore$ (30. 3.); and because LN is parallel to DS, the angle $ELN=LDS$ (29. 1.), $\therefore$ the right angled triangles ELN, LDS having all their angles equal, and the homologous sides EL, LD equal, are equal and similar (26. 1. and def. 1. 6.), $\therefore EN=LS$ and $NL=SD$; also in the parallelograms $NMOL$, $SOHD$, we have $NM=LO$, $NL=MO$, $DH=SO$, and $SD=OH$ (34. 1.), $\therefore NL=MO=SD=OH$. Let the *arc* $KF=A$, the *arc* $FE=B$, and the radius $CF=R$; then will the *arc* $KE=(KF+FE=)\ A+B$, and the *arc* $KD=(KF-$



$FD=KF-FE=)\ A-B$;

also FG is the sine, CG ... co-sine } of A. and EM is the sine, CM ... co-sine } of $\overline{A+B}$.

EL ... sine, CL ... co-sine } of B. DH ... sine, CH ... co-sine } of $\overline{A-B}$.

Because NL is parallel to CO, and FG to LO and the angles at G, O, and N right angles, the triangles CFG, CLO, and ELN are equiangular (29 and 32. 1.), consequently (4. 6.)

$$CF : FG :: CL : LO, \because LO = \left(\frac{FG.CL}{CF} =\right) \frac{\sin A.\cos B}{R}.$$

$$CF : CG :: EL : EN, \because EN = \left(\frac{CG.EL}{CF} =\right) \frac{\cos A.\sin B}{R}.$$

$$CF : CG :: CL : CO, \because CO = \left(\frac{CG.CL}{CF} =\right) \frac{\cos A.\cos B}{R}.$$

$$CF : FG :: EL : LN, \because LN = \left(\frac{FG.EL}{CF} =\right) \frac{\sin A.\sin B}{R}.$$

But EM $(=MN+EN=LO+EN)$, or $\sin \overline{A+B} = \frac{\sin A.\cos B+\cos A.\sin B}{R}$.

CM $(=CO-MO=CO-LN)$, or $\cos \overline{A+B} = \frac{\cos A.\cos B-\sin A.\sin B}{R}$.

DH $(=SO=LO-LS=LO-EN)$, or $\sin \overline{A-B} = \frac{\sin A.\cos B-\cos A.\sin B}{R}$.

CH $(=CO+OH=CO+LN)$, or $\cos \overline{A-B} = \frac{\cos A.\cos B+\sin A.\sin B}{R}$.

44. These formulæ for the sines and co-sines of the *arcs* $\overline{A \pm B}$ which are, it is plain, adapted to *any* radius R, may be simplified and rendered more convenient by putting $R=1$; they will then become

Formula 1. $\text{Sin } \overline{A+B} = \sin A.\cos B+\cos A.\sin B.$

2. $\text{Cos } \overline{A+B} = \cos A.\cos B-\sin A.\sin B.$

3. $\text{Sin } \overline{A-B} = \sin A.\cos B-\cos A.\sin B.$

4. $\text{Cos } \overline{A-B} = \cos A.\cos B+\sin A.\sin B.$

45. To find the sine and co-sine of multiple *arcs*, that is, if A be any *arc*, to find the sine and co-sine of nA.

Add the *first* and *third* of the above formulæ together, and in the sum let A be substituted for B, and B for A, and we shall have

$\sin \overline{B+A} + \sin \overline{B-A} = 2\cos A.\sin B$, that is,

$\sin \overline{B+A} = 2\cos A.\sin B - \sin \overline{B-A}$. (Y).

Add the *second* and *fourth* together, and substitute B for A, and A for B as before: then,

$\cos \overline{B+A} + \cos \overline{B-A} = 2 \cos A . \cos B$; that is,

$\cos \overline{B+A} = 2 \cos A . \cos B - \cos \overline{B-A}$ (Z)

Let $\overline{n-1}.A = B$; this value being substituted for B in the expressions Y and Z, we have the two following theorems for the sines and co-sines of multiple *arcs*, viz.

Theor. 1. $\sin nA = 2 \cos A . \sin \overline{n-1}.A - \sin \overline{n-2}A$.

2. $\cos nA = 2 \cos A . \cos \overline{n-1}\, A - \cos \overline{n-2}\, A$.

In which general theorems, if n be expounded by 1, 2, 3, 4, 5, &c. we have the formulæ for all particular multiple *arcs*, viz. if

$n=2. \begin{cases} 5. \ \sin 2A = 2 \cos A . \sin A \text{ (from theor. 1.)} \\ 6. \ \cos 2A = 2 \cos A . \cos A - \cos 0\ (=1) \text{ (theor. 2.)} \end{cases}$

$n=3. \begin{cases} 7. \ \sin 3A = 2 \cos A . \sin 2A - \sin A \text{ (theor. 1.)} \\ 8. \ \cos 3A = 2 \cos A . \cos 2A - \cos A \text{ (theor. 2.)} \end{cases}$

$n=4. \begin{cases} 9. \ \sin 4A = 2 \cos A . \sin 3A - \sin 2A \text{ (theor. 1.)} \\ 10. \ \cos 4A = 2 \cos A . \cos 3A - \cos 2A \text{ (theor. 2.)} \end{cases}$

$n=5. \begin{cases} 11. \ \sin 5A = 2 \cos A . \sin 4A - \sin 3A \text{ (theor. 1.)} \\ 12. \ \cos 5A = 2 \cos A . \cos 4A - \cos 3A \text{ (theor. 2.)} \end{cases}$

&c. &c. &c.

46. These formulæ may be continued to any length, and by means of them, the sine and co-sine of every degree and minute of the quadrant, may be computed, as will be shewn; but, having found the sines and co-sines to the end of the first 30 degrees by this method, those from 30° to 60° may be obtained by an easier process, by means of the following formula.

Add formulæ 1 and 3 (Art. 44.) together, and sine $\overline{A+B}$ $+ \sin \overline{A-B} = 2 \sin A . \cos B$; let $A = 30°$, then will sin. $A = \frac{1}{2}$ (cor. Art. 34); substitute these values of A and sin. A in the above expression, and it will become

$\sin \overline{30° + B} + \sin \overline{30 - B} = (2 \times \frac{1}{2} \times \cos B =) \cos B$;

∵ *Formula* 13. $\sin \overline{30+B} = \cos B - \sin \overline{30-B}$.

47. The tangents of two unequal *arcs* A and B being given, to find the tangents and co-tangents of their sum and difference.

It has been shewn (Art. 35.), that when radius $=1$, the tangent of any *arc* $= \dfrac{\text{sine}}{\text{co-sine}}$: wherefore, by substituting for

the sine and co-sine their respective values as given in the formulæ, Art. 44. we shall have

Formula 14. $\text{Tan } \overline{A+B} = \frac{\sin \overline{A+B}}{\cos \overline{A+B}} =$
$\frac{\sin A. \cos B + \cos A. \sin B}{\cos A. \cos B - \sin A. \sin B}.$

F. 15. $\text{Tan } \overline{A-B} = \frac{\sin \overline{A-B}}{\cos \overline{A-B}} = \frac{\sin A. \cos B - \cos A. \sin B}{\cos A. \cos B + \sin A. \sin B}.$

If both terms of the right hand fractions be divided by *cos A. cos B*, they will become

F. 16. $\text{Tan } \overline{A+B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A. \sin B}{\cos A. \cos B}} = \frac{\tan A + \tan B}{1 - \tan A. \tan B}$ (Art. 35.)

F. 17. $\text{Tan } \overline{A-B} = \frac{\frac{\sin A.}{\cos A.} - \frac{\sin B}{\cos B}}{1 + \frac{\sin A. \sin B}{\cos A. \cos B}} = \frac{\tan A - \tan B}{1 + \tan A. \tan B}$ (Art. 35.)

F. 18. $\text{Cotan } \overline{A+B} = \frac{\cos \overline{A+B}}{\sin \overline{A+B}}$ (Art. 35.) $= \frac{1 - \tan A. \tan B}{\tan A + \tan B}.$

F. 19. $\text{Cotan } \overline{A-B} = \frac{\cos \overline{A-B}}{\sin \overline{A-B}} = \frac{1 + \tan A. \tan B}{\tan A - \tan B}.$

48. To find the tangents and co-tangents of multiple *arcs*; that is, if *A* be any *arc*, to find the tangent and co-tangent of *nA*.

Since $\tan \overline{A+B} = \frac{\tan A + \tan B}{1 - \tan A. \tan B}$. (Art. 47.) *First*, let $B = A$, then

F. 20. $\text{Tan } 2A = (\tan \overline{A+B} =)\ \frac{2 \tan A}{1 - \overline{\tan A}|^2}.$

F. 21. $\text{Co-tan } 2A = (\frac{1}{\tan 2A}$ Art. 35. $=)\ \frac{1 - \overline{\tan A}|^2}{2 \tan A} = \frac{1}{2 \tan A} - \frac{\overline{\tan A}|^2}{2 \tan A} =$ (Art. 35. analogy 5.) $\frac{1}{2}$ co-tan $A - \frac{1}{2}$ tan A.

Secondly. Let $B = 2A$, then will

$$F. 22. \text{ Tan } 3A = \frac{\tan A + \tan 2A}{1 - \tan A . \tan 2A} = \frac{\tan A + \frac{2 \tan A}{1 - \overline{\tan A}|^2}}{1 - \frac{2 \overline{\tan A}|^2}{1 - \overline{\tan A}|^2}} = \frac{3 \tan A - \overline{\tan A}|^3}{1 - 3 \overline{\tan A}|^2}.$$

$$F. 23. \text{ Co-tan } 3A = \left(\frac{1}{\tan 3A} \text{ Art. 35.} =\right) \frac{1 - 3 \tan A^2}{3 \tan A - \overline{\tan A}|^3}.$$

In like manner,

$$F. 24. \text{ Tan } 4A = \frac{4 \tan A - 4 \overline{\tan A}|^3}{1 - 6 \overline{\tan A}|^2 + \overline{\tan A}|^4}.$$

$$F. 25. \text{ Co-tan } 4A = \frac{1 - 6 \overline{\tan A}|^2 + \overline{\tan A}|^4}{4 \tan A - 4 \overline{\tan A}|^3}.$$

&c. &c.

49. These formulæ may be extended to every minute of the quadrant; but although it seemed necessary to shew how the tangents and co-tangents of multiple *arcs* are expressed in terms of the tangents of the component *arcs* themselves, yet we have shewn how to compute the tangents and co-tangents for the first 45° by means of the sines and co-sines, which is in many respects preferable to the above method. The tangents and co-tangents of *arcs* above 45°, may be found by a very easy process, the formula for which is deduced as follows:

It appears from formulæ 16 and 17, Art. 47. that

$\text{Tan } \overline{A \pm B} \frac{\tan A \pm \tan B}{1 \mp \tan A . \tan B}$; let $A = 45°$, then (Art. 16. cor.) $\tan . A = 1$.

Hence, $\tan \overline{45° + B} = \frac{1 + \tan B}{1 - \tan B}$, and $\tan \overline{45° - B} = \frac{1 - \tan B}{1 + \tan B}$.

Subtract the latter from the former, and

$$\text{Tan } \overline{45° + B} - \tan \overline{45 - B} = \frac{1 + \tan B}{1 - \tan B} - \frac{1 - \tan B}{1 + \tan B} = \frac{\overline{1 + \tan B}|^2 - \overline{1 - \tan B}|^2}{1 - \overline{\tan B}|^2} = \frac{4 \tan B}{1 - \overline{\tan B}|^2};$$

but since $\tan 2B = \frac{2. \tan B}{1 - \overline{\tan B}|^2}$ (formula 20. Art. 48); $\because 2 \tan 2B = \frac{4. \tan B}{1 - \overline{\tan B}|^2}$,

for this fraction substitute its equal (2 tan 2B) in the last equation but one, and we shall have tan $\overline{45°+B}$—tan $\overline{45°-B}$= 2 tan. 2B; hence arises

Formula 26. Tan $\overline{45°+B}$=tan $\overline{45°-B}$+2 tan 2B [m].

THE METHOD OF CONSTRUCTING A TABLE OF SINES, TANGENTS, SECANTS, AND VERSED SINES.

50. In the preceding articles the methods of deriving expressions for the sines, co-sines, tangents, &c. of the sum, difference, and multiples of *arcs* in terms of the sines, co-sines, &c. of the *arcs* themselves, have been shewn; but before we can employ these formulæ in the actual construction of the trigonometrical canon, in which the numerical values of the sine, tangent, &c. of *arcs* for every minute of the quadrant are usually exhibited, it will be necessary to compute the sine and co-sine of 1 minute, and from these we shall be able, by means of what has already been proved, to determine not only the numerical values of the rest of the sines and co-sines, but likewise those of the tangents, co-tangents, secants, co-secants, versed sines, and co-versed sines, which constitute the entire canon.

51. *To find the sine and co-sine of an arc of* 1′, *the radius being unity.*

It has been shewn (part 8. p. 231, 232.) that if the radius of a circle be unity, the semi-circumference will be 3.1415926535898 nearly; this semi-circumference consists of 180 degrees, each degree being 60 minutes; that is, of (180×60=) 10800 minutes; $\therefore \frac{3.1415926535898}{10800}$=.0002908882086=the length of an *arc* of 1′, the radius being unity.

But in a very small *arc*, as that of 1′, the sine coincides indefinitely near with the *arc* [n]; wherefore the above number

[m] The trigonometrical formulæ, introduced into this work, are those only which are necessary for the construction of a table of sines, tangents, &c. Several of the French and German mathematicians have excelled in this species of investigation, and produced a great variety of theorems suited to every case in Trigonometry. The English reader will find a collection of formulæ, applicable to the most delicate investigations in Mechanics, Astronomy, &c. in Mr. Bonnycastle's *Treatise on Plane and Spherical Trigonometry*, London, 1806.

[n] In Simpson's *Doctrine and Application of Fluxions*, part 2. p. 501. and

.0002908882, &c. may be taken for the length of $\overline{\text{the sine of } 1'}$. Wherefore also (Art. 35.) the co-sine of $1' = \sqrt{1 - \overline{\sin 1'}|^2} =$ ($\sqrt{.9999999153840}5$, &c.=) .99999996.

52. *Construction of the sines and co-sines from* 0 *to* 30°.

Since (Art. 51.) the sine of $1' = (.0002908882086$, &c. =) .0002909, which is its nearest value to seven places of decimals, and co-sine of $1' = .99999996$. Let A = an *arc* of $1'$, then the above numeral values being substituted respectively for sine and co-sine of $1'$ in formula 5. Art. 45. we shall have

By *Formula* 5. $\sin 2' = 2 \cos 1'. \sin 1' = 2 \times .99999996 \times .0002909 = .0005818$, *here the sine of* $2'$ *is found*

F. 6. $\text{Cos } 2' = 2 \cos 1'. \cos 1' - 1 = 2 \times .99999996 \times .99999996 - 1 = .9999998$, *here the co-sine of* $2'$ *is found.*

F. 7. $\text{Sin } 3' = 2 \cos 1'. \sin 2' - \sin 1' = 2 \times .99999996 \times .0005818 - .0002909 = .0008727$, *here the sine of* $3'$ *is found.*

F. 8. $\text{Cos } 3' = 2 \cos 1'. \cos 2' - \cos 1' = 2 \times .99999996 \times .9999998 - .99999996 = .9999996$, *here the co-sine of* $3'$ *is found.*

F. 9. $\text{Sin } 4' = 2 \cos 1'. \sin 3' - \sin 2' = 2 \times .99999996 \times .0008727 - .0005818 = .0011636$.

F. 10. $\text{Cos } 4' = 2 \cos 1'. \cos 3' - \cos 2' = 2 \times .99999996 \times .9999996 - .9999998 = .9999993$.

F. 11. $\text{Sin } 5' = 2 \cos 1'. \sin 4' - \sin 3' = .0014544$.

F. 12. $\text{Cos } 5' = 2 \cos 1'. \cos 4' - \cos 3' = .9999989$. And in this manner proceed to find the sine and co-sine of every minute as far as 30°.

52. B. *To find the sines and co-sines from* 30° *to* 60°.

By formula 13. Art. 46. $\sin \overline{30° + B} = \cos B - \sin \overline{30 - B}$.

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in Vince's *Fluxions*, p. 220. it is shewn that (radius being 1,) the sine of any arc $A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$, = (in the present instance)

$$.0002908882086 - \frac{\overline{.0002908882086}|^3}{6} + \frac{\overline{.0002908882086}|^5}{6.4.5} - \frac{\overline{.0002908882086}|^7}{120.6.7}$$

+ &c. =.0002908881676, &c. = the sine of $1'$, which differs from the above expression for the length of the *arc* of $1'$ by only .000000000141; that is, the *arc* of $1'$ and its sine, coincide to 9 decimal places inclusive, therefore the sine of $1'$ to 7 places of decimals (the number to which the tables are usually computed) exactly coincides with its *arc*.

° Let $B=1$, then sin 30° 1′=cos 1′−sin 29° 59′=.99999996−.4997481=.5002519.

$B=2'$ .Sin 30° 2′=cos 2′−sin 29° 58′=.9999998−.4994961=.5005037.

$B=3'$ .Sin 30° 3′=cos 3′−sin 29° 57′=.5007556.

$B=4'$ .Sin 30° 4′=cos 4′−sin 29° 56′=.5010073.

$B=5'$ .Sin 30° 5′=cos 5′−sin 29° 55′=.5012591.

&c. &c. &c.

53. Having computed the sines in this manner as far as 60°, the co-sines from 30° to 60° will likewise be known; the co-sine of any *arc above* 30° being the same as the sine of an *arc* as much *below* 60°.

Thus, cos 30° 1′=sin 59° 59′=.8658799.
cos 30° 2′=sin 59° 58′=.8657344.
cos 30° 3′=sin 59° 57′=.8655887.
cos 30° 4′=sin 59° 56′=.8654430.
&c. &c. &c.
cos 60° =sin 30° =.5000000.

54. *To find the sines and co-sines from 60° to 90°.*

The sine of any *arc* above 60° is the same as the co-sine of an *arc* at *the same distance* below 30°; and in like manner, the co-sine of an *arc* above 60° is the same as the sine of an *arc* equally below 30°: thus,

Sin 60° 1′=cos 29° 59′=.8661708. cos 60° 1′=sin 29° 59′=.4997481.

Sin 60° 2′=cos 29° 58′	cos 60° 2′=sin 29° 58′
Sin 60° 3′=cos 22° 57′	cos 60° 3′=sin 29° 57′
&c.	&c.

55. *To find the versed sines and co-versed sines of the quadrant.*

In any *arc* less than 90° the versed sine is found by subtracting the co-sine from radius (cor. Art. 14.); and in *arcs* greater than 90°, it is found by adding the co-sine to radius: thus,

° The learner is supposed (in this and the following articles,) to have computed all the preceding sines, co-sines, tangents, &c.; if he has not, he must, in order to work the examples, take them from a table. By means of the formulæ here given, any natural sine, tangent, secant, &c. in the table, which is supected to be wrong, may be examined, and if necessary, corrected.

ver. sin $1'=1-\cos 1'=(1-.99999996=)$ .00000004
ver. sin $2'=1-\cos 2'=(1-.9999998=)$ .0000002
ver. sin $3'=1-\cos 3'=(1-.9999996=)$ .0000004.
ver. sin $4'=1-\cos 4'=0000007$
ver. sin $5'=1-\cos 5=.0000011$
&c. &c.
ver. sin $90^\circ\ 1'=1+\cos 89^\circ\ 69'=1.0002909$
ver. sin $90^\circ\ 2'=1+\cos 89^\circ\ 58'=1.0005818$
&c. &c.

Versed sines for *arcs* greater than 90, do not occur in the common tables.

56. The co-versed sine is found by subtracting the sine from the radius (cor. Art. 15,); thus,

co-versed sin $1'=1-\sin 1'=(1-.0002909=)$ .9997091
co-versed sin $2'=1-\sin 2'=(1-.0005818=)$ .9994182
co-versed sin $3'=1-\sin 3'=(1-.0008727=)$ .9991273
&c., &c.

57. *To find the tangents and co-tangents from* 0° *to* 45°.

By Art. 35. anal. 1. it appears that the tangent of any *arc*

$$A=(\text{radius being }1.)=\frac{\text{sine}}{\text{co-sine}}$$

$$\because \left.\begin{array}{l}\tan 1' \\ \text{co-tan}\ 89^\circ\ 59'\end{array}\right\}=\frac{\sin 1'}{\cos 1'}=\left(\frac{.0002909}{.99999996}=\right).0002909$$

$$\left.\begin{array}{l}\tan 2' \\ \text{co-tan}\ 89^\circ\ 58'\end{array}\right\}=\frac{\sin 2'}{\cos 2'}=\left(\frac{.9005818}{.9999998}=\right).0005818$$

$$\left.\begin{array}{l}\tan 3' \\ \text{co-tan}\ 89^\circ\ 57'\end{array}\right\}=\frac{\sin 3'}{\text{co-sin}\ 3'}=\left(\frac{.0008727}{.9999996}=\right).0008727$$

$$\left.\begin{array}{l}\tan 4' \\ \text{co-tan}\ 89^\circ\ 56'\end{array}\right\}=\frac{\sin 4'}{\text{co-sin}\ 4'}=\left(\frac{.0011636}{.9999993}=\right).0011636$$

&c. &c.

And proceed in this manner to 45°.

58. *To find the tangents and co-tangents from* 45° *to* 90°.

Because (formula 26. Art. 49.) the tangents of $\overline{45^\circ+B}=$ tan. $\overline{45^\circ-B}+2$ tan. $2B$; therefore if

$$B=1',\ \text{then}\left\{\begin{array}{l}\tan 45^\circ\ 1' \\ \text{co-tan}\ 44^\circ\ 59\end{array}\right\}=\tan\ 44^\circ\ 59'+2\ \tan\ 2'=$$

$(.9994184+2\times.0005818=)$ 1.0005820.

$B=2' \ldots \left\{\begin{array}{l}\tan 45^\circ\ 2' \\ \text{co-tan } 44^\circ\ 58'\end{array}\right\} = \tan 44^\circ\ 58' + 2 \tan 4' =$
1.0011642.

$B=3' - \left\{\begin{array}{l}\tan 45^\circ\ 3' \\ \text{co-tan } 44^\circ\ 57'\end{array}\right\} = \tan 44^\circ\ 57' + 2 \tan 6' =$
1.0017469.

$B=4 \ldots \left\{\begin{array}{l}\tan 45^\circ\ 4' \\ \text{co-tan } 44.56.\end{array}\right\} = \tan 44^\circ\ 56' + 2 \tan 8' =$
1.0023298. &c. &c.

And in this manner the tangent of every succeeding minute of the remainder of the quadrant, must be found.

59. *To find the secants and co-secants of the quadrant.*

By the second analogy Art. 35. we have $\sec A = \frac{1}{\cos A}$ the radius being unity; whence if

$A=1'$, then $\left\{\begin{array}{l}\sec 1' \\ \text{co-sec } 89^\circ\ 59'\end{array}\right\} = \frac{1}{\cos 1'} = \left(\frac{1}{.99999996} =\right)$
1.00000004.

$A=3' \ldots \left\{\begin{array}{l}\sec 3' \\ \text{co-sec } 89^\circ\ 57'\end{array}\right\} = \frac{1}{\cos 3'} = \left(\frac{1}{.9999996} =\right)$
1.0000004.

$A=5' \ldots \left\{\begin{array}{l}\sec 5' \\ \text{co-sec } 89^\circ\ 55'\end{array}\right\} = \frac{1}{\cos 5'} = \left(\frac{1}{.9999989} =\right)$
1.0000011.

$A=7' \ldots \left\{\begin{array}{l}\sec 7' \\ \text{so-sec } 89^\circ\ 53'\end{array}\right\} = \frac{1}{\cos 7'} = 1.0000021$
&c. &c.

60. By this method the secants and co-secants of every minute of the quadrant may be computed, but it is necessary to employ it only for the *odd* minutes; the secants and co-secants of the *even* minutes may be obtained by a process which is somewhat more easy; as follows

By art. 41. $\tan A + \sec A = \text{co-tan } \frac{1}{2}\,\overline{90-A}$.

$\therefore \sec A = \text{co-tan } \frac{1}{2}\,\overline{90-A} - \tan A$.

Let $A=2'$, then $\left\{\begin{array}{l}\sec 2' \\ \text{co-sec } 89^\circ\ 58'\end{array}\right\} = \left(\text{co-tan } \frac{89\ 58'}{2} - \tan 2'\right.$
$=)$ co-tan $44^\circ\ 59' - \tan 2' = (1.0005819 - .0005818 =)\ 1.0000001$.

$$A=4' \ldots \left.\begin{cases}\sec 4' \\ \text{co-sec } 89^\circ\ 56'\end{cases}\right\}=\text{co-tan } 44^\circ\ 58=\tan 4'$$

$=(1.0011642-.0011636=)$ 1.0000006.

$$A=6' \ldots \left.\begin{cases}\sec 6' \\ \text{co-sec } 89^\circ\ 54'\end{cases}\right\}=\text{co-tan } 44^\circ\ 57'-\tan 6'$$

$=(1.0017469-.0017453=)$ 1.0000016.

$$A=8' \ldots \left.\begin{cases}\sec 8' \\ \text{co-sec } 89^\circ\ 52'\end{cases}\right\}=\text{co-tan } 44^\circ\ 56'-\tan 8'$$

$=1.0000027$.

&c. &c.

61. The numbers thus computed are called *natural* sines, tangents, &c. they are computed for every degree and minute of the quadrant, and arranged in eight columns, titled at the top and bottom; these together constitute the table of natural sines, tangents, &c. directions for the use of which are given in the introduction to every system of trigonometrical tables [p].

OF THE TABLE OF LOGARITHMIC SINES, TANGENTS, &c.

62. The *logarithmic* or *artificial* sines, tangents, &c. are the logarithms of the sines, tangents, &c. computed to the radius $\overline{10}|^{10}=10000000000$; for since the sines, co-sines, and many of the versed sines and tangents computed to the radius 1 are proper fractions, their logarithms will have a negative index, (vol. 1. page 287.) but by assuming the above number for radius, these fractions become whole numbers, their logarithms affirmative, and the figures expressing any sine, tangent, &c. will be the same in both cases, as likewise their logarithms, excepting the indices, which (as we have observed) will be *frequently* negative in the former case, but *always* affirmative in the latter; therefore, in order to find the logarithm of the sine of an *arc*, calculated to the radius $\overline{10}|^{10}$, we must add 10 to the index of the logarithm of the same sine to the radius 1: for, *let* $r=$ *the radius*, $s=$ *the sine of any arc to rad.* r, $R=$ *a different radius*, $S=$ *the sine of an arc* (*to rad.* R) *similar to the former, then* (*Art.* 38.)

[p] For an account of the tables of sines, tangents, &c. with ample directions to assist the learner in their use, see Dr. Hutton's *Mathematical Tables*, 2 *edit.* p. 151, 152.

$r : R :: s : S$; *which if* $r=1$ *and* $R=\overline{10}|^{10}$, *becomes* $1 : \overline{10}|^{10} :: s : S$, $\because S=\overline{10}|^{10}\times s$, $\because$ *log.* $S=10\times$ *log.* $10+$ *log.* $s=$(*since log.* $10=1$) $10+$ *log.* s. Q. E. D.

EXAMPLES.—1. To find the logarithmic sine of 1′.

To log. of .0002909 (=*sin* 1′) = −4.4637437
Add 10
The sum is .... 6.4637437 = *the log. sine of* 1′ *to radius* 10000000000.[q]

2. To find the logarithmic tangent of 2°. 35′=

To log. of .0451183 (=*tan* 2° 35′) = −2.6543527
Add 10
The sum .... 8.6543527 *is the log. tangent of* 2°, 35′.

3. To find the logarithmic secant of 7°. 5′:

The log. of 1.0076908 (=*sec* 7° 5′) = 0.0033273
Add 10
The log. secant of 7° 5′ = 10.0033273

4. To find the logarithmic versed sine of 20° 12′.

To log. of .0615070 (=*ver. s. of* 20° 12′) = −2.7889245
Add 10
The log. versed sine of 20° 12′ = 8.7889245

In this manner the logarithmic sines, co-sines, tangents, &c. are computed; viz. by adding 10 to the index of the logarithm of the natural sine, co-sine, tangent, &c. respectively corresponding to the radius 1[r].

Having shewn the method of computing the trigonometrical canon, both in natural numbers and logarithms, the next thing to be done is to demonstrate the propositions on which the practical part of trigonometry is founded.

THE FUNDAMENTAL THEOREMS OF PLANE TRIGONOMETRY.

63. In a right angled triangle the hypothenuse : is to either of the sides :: as radius : to the sine of the angle opposite to that side.

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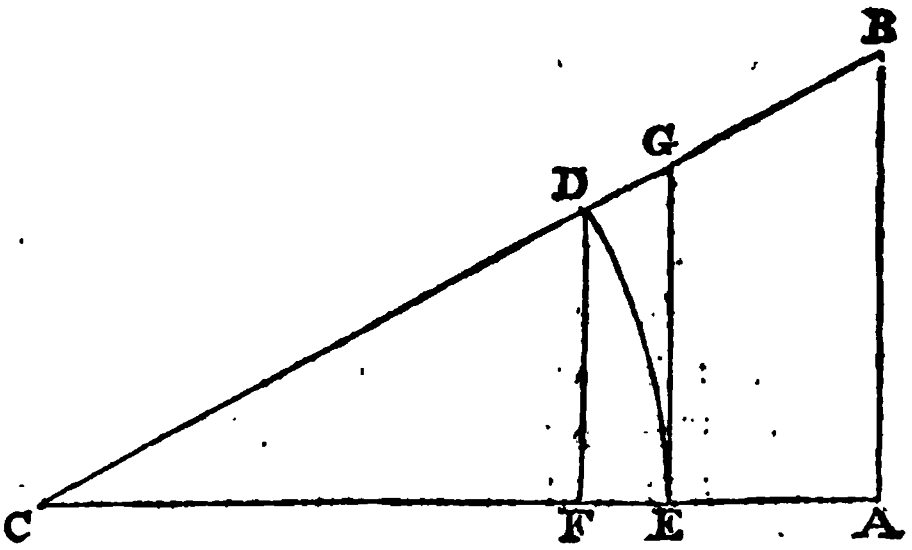
[q] By the preceding rules any logarithmic sine, tangent, secant, &c. in the table, suspected to be inaccurate, may be examined, and the error (if any should be found) corrected.

[r] The log. sine of 1′ (as here given) exceeds the truth by .0000176 because the sine of 1′ is only .000290888 and not .0002909. See Art. 51.

Let ACB be a triangle, right angled at A; from C as a centre with any radius CD describe a circle DE, and draw DF perpendicular to CA.

Because DF is parallel to BA (28. 1.)

$$\left.\begin{array}{l} CB : BA :: CD : DF \text{ and} \\ CB : CA :: CD : CF \end{array}\right\} (2.6.)$$



But DF is the sine of the angle C (Art. 12.), and CF is the cosine of the angle C (Art. 13), or the sine of the angle ($CDF=$) B; ∵ hyp. CB : side BA :: radius (CD) : sin ang. C (DF) opposite to BA; in like manner hyp. CB : side CA :: radius (CD) : sin. ang. B (CF) opposite to CA. Q. E. D.

64. If CD be the radius to which the trigonometrical canon is computed, then will DF be the sine of C, and CF the sine of B, as actually exhibited in the canon; and therefore, having the hypothenuse CB, and one side BA, of a right angled triangle given, the angle C (opposite BA) may be found, for CB : BA :: tabular radius : tabular sine of C, which sine being found in the table, the angle of which it is the sine, will be known.

Hence, the angle C being known, the angle $B=90°-C$ is likewise known.

65. **In a right angled triangle, one of the sides about the right angle : is to the other :: as radius : to the tangent of the angle opposite the latter side.**

About the angular point C, of the triangle ABC, with any radius CE, describe the *arc* DE as before, and draw EG at right angles to CA (11. 1.) meeting CB in G, EG will be the tangent of the angle C (Art. 16.) ∵ CA : AB :: CE : EG (4. 6.); that is, side CA : side AB :: radius : tan. ang. C.

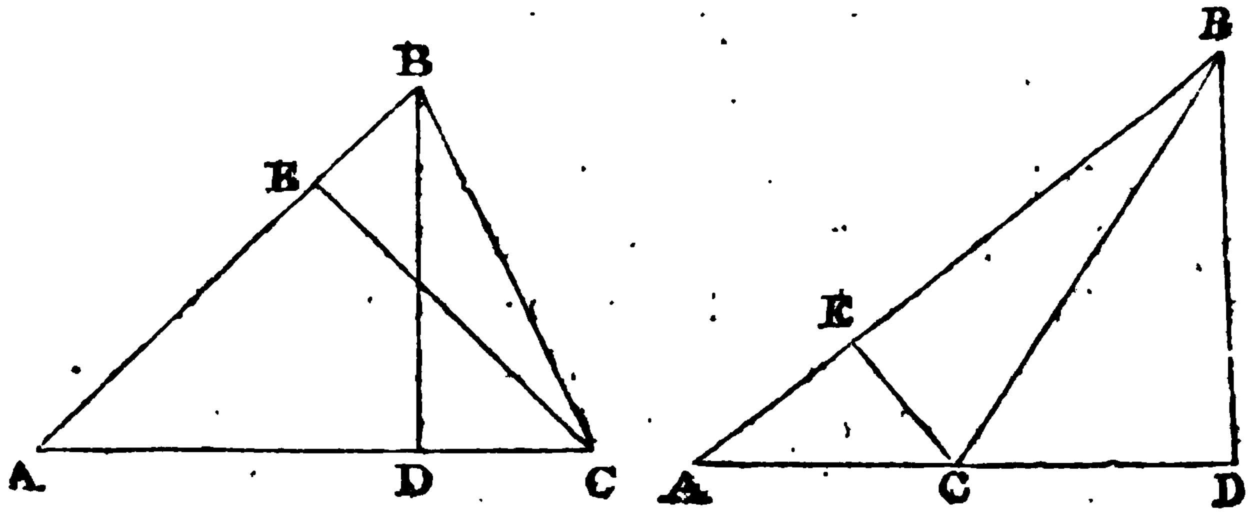
In like manner, if from B as a centre with the radius BA a circle be described, AC will be the tangent of the angle B; and it may in like manner be shewn, that BA : AC :: radius : tan. ang B. Q. E. D.

66. If CE be the radius to which the canon is computed, EG will be the tabular tangent of C; wherefore, since $CA : AB :: CE : EG$, we have only to find EG in the tangents, and its corresponding angle C will be known; wherefore the two sides about the right angle of any right angled triangle being given, the angle C, and likewise the angle B ($=90°-C$,) may be found.

67. The sides of any plane triangle are to each other as the sines of their opposite angles.

Let ABC be a triangle, from B draw BD perpendicular to AC produced if necessary; and CE perpendicular to AB.

If a circle be described from B as a centre, with the radius BC, then it is evident that CE will be the sine of the angle ABC; and if from the centre C, with the same radius, a circle be described, BD will be the sine of the angle BCA (Art. 12.);



wherefore, since the angle A is common to the right angled triangles AEC, ADB, these triangles are equiangular (32. 1.), and $AB : BD :: AC : CE$ (4. 6.) $\therefore AB : AC :: BD : CE$ (16. 5.); that is, side AB : side AC :: sin. ang. ACB opposite AB : sin. ang. ABC opposite AC. Q. E. D.

In the case in which the perpendicular BD falls without the triangle ABC, BD is *actually* the sine of the exterior angle BCD; but BCA is the supplement of BCD (13. 1. and Art. 8.) and since the sine of an angle is likewise the sine of its supplement (cor. Art. 12.) BD is therefore the sine of the angle BCA.

68. Hence, if we have two sides AB, AC of any triangle given, and likewise an angle ACB opposite (AB) one of them; the angle ABC opposite the other given side (AC) may be found; and thence the remaining angle A. For since $AB : AC ::$ sin. ang. ACB : sin. ang. ABC, the three first terms being

given, the fourth, or sine of ABC, and consequently the angle ABC is known; whence also the angle $A=180^\circ-\overline{ACB+ABC}$ is known. Lastly, from the two given sides AB, AC, and the three angles which we have found, the third side BC will be obtained, for invertendo, sin. ang. ABC : sin. ang. BAC : : side AC : side BC.

69. If half the difference of two quantities be *added to* half their sum, the result will be the greater of the two proposed quantities; but if half the difference be *taken from* half their sum, the result will be the less.

Thus, let A and B be two quantities, of which A is the greater; S=their sum, D=their difference.

$$\left.\begin{array}{l}\text{Then } A+B=S \\ \text{And } \;\; A-B=D\end{array}\right\}\text{ by hypothesis.}$$

Their sum $2A=S+D$, $\because A=\frac{S+D}{2}=\frac{S}{2}+\frac{D}{2}$.

Their difference $2B=S-D$, $\because B=\frac{S-D}{2}=\frac{S}{2}-\frac{D}{2}$ Q.E.D.

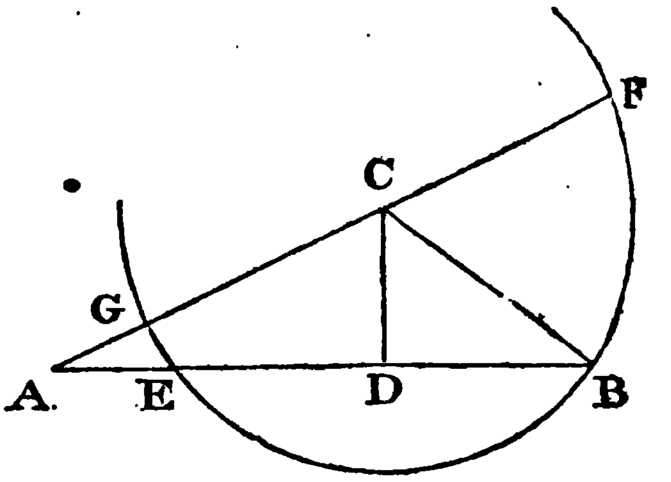
Cor. Hence, if from $(A=)\ \frac{S}{2}+\frac{D}{2}$ we take $\frac{S}{2}$, the remainder is $\frac{D}{2}$; that is, "if half the sum be subtracted from the greater, the remainder is half the difference."

70. If within a triangle, a perpendicular be drawn from the opposite angle to the base, then will the base : be to the sum of the other two sides : : as the difference of these sides : to the difference of the segments of the base.

Let ABC be a triangle, having the straight line CD drawn from the angle C perpendicular to the base AB; then will AB : $AC+CB$: : $AC-CB$: $AD-DB$.

From C as a centre with the distance CB the least of the two sides, describe the circle EBF, cutting CB in B, and AC produced in G and F; then because $CF=CB$ (15 def. 1.) $AF=AC$

$+CB=$the sum of the sides; and because $CG=CB$, $AC-CB=(AC-CG=)\ AG=$the difference of the sides. Also, since $DE=DB$ (3.3.), $AD-DB=(AD-DE=)\ AE=$the difference of the segments (AD and DB) of the base.

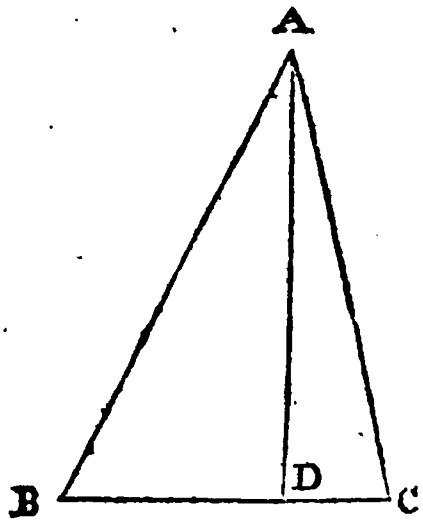


Because from the point A without the circle, AB and AF are drawn cutting the circle, $AB.AE=AF.AG$ (cor. 36.3.), ∵ $AB : AF :: AG : AE$ (16.6.); that is, the base : sum of the sides :: difference of the sides : difference of the segments of the base. Q. E. D.

When the three sides of a triangle are given, the angles are found by this proposition.

71. In a plane triangle, twice the rectangle contained by any two sides, is to the difference of the sum of the squares of these two sides and the square of the base, as radius to the co-sine of the angle contained by the two sides.

Let ABC be a triangle $2AB.BC : \overline{AB}^2+\overline{BC}^2-\overline{AC}^2 ::$ radius : co-sine of ABC. Draw AD perpendicular to BC (produced if necessary), then $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2+2CB.BD$ (13.2), ∵ $\overline{AB}^2+\overline{BC}^2-\overline{AC}^2=2CB.BD$; but $2CB.BA : 2CB.BD :: AB : BD$ (1.6.); that is, twice the rectangle contained by the sides : is to the difference of the sum of the squares of the sides, and the square of the base :: as AB : to BD; but B being the centre, and AB radius, BD will be the co-sine of the angle ABC (Art. 13.), ∵ twice the rectangle contained by the sides, is to the difference of the sum of the squares of these two sides and the square of the base, as radius, to the co-sine of the angle contained by the two sides; and the same may in like manner be proved when the angle at B is obtuse, by using the 12th proposition of the second book of Euclid, instead of the 13th. Q. E. D.

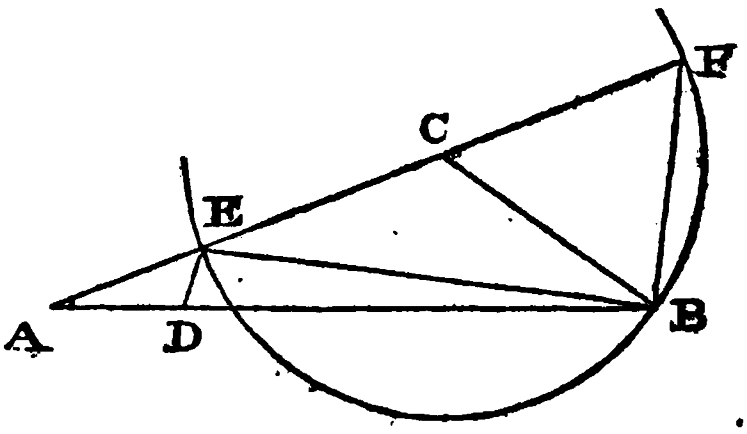


When the three sides *only* of a plane triangle are given,

the angles may be found by means of this proposition, without letting fall a perpendicular, as in the preceding article.

72. In a plane triangle, the sum of any two sides : is to their difference : : as the tangent of half the sum of the angles at the base : to the tangent of half the difference.

Let ABC be a triangle, from C as a centre with the least side CB as radius, describe the circle EBF; produce AC to F, join BE, BF, and draw ED perpendicular to EB.



Because $CE=CF=CB$, $AF=(AC+CF=)\ AC+CB=$ *the sum of the sides*, and $AE=(AC-CE=)\ AC-CB=$ *difference of the sides.* Also $FCB=CBA+CAB$ (32. 1.) $=$the sum of the angles at the base, $\because$ $FEB=(\frac{1}{2}FCB$ by 20. 3.$=$) *half the sum of the angles at the base.* And since $CE=CB$, the angle $CEB=CBE$ (6. 1.); but $CEB=CAB+EBA$ (32. 1.); $\because$ $CBE=CAB+EBA$; to each of these equals add EBA, $\because$ $(CBE+EBA=)\ CBA=CAB+2EBA$ or $CBA-CAB=2EBA$; that is, $2EBA=$the difference of the angles (CBA, CAB) at the base, $\because$ $EBA=$ *half the difference of the angles at the base.* Now since EBF is a right angle (31. 3.), and BED a right angle by construction, if from E as a centre with the radius EB a circle be described, it is evident that FB is the tangent of FEB (Art. 16.); that is, FB *is the tangent of half the sum of the angles* (CAB, CBA) *at the base*; and if from B as a centre with the same radius (EB) a circle be described, it will be equally plain that ED is the tangent of EBA; that is, ED is *the tangent of half the difference of the angles* (CAB, CBA) *at the base.* Again, because ED is parallel to FB (27. 1.), and the angle A common, the two triangles AFB, AED are equiangular (29. 1.), $\because$ $AF : FB : : AE : ED$ (4. 6.) and $AF : AE : : FB : ED$ (16. 5.); that is, the sum of the sides : is to their difference : : as the tangent of half the sum of the angles at the base : to the tangent of half their difference. Q. E. D.

When two sides and the included angle are given, the remaining angles may be found by this proposition with the help of Art. 69.

SOLUTION OF THE CASES OF PLANE TRIANGLES.

73. There are three ways of solving trigonometrical problems, viz. by *geometrical construction*, by *arithmetical computation*, and *instrumentally*, or by the scale and compasses. The first of these methods has been already explained in part 8. under the head of *Practical Geometry*; the second consists in the application of the principles laid down in the foregoing theorems, by the help of either natural numbers, or logarithms; and by the third, the proportions are worked with a pair of compasses on the Gunters' scale [r]; the method of doing which will be explained in the following examples, where the conditions are exhibited in the form of a Rule of Three stating, having either the *first* and *second* terms, or the *first* and *third*, always of the same kind.

74. *When the first and second terms are of the same kind.*

Extend the compasses from the first term to the second, on that line of the Gunter which is of the same name with these terms; this extent will reach from the third term to the fourth, on the line which is of the same name with the third and fourth.

75. *When the first and third terms are of the same kind.*

Extend the compasses (on the proper line) from the first to the third; that extent will reach (on the proper line) from the second to the fourth; observing in all cases, that when the proportion is *increasing*, the extent must be taken *forwards* on

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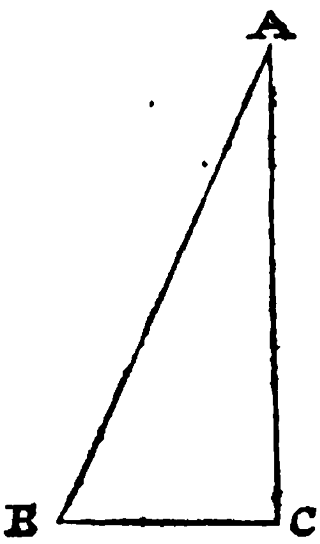
[r] This scale was invented by the Rev. Edmund Gunter, B.D. professor of Astronomy at Gresham College, probably about the year 1624; it is a broad flat ruler two feet in length, on which are laid down (besides all the lines common to the plane scale) logarithmic lines of numbers, sines, versed sines, tangents, meridional parts, equal parts, sine rhumbs, and tangent rhumbs; that is, the actual lengths (taken on a scale of equal parts) are expressed by the figures constituting the logarithms of the quantities in question. With these logarithmic scales, all questions relating to proportion in numbers may be solved, for the compasses being extended from the first term to the second or third, that extent will reach from the second, or from the third to the fourth, according as the first and second, or first and third terms are of the same kind. For an ample description of this scale, see Robertson's *Elements of Navigation, vol.* 1. *p.* 114. *4th. edit.* likewise Mr. Donne's directions usually sold with his improved scale; and for an account of the improvements by Mr. Robertson, see a tract on the subject, published in 1778, by William Mountaine, Esq. F. R. S.

the scale, but when the proportion is *decreasing*, it must be taken *backwards*.

SOLUTION OF RIGHT ANGLED TRIANGLES.

76. *Case* 1. Given the hypothenuse AB, and one side AC, of a right angled triangle; to find the remaining side BC, and the angles A and B [s].

Because $\overline{BC}|^2+\overline{AC}|^2=\overline{AB}|^2$ (47. 1.) $\because$ $\overline{BC}|^2=\overline{AB}|^2-\overline{AC}|^2$, and $BC=\sqrt{\overline{AB}|^2-\overline{AC}|^2}$, *whence BC is found*. Likewise (Art. 63.) hyp. AB : side AC :: radius : sin. angle B; that is, sin $B=\frac{AC\times\text{radius}}{AB}$; or by logarithms [t], log. sin B $=$log. $AC+$log. rad.$-$log. AB; *whence the angle B is found*, both by natural numbers and logarithms.



Lastly, since the three angles of any triangle are equal to two right angles (32. 1.) $=180^\circ$, and the angle C (a right angle)$=90^\circ$, $\because$ $B+A=$ $(180^\circ-C=180^\circ-90^\circ=)$ 90°; but the angle B has been found, $\because$ $A=90-B$ is likewise known [u].

By a similar process AB and BC being given, AC and the angles B and A may be found.

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[s] Before you begin to work any question in Trigonometry, you must draw a sketch resembling, as nearly as you can guess, the figure intended; placing letters at the angles, and each number given in the question opposite the side or angle to which it belongs; some authors mark the given sides and angles by a small stroke, drawn across the given side, or issuing from the given angle; the unknown parts they mark with a cipher (0).

[t] It must be remembered, that multiplication of natural numbers is performed by the *addition* of their logarithms, division by *subtraction*, involution by *multiplication*, and evolution by *division*; if these particulars be kept in mind, there will be no difficulty in solving trigonometrical problems by logarithms, see vol. 1. part. 3.

[u] The angle A may be found *in the same page* of the table in which B is found; thus, if the degrees and minutes contained in B be found at the *top* and on the *left hand* respectively, of the page, those contained in A will be found at the *bottom* and on the *right*; viz. the degrees at the *bottom* of the page, and the minutes on the *right hand*, in a line with the minutes in B.

EXAMPLES.—1. Given the hypothenuse $AB=120$, and the perpendicular $AC=95$, to find the base BC and the angles A and B.

By construction.

Draw any straight line BC, at C draw CA perpendicular to BC, and make it equal to 95 taken from any convenient scale of equal parts; from A as a centre with the radius 120 taken from the same scale, cross CB in B, and join AB. Take the length of CB in the compasses, and apply it to the above-mentioned scale, and it will be found to measure 73 nearly; next measure the angles A and B by the scale of chords or the protractor, and they will be known, viz. $A=38°$ and $B=52°$, nearly [x].

By calculation.

First, to find BC. We have $BC=\sqrt{\overline{AB}|^2-\overline{AC}|^2}=(\sqrt{\overline{120}|^2-\overline{95}|^2}=\sqrt{5375}=)$ 73.3143, &c. [y]

Secondly, to find the angle B. We have $\sin B=\frac{AC\times \text{rad}}{AB}=(\frac{95\times 1}{120}=).7916666$ the natural sine of B, and the nearest angle in the table corresponding with this sine is $52°\ 20'$ [z]; wherefore the angle $B=52°\ 20'$, and $A=(90°-B=90°-52°\ 20'=)\ 37°\ 40'$.

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[x] The sides and angles of triangles are very expeditiously determined both by the plane scale and the Gunter, but these methods are not to be depended on in cases where accuracy is required; they are nevertheless useful where great exactness is no object, and as convenient checks on the method of calculation.

[y] The side BC may likewise be found trigonometrically, after the angle A has been found; thus (Art. 63.) $AB : BC :: \text{rad} : \sin A$, $\therefore BC=\frac{AB.\sin A}{\text{rad}}$; this solution may be performed by the Gunter; thus, extend on the sines from $90°$ to $37°\frac{2}{3}$, this extent will reach on the numbers from 120 to $73\frac{1}{3}=BC$ nearly.

[z] This, although it is the angle which has the nearest sine in the table to the above, is not perfectly exact; the natural sine of $52°\ 20'$ being only .7915792 which is less than .7916666 by .0000874; now the sine of $52°\ 21'$ exceeds that of $52°\ 20'$ by 1777, therefore our angle $52°\ 20'$ is too small by $\frac{874}{1777}$ of a minute; that is, by $29''\ \frac{907}{1777}$: whence, in strict exactness, angle $B=52°\ 20'\ 29''\ \frac{907}{1777}$, and angle $A=37°\ 39'\ 30''\ \frac{870}{1777}$.

The same by logarithms. Since log. sin B=log. AC+log. rad.—log. AB, ∵ to log. AC=log. 95= 1.9777236

Add log. radius =log. 10000000000=10.0000000

And from the sum = ... 11.9777236

Subtract log. AB=log. 120 = ... 2.0791812

* Remains log. sin B=52° 20′= 9.8985424

Whence angle A=(90°—B=) 37° 40′ as before.

Instrumentally, by the Gunter.

Extend the compasses from 120 to 95 on the line (of numbers) marked *Num.* that extent will reach from (radius) 90° on the line (of sines) marked *sin.* to 52°$\frac{1}{3}$=52° 20′=the angle B. We cannot find the side BC by this method, without anticipating case 4.

2. In the right angled triangle ABC, given the hypothenuse AB=135, and the perpendicular AC=108, required the base

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* An observation similar to that in the preceding note occurs here: the log. sine in the table which is the nearest to the above, is that of 52° 20′, viz. 9.8984944, but this is less than the above, being too small by 480, wherefore 52° 11′ is too little for the angle B: now the difference between the log. sine of 52° 20′, and that of 52° 21′ is 975, whence the above value of B is $\frac{480}{975}$ of a minute, or 29″$\frac{7}{13}$ too small; that is, the angle B=52° 20′ 29″$\frac{7}{13}$, and A=37° 39′ 30″$\frac{6}{13}$ by this mode of calculation.

It is worth while to observe, that the difference of about $\frac{1}{38}$ of a second between this result, and that in the foregoing note, arises from the circumstance of the logarithms, as well as the sines, being approximations, and not absolutely exact.

When the sine, tangent, &c. found by operation is not in the table, 1. take the nearest from the table, and find the difference between that and the one found by operation; call this difference *the numerator*. 2. Find the difference of the *next greater* and *next less* than that found by operation, and call this difference *the denominator*. 3. Multiply the numerator by 60 and divide the product by the denominator, the quotient will be *seconds*, which must be added to, or subtracted from the degrees and minutes corresponding to the nearest tabular number, according as that number is less or greater than the number found by operation.

This rule will serve both for natural and logarithmic sines, tangents, &c. and likewise for the logarithms of numbers, observing in the latter case (instead of multiplying by 60) to subjoin a cipher to the numerator, and having divided by the denominator, the first quotient figure must occupy *one place to the right* of the right hand figure in the nearest tabular number, and be added, or subtracted, according as that number is too little, or too great.

BC, and the angles A and B? *Ans.* $BC=81$, *ang.* $A=36°\ 52'$, *ang.* $B=53°\ 8'$.

3. Given $AB=291$, $BC=216$, required the remaining side and angles? *Ans.* $AC=195$, *ang.* $A=42°\ 5'$, *ang.* $B=47°\ 55'$.

77. *Case* 2. Given the two sides AC and CB, to find the hypothenuse AB and the angles A and B.

First, (47.1.) $AB=\sqrt{\overline{AC}|^2+\overline{CB}|^2}$; *whence AB is found.*

Secondly, (Art. 65.) $AC : CB ::$ radius : tangent ang. A; or $\tan A=\frac{CB\times\text{radius}}{AC}$; and by logarithms, log. tan. A=log. BC +log. rad.—log. AC, ∵ *the angle A is found*, both by natural numbers and logarithms, and *the angle* $B=90°-A$ *is likewise found.*

EXAMPLES.—1. Given the side $AC=123$, and the side $CB=132$, to find the hypothenuse AB and the angles A and B.

By calculation[b].

First, $AB=\sqrt{\overline{AC}|^2+\overline{CB}|^2}=\sqrt{\overline{123}|^2+\overline{132}|^2}=\sqrt{32553}=$ 180.424.

Secondly, by natural numbers, $\tan A=\frac{CB\times\text{rad.}}{AC}=\frac{132}{123}=$ 1.0731707 =natural tangent of $47°\ 1'$=ang. A, ∵ ang. $B=(90°-A=)\ 90°-47°\ 1'=42°\ 59'$.

Thirdly, by logarithms, log. tan. A=log. CB+log. rad.—log. AB ∵ to log. CB 132= 2.1205739
Add log. radius 10000000000=10.0000000
And from the sum = ... 12.1205739
Subtract log. AB 123 = ... 2.0899051
Remains log. tan. ang. $A=47°\ 1'$=10.0306688
And ang. $B=90°-A=42°\ 59'$ as before.

Instrumentally.

Extend the compasses from 123 to 132 on the line (of numbers) marked *Num.* this extent will reach from (radius =) 45° on the line (of tangents) marked *Tan.* to $47°\ 1'$=the angle A.

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[b] In this and the following cases of right angled triangles, the construction is purposely omitted, it being perfectly easy and obvious, from what has been given on the subject in the *Practical Geometry*, near the end of part 8.

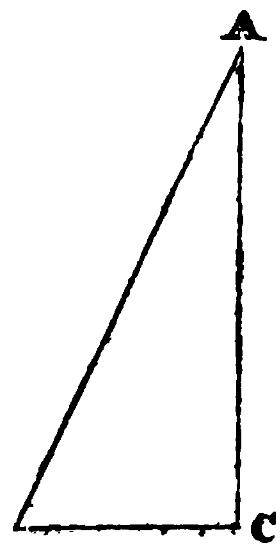
The side AB is not found *instrumentally* for a reason similar to that before given.

2. The perpendicular $AC=200$, and the base $BC=110$ of a right angled triangle ABC being given, required the hypothenuse AB, and the angles A and B? *Ans.* $AB=228.254$, *ang.* $A=28^\circ\ 49'$, *ang.* $B=61^\circ\ 11'$.

3. Given $AC=4$, and $BC=3$, to find AB, and the angles A and B. *Ans.* $AB=5$, *ang.* $A=36^\circ\ 52'$, *ang.* $B=53^\circ\ 8'$.

78. *Case* 3. The hypothenuse AB and the angle B being given, to find the sides AC, CB, and the angle A.

First, since the angle at B is given, the angle $A=90^\circ-B$.



Secondly, (Art. 63.) $AB : AC ::$ radius : sin ang. B ∴ $AC=\frac{\sin B.AB}{\text{radius}}$; and log. AC $=$log. sin $B+$log. $AB-$log. radius; whence AC *is found* both by natural numbers and logarithms.

Thirdly, $\overline{AB}|^2=\overline{AC}|^2+\overline{CB}|^2$ (47. 1.) ∴ $\overline{CB}|^2=\overline{AB}|^2-\overline{AC}|^2$ and $CB=\sqrt{\overline{AB+AC}.\overline{AB-AC}}$ (cor. 5. 2.); also log. $CB=\frac{\log.\ \overline{AB+AC}+\log.\ \overline{AB-AC}}{2}$; ∴ CB *is found,* both by natural numbers and logarithms.

EXAMPLES.—1. Given the hypothenuse $AB=165$, and the angle $B=35^\circ\ 30'$, to find the sides AC, CB, and the angle A.

By calculation.

First, ang. $A=90^\circ-B=(90^\circ-35^\circ\ 30'=)\ 54^\circ\ 30'$.

Secondly, (by natural numbers) $AC=\frac{\sin B.AB}{\text{rad}}=$ (since rad$=1$, sin $35^\circ\ 30'\times AB=$) $.580703\times165=95.815995$; but the same may be done more readily by logarithms; thus, because log. $AC=$log. sin $B+$log. $AB-$log. rad.

∴ To log. sin B. or $35^\circ\ 30'=$	9.7639540
Add log. AB. or $165=$	2.2174839
And from their sum=	11.9814379
Subtract log. radius=	10.0000000
Remains log. AC 95.816=	1.9814379

Thirdly, $CB=\sqrt{\overline{AB+AC}.\overline{AB-AC}}=$ ($\sqrt{\overline{165+95.816}\times\overline{165-95.816}}=\sqrt{260.816\times 69.184}=\sqrt{18044.294144}=$) 134.329.

The same by logarithms, $\log. CB=\dfrac{\log.\overline{AB+AC}+\log.\overline{AB-AC}}{2}$;

that is, to log. $\overline{AB+AC}$, or 260.816= 2.4163342
Add log. $\overline{AB-AC}$, or 69.184= 1.8400057
The sum divided by 2 .... 2)4.2563399
Gives the log. of $CB=134.329$ 2.1281699

Instrumentally.

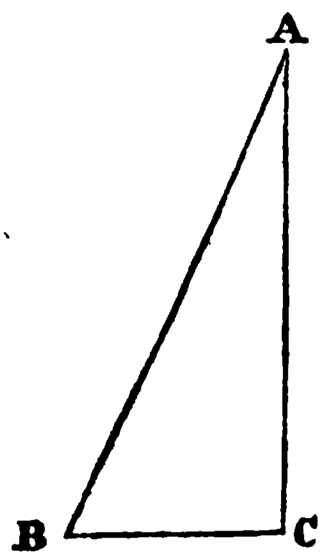
1. Extend from (radius or) 90° to 35° 30′ (=ang. B) on the line of sines; this extent will reach from 165 (backwards) to about $95\frac{8}{10}$ on the line of numbers, for the side AC (opposite the ang. B.)

2. Extend on the line of sines, from 90° to 54° 30′ (comp. B.); this extent will reach on the lines of numbers from 165 to about $134\frac{3}{10}$ for the side CB.

Ex.—2. Given the hypothenuse $AB=25$, and the angle $B=49°$, to find the sides AC, CB, and the angle A? *Ans.* $AC=18.893$, $CB=16.4017$, *ang.* $A=41°$.

3. Given $AB=100$, and the angle $A=45°$, to find the rest? *Ans.* $BC=AC=70.7108$, *ang.* $B=45°$.

79. *Case* 4. One side AC, and its adjacent angle A being given, to find the other sides AB, BC, and the remaining angle B.



First, angle $B=90°-A$.

Secondly, because (Art. 67.) $AC : CB :: \sin B : \sin A$, $\because CB=\dfrac{\sin A.AC}{\sin B}$; and $\log. CB=\log. \sin A+\log. AC-\log. \sin B$.

Thirdly, because (Art. 63.) $AB : AC ::$ radius : $\sin B$, $\because AB=\dfrac{AC.\ \text{rad}}{\sin B}$; also $\log. AB=\log. AC+\log. \text{rad.}-\log. \sin B$.

Examples.—1. Given the perpendicular $AC=1023$, and the angle $A=12°\ 45'$; to find the angle B, and the remaining sides AB, BC.

By calculation.

First, ang. $B=90^{\circ}-A=(90^{\circ}-12^{\circ}\,45'=)\,77^{\circ}\,15'$.

Secondly, $CB=\dfrac{\sin A.AC}{\sin B}=\dfrac{.2206974\times 1023}{.9753423}=231.4812$;

and by logarithms, log. CB=log. sin A+log. AC−log. sin B; that is, to log. sin A. 12° 45′= 9.3437973

Add log. AC 1023=	3.0098756
From the sum = ...	12.3536729
Subtract log. sin B 77° 15′=	9.9891571
Gives log. CB. 231.4812=	2.3645158

Thirdly, $AB=\dfrac{AC.\text{rad.}}{\sin B}=\dfrac{1023\times 1}{.9753423}=1048.862$.

And by logarithms, log. AB=log. AC+log. rad−log. sin B; that is, to log. AC 1023= 3.0098756

Add log. radius=	10.0000000
And from the sum=	13.0098756
Subtract log. sin B 77° 15′= . . .	9.9891571
Gives log. AB 1048.862=	3.0207185

Instrumentally.

1. To find CB, extend from (sin B, to sin A, that is, from) sin 77°¼ to sin 12¾; this extent will reach on the line of numbers from (AC) 1023 to 231½.

2. To find AB, extend from (sin B to radius, that is, from) 77°¼ to 90° on the sines; this extent will reach from 1023 to about 1049 on the numbers.

Ex.—2. Given the perpendicular AC=400, and the angle A=47° 30′, to find the hypothenuse AB, the base BC, and the angle B? *Ans.* AB=592.072, BC=436.52, *ang.* B=42° 30′.

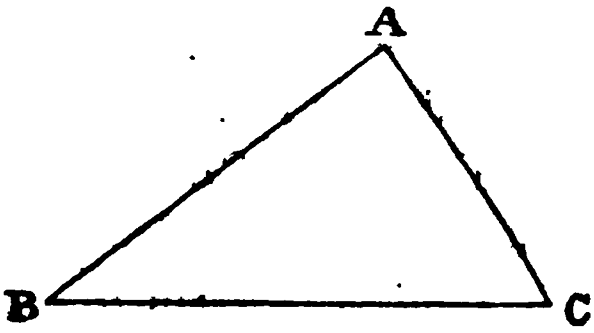
3. Given AC=82, ang. A=33° 13′, to find the rest? *Ans.* AB=97.9, CB=53.69, *ang.* B=26° 47′.

SOLUTION OF THE CASES OF OBLIQUE ANGLED TRIANGLES.

The foregoing calculations are effected both by natural numbers and logarithms, serving as a useful exercise for the learner; but principally to shew, that both methods terminate in the same result.

Trigonometrical operations are however seldom performed by the natural numbers, and therefore, in the following cases, we shall employ only the logarithmic process.

80. *Case* 1. Let there be given the two angles B and C, and the side AC opposite to one of them; to find the angle A, and the sides AB and BC.



First, the angles B and C being given, and $A=180^{\circ}-\overline{B+C}$, *the angle A will be known.*

Secondly, (Art. 67.) $AC : AB :: \sin B : \sin C \therefore AB=\frac{AC.\sin C}{\sin B}$; or by logarithms, log. $AB=$log. $AC+$log. sin $C-$log. sin B; $\therefore$ *AB is known.*

Thirdly, (Art. 67.) $AC : CB :: \sin B : \sin A \therefore CB=\frac{AC.\sin A}{\sin B}$. By logarithms, log. $CB=$log. $AC+$log. sin $A-$log. sin B; $\therefore$ *CB is known.*

EXAMPLES.—1. Given the angle $B=46^{\circ}$, the angle $C=59^{\circ}$, and the side AC (opposite B)$=120$; to find the angle A and the sides AB, BC.

By construction.

From any scale of equal parts take $AC=120$, at C make the angle $ACB=59$, and at A make the angle $CAB=(180^{\circ}-\overline{B+C}=180^{\circ}-\overline{46^{\circ}+59^{\circ}}=)$ 75°; then take the length of AB, and of BC respectively in the compasses, and apply them to the above-mentioned scale, and AB will$=143$, $BC=161$.

By computation.

1. Log. $AB=$log. $AC+$log. sin $C-$log. sin B

$\therefore$ To log. AC $120=$	2.0791812
Add log. sin C 59	9.9330656
And from the sum$=$	12.0122468
Subtract log. sin B $46^{\circ}=$	9.8569341
Remains log. AB $142.9845=$ ...	2.1553127

2. Log. CB=log. AC+log. sin A—log. sin B.

∵ To log. AC 120=	2.0791812
Add log. sin A 75°=	9.9849438
And from the sum=	12.0641250
Subtract log. sin B 46°=	9.8569341
Remains log. CB 161.1354	2.2071909

Instrumentally.

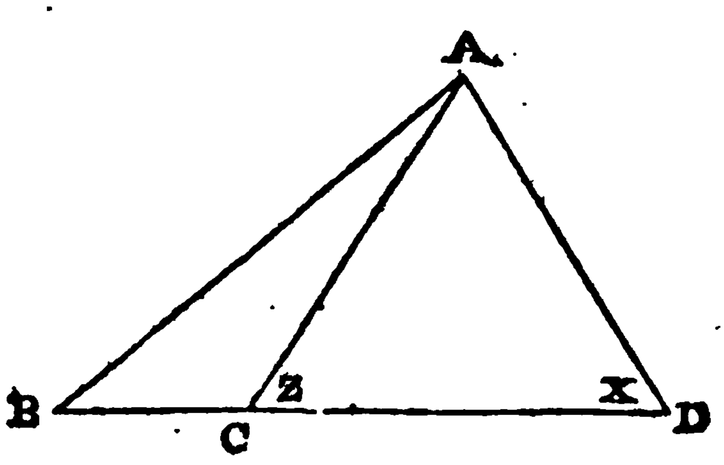
1. Extend on the sines from 46° (ang. B), to 59° (ang. C); this extent will reach on the numbers from 120 (AC), to about 143 (AB).

2. Extend from 46° to 75° on the sines; this extent will reach from 120 (AC), to about 161 (CB), on the numbers.

Ex. 2. Given the angle A=58° 43′, the angle C=74° 7′, and the side AB=610; to find the angle B, and the sides AC, CB? *Ans. ang.* B=47° 10′, AC=465.08, CB=542.

3. Given the side AB=1075, the angle B=34° 46′, and the angle C=22° 5′; to find the rest? *Ans.* BC=2394, AC=1630.5, *ang.* A=123° 9′.

81. *Case* 2. Let there be given the two sides AB, AC, and the angle B, opposite AC; to find the angle BAC and C, and the remaining side BC.



First, (*Art.* 67.) AC : AB : : sin B : sin C; ∵ sin $C=\frac{AB.\sin B}{AC}$ [*]; which by logarithms is, log. sin C= log. AB+log. sin B—log. AC; ∵ *angle* C *is known.*

Secondly, angle BAC $=180-\overline{B+C}$, ∵ *angle BAC is known.*

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[*] This case will be always ambiguous when the given angle B is acute, and AB greater than AC, (as in the first example); for the above expression is the sine of both $AxB=Azx$, or of its supplement AzB (for the sine of an angle and the sine of its supplement are the same, by cor. Art. 12.); consequently the angle A will be either BAx or BAz, according as the angle AxB, or its supplement AzB be taken; and the corresponding value of BC will be either Bx or Bz. But if the given angle be either obtuse, or a right

Thirdly, (Art. 67.) $AC : BC :: \sin B : \sin BAC$, $\because BC = \frac{AC . \sin BAC}{\sin B}$; that is, by logarithms, log. BC=log. AC+log. sin BAC−log. sin B; $\because$ *BC is known.*

EXAMPLES.—1. Given $AB=204$, $AC=145$, and the angle $B=35°$; to find the side BC and the angles BAC and C.

By construction.

Draw AB and make it=204 by any scale of equal parts and make the angle $B=35°$; from A as a centre with the radius (AC=) 145 taken from the same scale, cross BC in z and x; join Az, Ax, either of which will be AC; then will Bz or Bx be the value of BC, these being measured by the above scale, will be $Bz=81\frac{1}{7}$, and $Bx=252\frac{1}{7}$ for the values of AC; also by the scale of chords, or protractor, $BAx=91°$, $BAz=19°$, for the corresponding values of BAC; likewise $AxB=54°$, $AzB=126°$, for those of C.

By calculation.

To find the angle C.

Because log. sin C=log. AB+log. sin B−log. AC;

$\because$ To log. AB 204= 2.3096302
Add log. sin B 35°= 9.7585913
From this sum= 12.0682215
Subtract log. AC 145= 2.1613680

$$\text{Remains log. sin } C \begin{Bmatrix} = 53°\ 48' \\ \text{or its supp.} \\ \text{viz. } 126°\ 12' \end{Bmatrix} = 9.9068535$$

Next, to find the angle BAC.

$$\textit{First, } B+C = \begin{Bmatrix} 35° + 53°\ 48' \\ \text{or} \\ 35° + 126°\ 12' \end{Bmatrix} = \begin{Bmatrix} 88°\ 48' \\ \text{or} \\ 161°\ 12' \end{Bmatrix}$$

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angle, each of the remaining angles will be acute (32. 1.); therefore when the angle B is either obtuse, or a right angle, C *must* be acute; consequently when B is not less than a right angle, no ambiguity can possibly take place.

If the angle B (in any proposed example under this case) be either acute, obtuse, or a right angle, and AC greater than AB, there is no ambiguity; but it must be remarked, that if AC be less than $AB \times$ nat. sin B (or the perpendicular drawn from A to the base BC,) the question is impossible.

$$\text{angle } BAC=180-\overline{B+C}=\left\{\begin{array}{c}180^\circ-88^\circ\, 48'\\ \text{or}\\ 180^\circ-161^\circ\, 12'\end{array}\right\}=\left\{\begin{array}{c}91^\circ\, 12'\\ \text{or}\\ 18^\circ\, 48'\end{array}\right.$$

Lastly, to find the side *BC*.

Since log. BC=log. AC+log. sin BAC−log. sin B.

If $BAC=91^\circ\, 12'$

To log. AC 145= 2.1613680

Add log. sin $BAC\left\{\begin{array}{c}91^\circ\, 12'\\ \text{or its sup.}\\ 88^\circ\, 48'\end{array}\right\}$ 9.9999047

And from the sum= 12.1612727

Subtract log. sin B 35°= 9.7585913

Remains log. BC=252.744= . . . 2.4026814

If $BAC=18^\circ\, 48'$

To log. AC 145= 2.1613680

Add log. sin BAC 18° 48′= 9.5082141

And from the sum= 11.6695821

Subtract log. sin B 35°= . . . 9.7585913

Remains log. BC=81.4687= 1.9109908

Instrumentally.

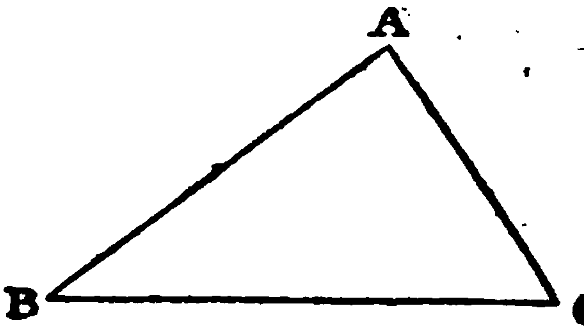
To find the angle C. Extend the compasses from 204 to 145 on the line of numbers, that extent will reach, on the sines from 35° to 53° 48′, the supplement of which is 126° 12′, either of these is the angle *C*.

To find the side BC. Extend on the sines from 35° to 88° 48′, that extent will reach on the numbers from 145 to 253; or extend on the sines from 35° to 18° 48′, this will reach from 145 to $81\frac{1}{2}$ on the line of numbers.

Ex.—2. Given the side AB=266, BC=179, and the angle C=107° 40′; to find AC, and the angles A and B? *Ans.* AC=149.8. *ang.* A=39° 53′, *ang.* B=32° 27′.

3. Given AC=236, BC=350, and the angle B=38° 40′; required the rest? *Ans.* AB=184.47, *or* 362.04, *ang.* A= 67° 54′, *or* 112° 6′, *ang.* C=73° 26′ *or* 29° 14′.

82. *Case* 3. Let the two sides BA, AC, and the included angle A, be given; to find the side BC and the angles B and C.



Let $AB > AC$, then (18. 1.) the ang. $C > B$; and since $B+C = 180^\circ - A$ (32. 1.) $\frac{1}{2}\overline{C+B} = \frac{1}{2} \times \overline{180^\circ - A} = 90^\circ - \frac{1}{2}A$; $\therefore$ $\frac{1}{2}\overline{C+B}$ *is known.*

But (Art. 72.) $AB+AC : AB-AC$ (:: tan $\frac{1}{2}\overline{C+B}$: tan $\frac{1}{2}\overline{C-B}$) :: tan $\overline{90-\frac{1}{2}A}$: tan $\frac{1}{2}\overline{C-B}$, $\because$ tan $\frac{1}{2}\overline{C-B} = \dfrac{\overline{AB-AC}.\tan \overline{90-\frac{1}{2}A}}{AB+AC}$; by logarithms, log. tan $\frac{1}{2}\overline{C-B}$ log. $\overline{AB-AC}$+log. tan $\overline{90-\frac{1}{2}A}$—log. $\overline{AB+AC}$ $\therefore$ $\frac{1}{2}\overline{C-B}$ *is known.*

Whence (Art. 69.) the greater angle $C = \frac{1}{2}\overline{C+B} + \frac{1}{2}\overline{C-B}$, and the less, viz. $B = \frac{1}{2}\overline{C+B} - \frac{1}{2}\overline{C-B}$; $\therefore$ *the angles C and B are known.*

Lastly, (Art. 67.) $AB : BC :: \sin C : \sin A$, $\therefore$ $BC = \dfrac{AB.\sin A}{\sin C}$; by logarithms, log. BC=log. AB+log. sin A—log. sin C $\therefore$ BC is known.

EXAMPLES.—1. Given $AB=20$, $AC=30$, and the angle $A = 80^\circ$; to find the side BC and the angles B and C.

By construction.

Make $AB=20$ by any scale of equal parts, at A (with the scale of chords or protractor) make the angle $BAC=80^\circ$, and make $AC=30$, by the above scale of equal parts, join BC; then the angles B and C, and the side BC being measured, will be as follows; viz. *ang.* $B=63^\circ\ 24'$, *ang.* $C=36^\circ\ 36'$, side $BC=33$, nearly.

By calculation.

$\frac{1}{2}\overline{B+C} = 90^\circ - \frac{1}{2}A = (90^\circ - 40^\circ =)\ 50^\circ$; this being known, in order to find $\frac{1}{2}\overline{B-C}$, we have log. tan $\frac{1}{2}\overline{B-C}$=(log. $\overline{AC-AB}$+log. tan $\overline{90^\circ-\frac{1}{2}A}$—log. $\overline{AC+AB}$=) log. 10+log. tan 50°—log. 50.

$\therefore$ To log. 10=	1.0000000
Add log. tan 50°=	10.0761865
From the sum=	11.0761865
Subtract log. 50=	1.6989700
Remains tan $\frac{1}{2}\overline{B-C}$ $13^\circ\ 24'\frac{1}{2}$	9.3772165

Also $\frac{1}{2}\overline{B+C}+\frac{1}{2}\overline{B-C}=50^\circ+13^\circ\ 24'\frac{1}{2}=63^\circ\ 24'\frac{1}{2}=$ angle B.
And $\frac{1}{2}\overline{B+C}-\frac{1}{2}\overline{B-C}=50^\circ-13^\circ\ 24'\frac{1}{2}=36^\circ\ 35'\frac{1}{2}=$ angle C.
Lastly, log. $BC=$ log. $AB+$ log. sin $A-$ log. sin C;

∵ To log. AB 20=	1.3010300
Add log. sin A 80°=	9.9933515
From the sum = ...	11.2943815
Subtract log. sin C 36° 35′$\frac{1}{2}$ . .	9.7753250
Remains log. BC 33.0412= . .	1.5190565

Instrumentally.

For the first proportion, extend from 50 to 10 on the numbers; this extent will reach on the tangents from 50° (the contrary way, because the tangents above 45° are set back again [d]) to about 8°$\frac{1}{2}$, that is, from 45° to 13°$\frac{1}{2}$.

Extend, for the second proportion, from 36° 36′ to 80° on the sines; this extent will reach from 20 to about 33 on the numbers.

Ex.—2. Given the side $AB=215$, the side $AC=478.8$, and the included angle $A=34^\circ\ 46'$; to find BC, and the angles B and C? *Ans.* $BC=326.1$, *ang.* $B=123^\circ\ 9'$, *ang.* $C=22^\circ\ 5'$.

3. Given $AB=116$, $AC=87$, and the angle $A=115^\circ\ 37'$; required the rest? *Ans.* $BC=172.5$, *ang.* $B=27^\circ\ 3'$, *ang.* $C=37^\circ\ 20'$.

83. *Case* 4. Let the three sides AB, BC, and CA, of the triangle ABC be given; to find the three angles A, B, and C.

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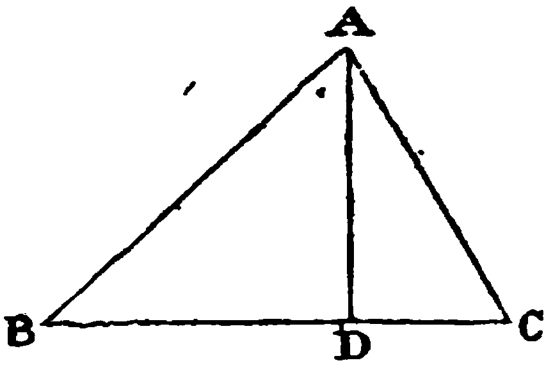
[d] When the ratio to be measured is in the tangents, and one of the terms below, and the other above 45°; having taken the extent of the two former terms on the numbers, &c. as the case may be, apply this distance on the tangents, from 45° downwards (to the left) and let the foot of the compasses rest on this point, which for distinction we will call z; with one foot on z, bring the other foot from 45°, to the given term of the ratio; apply the distance (of z from the given term) from 45° downwards, then, one foot of the compasses being on 45, the other will (with this extent) exactly reach the term required to be found.

First, by letting fall a perpendicular AD.



Let BA be the greater side, AC the less, and BC the base; then (Art. 70.) $BC : BA+AC :: BA-AC : BD-DC$, $\because BD-DC = \frac{\overline{BA+AC}.\overline{BA-AC}}{BC}$; and log. $\overline{BD-DC} = \text{log.}\ \overline{BA+AC} + \text{log.}\ \overline{BA-AC} - \text{log.}\ BC \because BD-DC$ *is known.* But $BD+DC$ ($=BC$) *is given,* $\because$ *the halves of these are likewise known.*

But (Art. 69.) $\begin{cases} BD = \frac{BD+DC}{2} + \frac{BD-DC}{2}, \text{ and} \\ DC = \frac{BD+DC}{2} - \frac{BD-DC}{2} \end{cases}$

$\because$ *the segments* BD, DC *are known.*

Now in the right angled triangle ABD we have AB, BD and the right angle ADB given.

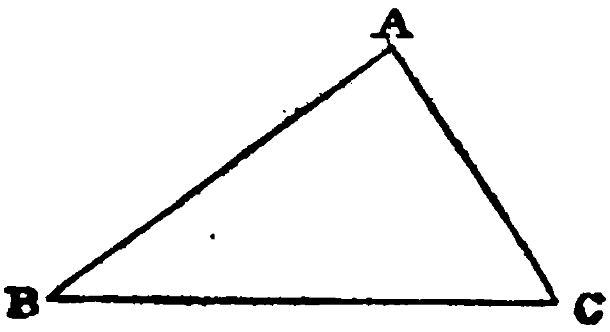
$\because$ (Art. 63.) $AB : BD ::$ rad. : sin BAD, or sin $BAD = \frac{BD.\text{rad}}{AB}$. In logarithms, log. sin $BAD = \text{log.}\ BD + 10 - \text{log.}\ AB$; $\because BAD$ is known, $\because$ also its complement; viz. *the angle ABC is known.*

And in the right angled triangle ADC we have AC, CD and the right angle ADC given, $\because$ as above, $CA : CD ::$ rad. : sin CAD, or sin $CAD = \frac{CD.\text{rad.}}{CA}$. By logarithms, log. sin $CAD = \text{log.}\ CD + 10 - \text{log.}\ CA \because CAD$, and consequently its complement, viz. *the angle C is known.*

Also $BAC = BAD + DAC$ *is known.*

The solution without a perpendicular.

By Art. 71. $2\,BA.AC : \overline{BA}|^2 + \overline{AC}|^2 - \overline{BC}|^2 ::$ radius : cos A



$\because \cos A = \frac{\text{rad.}\overline{BA}|^2 + \overline{AC}|^2 - \overline{BC}|^2}{2\,BA.AC}$

$=$ (since rad$=1$, see also cor. 5. 2.)

$\frac{\overline{BA}|^2 + \overline{AC+BC}.\overline{AC-BC}}{2\,BA.AC}$. By logarithms. log. cos $A = 10 + \text{log.}\ \overline{\overline{BA}|^2 + \overline{AC+BC}.\overline{AC-BC}} - \text{log.}\ 2\,BA + \text{log.}\ AC \because$ *the angle A is known;* and $B+C = 180° - A$, to find the angles B and C.

(Art. 82.) Log. tan. $\frac{1}{2}\overline{B-C}$=log. $\overline{AC-AB}$+log. tan. $\overline{90^\circ-\frac{1}{2}A}$ −log. $\overline{AC+AB}$; then $\left.\begin{array}{l}\frac{1}{2}\overline{B+C}+\frac{1}{2}\overline{B-C}=\text{ang. } B\\ \frac{1}{2}\overline{B+C}-\frac{1}{2}\overline{B-C}=\text{ang. } C\end{array}\right\}$ by Art. 69.

whence *the three angles A, B, and C, are known* *.

EXAMPLES.—1. Given the side $AB=12$, $AC=11$, and $BC=10$, to find the angles A, B, and C.

By construction.

1. Draw the straight line $BC=10$, taken from any convenient scale of equal parts, from B as a centre with the radius 12 describe an *arc*, and from C with the radius 11 cross the above *arc* in A, (both the latter distances being taken from the same scale with BC,) and join AB, AC.

2. Measure the angles by means of the scale of chords, or protractor, and they will be nearly as follows; viz. $A=51\frac{1}{4}$, $B=59^\circ\frac{1}{4}$, and $C=69^\circ\frac{1}{2}$.

By calculation.

First, let AD be perpendicular to BC; see the last figure but one. $BD+DC=BC=10$

$$BD-DC=\frac{\overline{BA+AC}.\overline{BA-AC}}{BC}=\frac{23\times 1}{10}=2.3.$$

$\because$ $BD=\frac{BD+DC}{2}+\frac{BD-DC}{2}=5+1.15=6.15$ the greater segment;

and $DC=\frac{BD+DC}{2}-\frac{BD-DC}{2}=5-1.15=3.85$ the less segment;

Then log. sin. BAD=log. BD+10−log. AB=0.7888751 +10−1.0791812=9.7096939; $\because$ ang. $BAD=30^\circ\ 50'$, and $\because$ its complement $B=59^\circ\ 10'$.

In like manner, log. sin. CAD=log. CD+10−log. CA= 0.5854607+10−1.0413927=9.5440680= $\because$ ang. $CAD=20^\circ$ $29'$; the complement of which is $69^\circ\ 31'$=the angle C.

Also the ang. $BAC=BAD+CAD=30\ 50'+20^\circ\ 29'=$ $51^\circ\ 19'$.

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* On having found the angle A, the remaining angles B and C may be found (perhaps more conveniently) by Art. 67. thus $BC : CA :: \sin A : \sin B=\frac{CA.\sin A}{BC}$; $\because$ *B is known*; whence also $C=180-\overline{A+B}$; $\because$ *C is likewise known.*

The solution without a perpendicular; see the last figure.

Natural $\cos A = \frac{\overline{BA}|^2 + \overline{AC+BC}.\overline{AC-BC}}{2\, BA.AC} = \frac{144+21}{264} =$ $.6250000$ ∵ angle $A = 51°\ 19'$; ∵ $C+B = 180° - 51°\ 19' = 128°\ 41'$, and $\frac{C+B}{2} = 64°\ 20'\frac{1}{2}$.

Log. tan. $\frac{C-B}{2} =$ log. $\overline{AB-AC}$ + log. tan. $64°\ 20'\frac{1}{2}$ − log. $\overline{AB+AC} = 0 + 10.3184222 - 1.3617278 = 8.9566944$ ∵ $\frac{C-B}{2} = 5°\ 10'\frac{1}{2}$.

∵ angle $C = \frac{C+B}{2} + \frac{C-B}{2} = 64°\ 20'\frac{1}{2} + 5°\ 10'\frac{1}{2} = 69°\ 31'$.

angle $B = \frac{C+B}{2} - \frac{C-B}{2} = 64°\ 20'\frac{1}{2} - 5°\ 10'\frac{1}{2} = 59°\ 10'$.

Instrumentally, first method.

1. Extend from 10 to 23 on the line of numbers; this extent will reach, on the same line, from 1 to $2\frac{3}{10}$, the difference of the segments of the base.

2. Extend from 12 to 6.15 on the numbers; this extent will reach on the sines from 90° (radius) to 30° 50′ $= BAD$, the complement of which is 59° 10′ = ang. B.

3. Extend from 11 to 3.85 on the numbers; that extent will reach from 90° to $20°\frac{1}{2}$ on the sines, the complement of which is $69\frac{1}{2} = C$.

Second method. 1. Extend from 264 ($= 2BA.AC$) to 165 ($= \overline{BA}|^2 + \overline{AC}|^2 - \overline{BC}|^2$) on the numbers; that extent will reach from 90° to $38\frac{2}{3}$ on the sines, the complement of which is $51\frac{1}{3}$ = angle A.

2. Extend on the numbers from 23 to 1; that extent will reach from $64°\frac{1}{3}$ to 45°; and back again to $5\frac{1}{6}$ on the tangents, for half the difference of the angles B and C.

Ex. 2. Given the three sides, viz. $AB = 100$, $AC = 40$, and $BC = 70.25$; to find the three angles? *Ans. ang.* $A = 33°\ 35'$, *ang.* $B = 18°\ 22'$, *ang.* $C = 128°\ 3'$.

3. Given $AB = 368.95$, $AC = 472$, and $BC = 700$, to find the angles? *Ans. ang.* $A = 112°\ 6'$, *ang.* $B = 38°\ 40'$, *ang.* $C = 29°\ 14'$.

THE APPLICATION OF PLANE TRIGONOMETRY TO THE FINDING OF THE HEIGHTS AND DISTANCES OF INACCESSIBLE OBJECTS.

The uses to which Plane Trigonometry may be applied are so various and extensive, that merely to point them out would require a very large volume; and to understand them, the student must be well acquainted with Geography, Astronomy, and the numerous branches of Natural Philosophy, of which this science forms a necessary part. At present we shall confine ourselves to one of its immediate and obvious applications, namely, that of determining the heights and distances of inaccessible objects.

The following instruments are used in this branch of mensuration, namely, a quadrant, a theodolite, a mariner's compass, a perambulator, Gunter's chain, measuring tapes, a measuring rod, station staves, and arrows; the description and uses of which are as follow:

84. THE QUADRANT [f] is an instrument for measuring angles in a vertical position; that is, to determine the angular altitude

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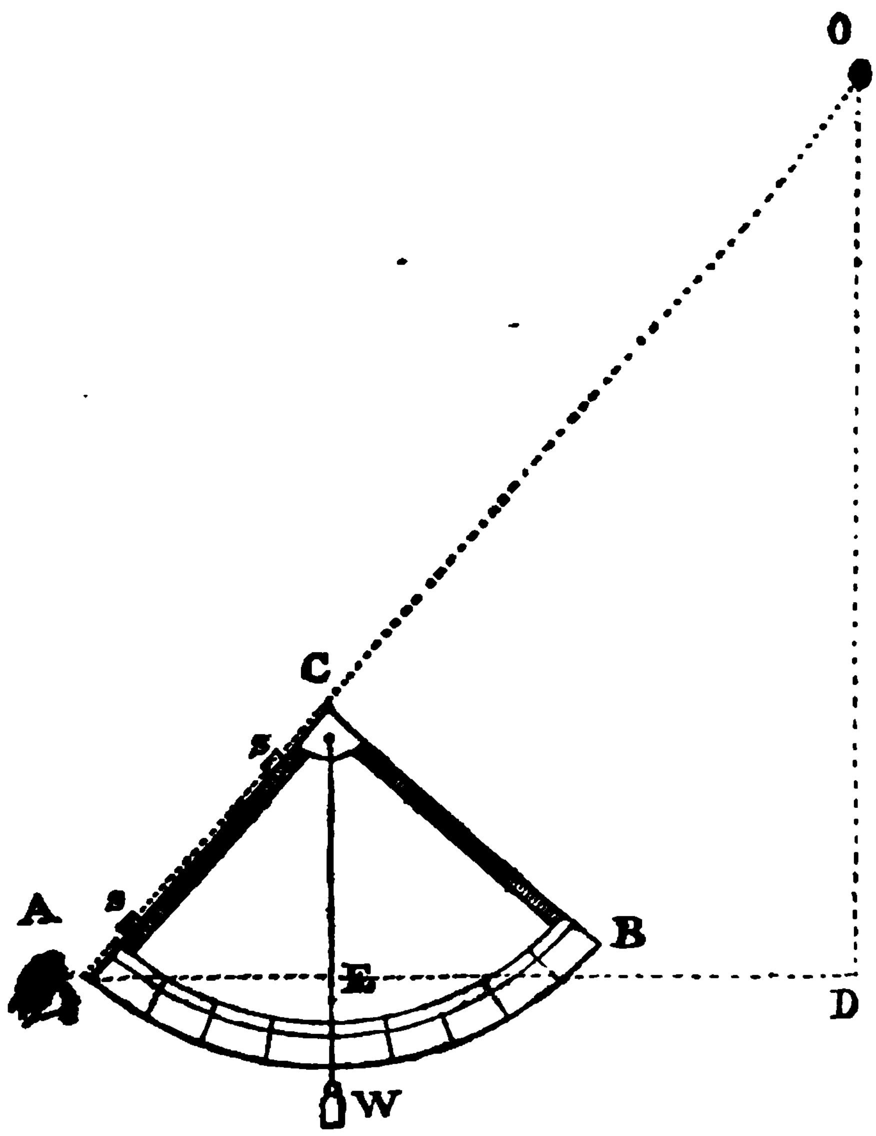
[f] Besides the common surveying quadrant, of which that described above is the simplest form, there are various other kinds, as the astronomical quadrant, the sinical quadrant, the herodictical quadrant, Davis's, Gunter's, Hadley's, Cole's, Collins's, Adams's, and some others. Quadrants may be had at any price from one to twelve guineas.

The height of an object may be taken in two senses, viz. 1. its perpendicular distance (in fathoms, yards, feet, &c.) from the ground; 2. its angular height, or the number of degrees contained in the angle at the eye of the observer, which the perpendicular height subtends; the former we have, for distinction, denominated *height*, the latter *altitude*.

of any proposed object. ABC is a quadrant, to the centre C of which the weight W is freely suspended, by means of the string CW; ss are two sights, through which the eye of an observer at A sees the object O.



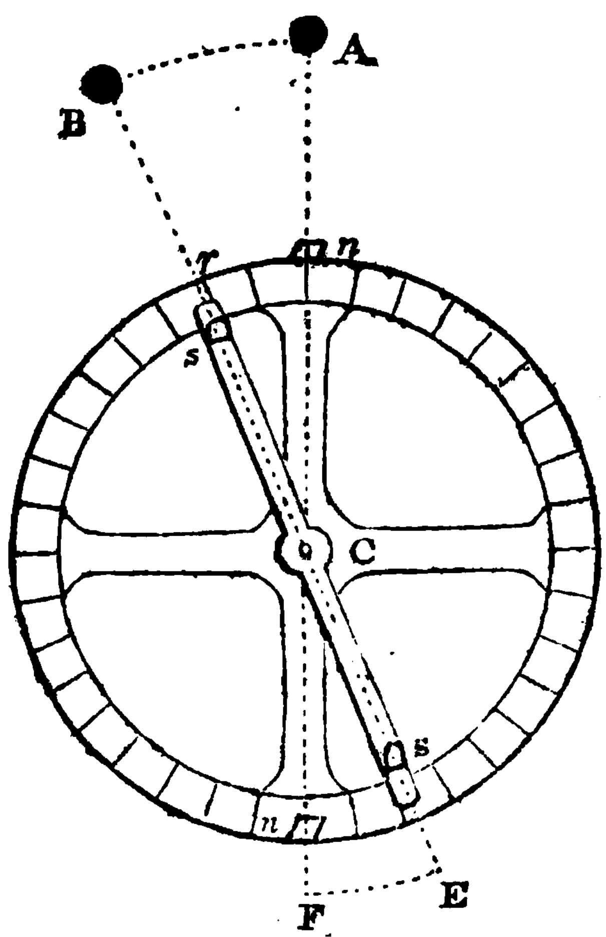
The *arc* AB of the quadrant is divided into degrees, which are subdivided into halves, quarters, or single minutes. In using this instrument, the observer turns it about the centre C, until the object O is visible through the sights ss; and as he turns it, the line CW, revolving freely about the centre C, moves along the circumference AB; when he sees the object O through the sights, the *arc* BW will be the measure of its angular altitude, that is, of the angle OAD.

Draw OD perpendicular, and AD parallel to the plane of the horizon; then because the angles at E and D are right angles and the angle A common, the triangles CAE, OAD are equiangular (32. 1.), ∵ the angle $ACE=AOD$; but $DOA+DAO=$ (a right angle$=$) ACB, from these equals take the equals $DOA=ECA$, and the remainder $DAO=ECB$. And since the *arc* BW is the measure of the angle ECB (Part 8. Art. 237.) it is likewise the measure of DAO, or of the angular altitude of the object O above the plane of the horizon.

85. THE THEODOLITE [s], in its simplest form, consists of a brass

[s] Some of the best theodolites are adapted to measuring *vertical* as well as *horizontal* angles to a single minute; being fitted with *vertical* arch, level, telescopic sights, and rack-work motions. The prices of theodolites are from two to forty guineas. The circumferentor is an instrument for measuring *horizontal* angles, chiefly used in wood lands, and its price is from two to five

circle of about a foot in diameter, having its circumference divided into 360 degrees, and these subdivided into halves, quarters, or minutes; the index *sCs* turns about the centre *C*, and has fixed on it two sights *s s*; there are likewise fixed on the circumference two sights *n n*; this circle is fixed in a horizontal position on three legs of a convenient height for making observations.



The theodolite is used for measuring the angular distances of objects situated on the plane of the horizon; thus.

Let *A* and *B* be two objects, place the instrument in such a position that one of them, as *A*, may be seen through the fixed sights *n* and *n* by an eye at *F*.

Turn the index *s s* about the centre *C*, until the other object *B* appears through the sights *s s* to an eye situated at *E*; then will the angle *ACB*, which is measured by the *arc nr*, be the angular distance of the given objects *A* and *B*.

86. THE MARINER'S COMPASS [h] is an instrument used for finding the position or bearings of objects with respect to the meridian, and for determining the course of a ship: what principally requires explanation is the card; it is a round piece of stiff pasteboard, having its circumference divided into thirty-two

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guineas. The semicircle is a much simpler and cheaper instrument than the theodolite, and serves very well for measuring angles on the plane of the horizon where very great accuracy is not required.

[h] The invention of the mariner's compass is usually ascribed to Flavio Gioia, an Italian, A.D. 1302; but it is stated by some authors that the Chinese had a knowledge of it as early as the year 1120 before Christ. The price of this useful instrument is from half-a-crown to twelve guineas.

equal parts, called *points*; a steel wire, called the needle, which has been rubbed with a load-stone, is fixed across the under side of the card from *N* to *S*, by which means (when the card is exactly balanced on its centre) the point *N* is directed to the north, and consequently the point *S* to the south, and each of the remaining points to its respective position in the horizon; in the centre of the card underneath, is fixed a finely polished conical brass socket, about one third of an inch deep.

The compass box is a basin of brass or wood, having a fine pointed steel needle fixed perpendicularly in its bottom: on the point of this, the above-mentioned socket in the bottom of the card being placed, the card is balanced and turns freely as impelled by the attractive force of the magnet. The box is suspended within a brass hoop or ring, by means of two gimbols placed on opposite sides, which serve as an axis, and admit free motion; and this hoop is in like manner suspended on the opposite sides of a square wooden box by gimbols, at 90° distance from the former, a contrivance intended to secure the horizontal position of the inner box and card, whatever may be the motion of the ship in which the compass is placed [1].

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[1] Those who cross forests, deserts, and uninhabited countries, find this instrument a necessary companion to direct them; they keep the compass always before them, and follow the direction of that point which indicates the situation of the place they wish to arrive at. The like method is employed in steering a ship, which is kept in such a position, that the proposed point may, of its own accord, stand in a direction towards the head of the ship. Note, N b E means *north by east*; N N E, *north north-east*; N E b N, *north-east by north*; N E, *north-east*, &c. &c. which will be easily understood.

87. *A table shewing the degrees and minutes that every point of the compass makes with the meridian* [k].

NORTH		Pts	Degrees		SOUTH	
N b E	N b W	1	11°	15′	S b E	S b W
N N E	N N W	2	22	30	S S E	S S W
N E b N	N W b N	3	33	45	S E b S	S W b S
N E	N W	4	45	0	S E	S W
N E b E	N W b W	5	56	15	S E b E	S W b W
E N E	W N W	6	67	30	E S E	W S W
E b N	W b N	7	78	45	E b S	W b S
EAST	WEST	8	90	0	EAST	WEST

Explanation.

In the preceding figure the line N S is called the *meridian line*; the two first columns of the table extend from *north* both ways to east and west, as the two last do from *south*; the two first points in the first and second columns make the same angle with the meridian line N S (11° 15′) reckoning from the north point, that the two first in the 5th and 6th columns do, reckoning from the south, and the like is evidently true of the points in any horizontal line of the table. The angles made by the points in the first and second columns with the meridian are therefore measured by the *arcs* intercepted between them and the north point, viz. the first column, on the *east* side of north; and the second on the *west*: in like manner the angles made by the points in the 5th and 6th columns with the meridian are measured by the respective *arcs* intercepted between them and the *south* point, those in the 5th column being on the *east* of south, and those in the sixth on the *west*: for example, N N E is 22° 30′ to the *east* of north, N N W is the same distance *west* of north; S S E is the same distance *east* of *south*, and S S W is the same distance *west* of *south*. In the third column each number denotes the distance from north or south of the points against which it stands; and the numbers in the fourth column shew the degrees and minutes of the *arc* intercepted between the north or south, and the points against which they stand.

88. *The use of the above Table.*

When a question is proposed in which the conditions require that lines should be drawn in given positions with the meridian expressed in points of the compass, the construction may be made with the greatest facility, by means of this table; to effect which this is the

RULE.—1. Describe a circle and draw the diameter *NS* for the meridian, *N* being the north point, *S* the south.

2. Take the degrees and minutes from the table which correspond with the points mentioned in the question, and measure *arcs* from the meridian equal to them.

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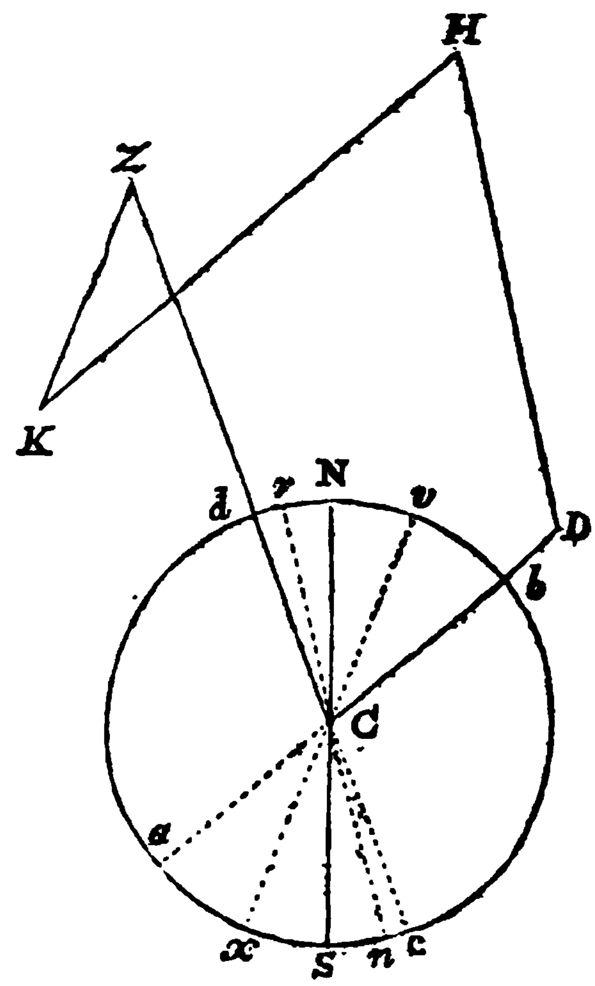
[k] The table is thus constructed: divide 360 (= the number of degrees in the circumference of a circle) by 32 (= the number of points in the compass,) and the quotient is $\frac{1}{32}$ part of the circumference = 11° 15′, or 1 point of the compass; this doubled is 22° 30′ for two points; its triple is 33° 45′ for three points, and so on.

3. Draw lines through the centre to the points thus measured, and construct your figure by drawing its sides respectively parallel to these, and each of its proper length taken from a scale of equal parts.

4. If the position of one of the lines be required, draw a line parallel to it through the centre of the circle, measure the angle this line makes with the meridian, then the point of the compass which stands opposite this measure will give the bearings or position required [1]; and its length, taken in the compasses, and applied to a scale of equal parts, will give its measure.

EXAMPLES.—1. A man intends to travel from C to Z which lies N N W from C 6 miles, but he must first call at D, which lies N E 3 miles, then at H, N b W from D 5 miles, and lastly at K, which is S W from H $4\frac{1}{2}$ miles; at H how far is he distant from Z, and what course must he travel to arrive there?



Here I first draw cCZ through the point d, distant 22° 30′ from N (answering to N N W); next I draw ab at 45° distance from N (answering to N E); next I draw rn at 11° 15′ distance on the left of N (answering to N b W); and since $aS = Nb = 45°$, it is plain that ab will be the S W as well as the N E line. I then take $CD = 3$, draw DH parallel to rn and make it $= 5$, whence I draw HK parallel to ab and make it $= 4\frac{1}{2}$, I then join KZ and find its measure to be $2\frac{1}{4}$ miles nearly, and its bearings (shewn by the parallel xv, the position of which is measured by the arc Nv)

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[1] The position, or bearings of a line may likewise be known by simply drawing a meridian from the given point, and measuring the angle which that line makes with it; the degrees contained in it being found in the table will shew the point of the compass required.

N $18^\circ\frac{1}{4}$ E [m], that is, N b E $7^\circ\frac{1}{4}$ E, or $7\frac{1}{4}$ degrees to the eastward of north by east.

2. *B* is 8 miles N W from *C*, and *A* 4 miles N from *B*; required the course and distance from *A* to *C*? *Ans. course* S $31^\circ\frac{1}{4}$ E. *Distance* 11 *miles*.

3. A ship sailed S E 12 leagues, N N E 20 leagues, and N N W 30 leagues; required her distance from the point sailed from, and her course back?

89. THE PERAMBULATOR [n], called also a *pedometer*, *waywiser*, and *surveying wheel*, is an instrument for measuring large distances on ground nearly level; it consists of a wheel $8\frac{1}{4}$ feet in circumference, which the measurer drives before him, by means of two handles, fixed at the end of a hollow shaft, terminating in two cheeks to receive the wheel, and in which its axis turns. The wheel goes over one pole of ground in every two revolutions, and its motion is communicated by the intervention of various clock-work movements within the shaft, to a dial, fixed near the handles, the index of which points out the distance passed over.

THE GUNTER'S CHAIN [o] is used to measure smaller distances than those to which the perambulator is applied; its length is 66 feet=22 yards=4 poles, and is divided into 100 links, each 7,92 inches in length. This is the most convenient instrument of any that has been contrived for measuring land, because 10

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[m] The bearings of two objects from each other may be estimated either in *degrees*, or *points*; degrees may be turned into points, or points into degrees, by referring to the table; thus, if an object bear 33° 45′ to the east of south, by turning to the table I find that the exact point of bearing is S E b S; if it bear 25° to the west of north, the bearing in *points* is N N W 2° 30′ W, that is, 2° 30′ west of N N W. Or the reckoning may be made to the nearest *quarter point*, thus N 14° 4′ W is N b W $\frac{1}{4}$ W; S 28° 7′$\frac{1}{2}$ E is S S E $\frac{1}{2}$ E; in like manner N 64° 41′ E is N E b E $\frac{3}{4}$ E, &c. &c.

[n] The price of this instrument is from five to ten guineas. The name *Pedometer* is likewise applied to an instrument of a watch size for the pocket, for ascertaining distances, either walking or riding, and costs from three to fifteen guineas. The perambulator, Gunter's chain, and tapes, will measure with sufficient exactness for most purposes where the ground is level, but where it is not, distances should be found by trigonometrical calculation.

[o] The Gunter's chain will cost from five to fourteen shillings, according to its strength, and the perfection of its workmanship.

chains in length, and one in breadth, (=100000 square links) make just an acre.

91. THE MEASURING TAPES [p] are of one, two, three, or four poles in length; they are applied to the same purposes as the chain, and, if kept dry, will measure with tolerable exactness.

92. THE MEASURING ROD may be of six, eight, or ten feet in length; it is divided into single feet, which are subdivided into halves and quarters, or into tenths of a foot, for the convenience of measuring small distances.

93. STATION STAVES or *prickets*, are staves of about five or six feet in length, having a small flag fixed at one end, the other end being sharpened to a point for fixing in the ground; these staves are used in measuring, for marking stations, which are required to be seen and distinguished at a distance.

94. THE ARROWS are of wood or iron, pointed at one end, and their use is to stick in the ground as a mark, at the end of every chain or other measure.

95. PROBLEMS.

Prob. 1. An observer at 113 feet distance from the foot of an obelisk, finds its angular altitude to be 40^0; required its height, that of the observer's eye above the plane of the horizon being 5 feet?

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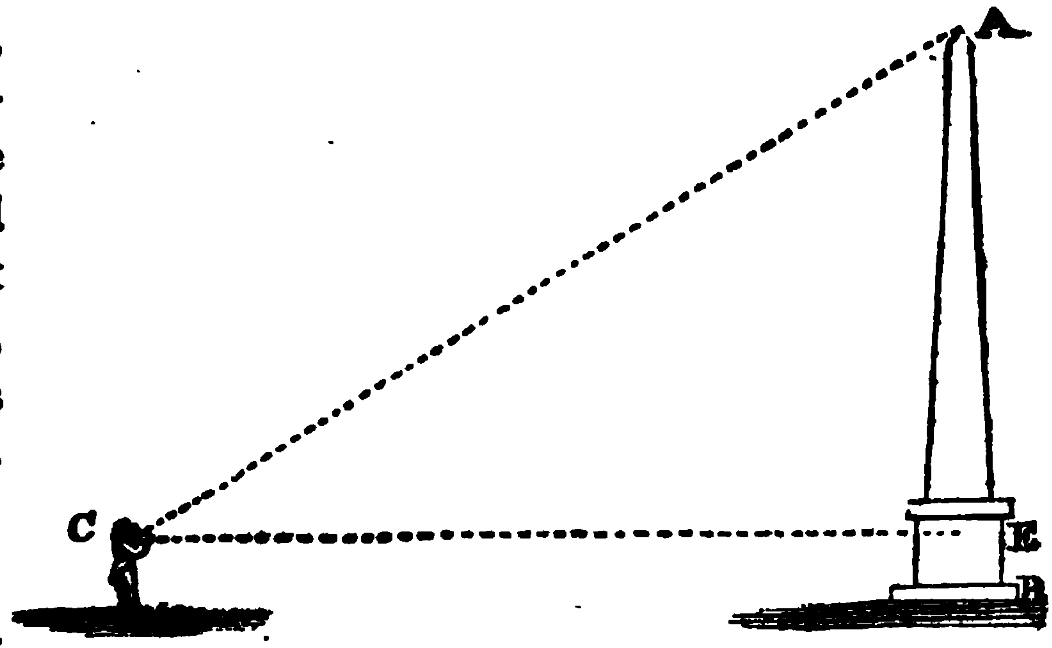
[p] These tapes are sold at the shops of the mathematical instrument makers, and cost from five to twelve shillings, according to their length.

The above instruments, at the prices we have mentioned, will perhaps be found too expensive for the student's pocket; in that case his own ingenuity may supply him with all that is necessary for measuring vertical and horizontal angles and distances. A theodolite may be made with a circular piece of stiff pasteboard, graduated and nailed (through its centre) on the top of a piece of mop-stick, the other end of the stick being sharpened to a point for fixing it in the ground. A quadrant likewise may be made of pasteboard, in like manner graduated, and having a piece of lead, or a stone, hung from its centre by a string. The chain or tapes may have their place supplied by a string previously measured, divided, and subdivided, according to the mind of the operator. The measuring rod may be made of any stick, of a proper length and thickness. The station staves may be made of sticks having one end pointed and the other split, for the purpose of holding a piece of white paper, and the arrows may be cut out of any hedge.

With apparatus of this kind, I have frequently known altitudes and distances determined, with sufficient exactness for any common purpose.

Note. In finding the height of objects, to the observed height must be added, that of the observer's eye above the plane of the horizon.

Let AB be the obelisk, CB the distance of the observer, and BE the height of his eye; then AE is the part required to be found.



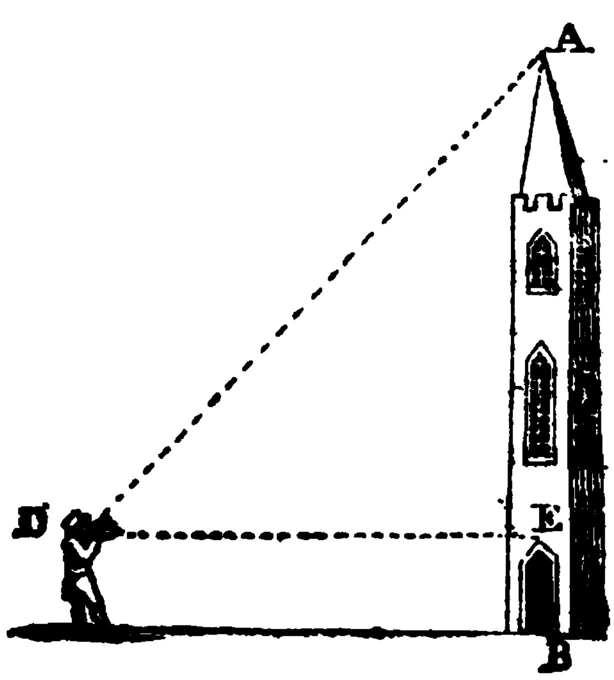
In the triangle ACE, we have given $CE=113$, the angle $ACE=40°$, consequently $CAE=(90-40=)$ $50°$, and the angle CEA a right angle; to find AE.

Now (Art. 67.) $CE : EA :: \sin A : \sin ACE$, $\because$ $EA=\frac{CE.\sin ACE}{\sin A}$, and $\log. EA=\log. CE+\log. \sin ACE-\log. \sin A$ $=2.0530784+9.8080675-9.8842540=1.9768919$, the natural number, corresponding to which is $94.8182=AE$, $\because$ $AE+EB=$ $94.8182+5=99.8182$ feet$=99$ feet 9 inches $\frac{1023}{1250}$=the height required.

Prob. 2. The angular altitude of a spire, known to be 137 feet high, is $51°$; now supposing the height of the observer's eye to be $5\frac{1}{2}$ feet, how far is he distant from the foot of the spire?

Note. In questions of this kind, the height of the eye must be subtracted from the given height, previous to the operation.

Here are given $AB=137$, $EB=5\frac{1}{2}$, $\because$ $AE=137-5\frac{1}{2}=$ 131.5, AED a right angle, and angle $ADE=51°$, $\because$ ang. $DAE=$ $(90°-51°=)$ $39°$ $\because$ (Art. 67.) $DE : EA :: \sin DAE : \sin ADE$ $\because$ $DE=\frac{EA.\sin DAE}{\sin ADE}=$ $\frac{131.5\times\sin 39°}{\sin 51°}$, $\because$ log DE=log



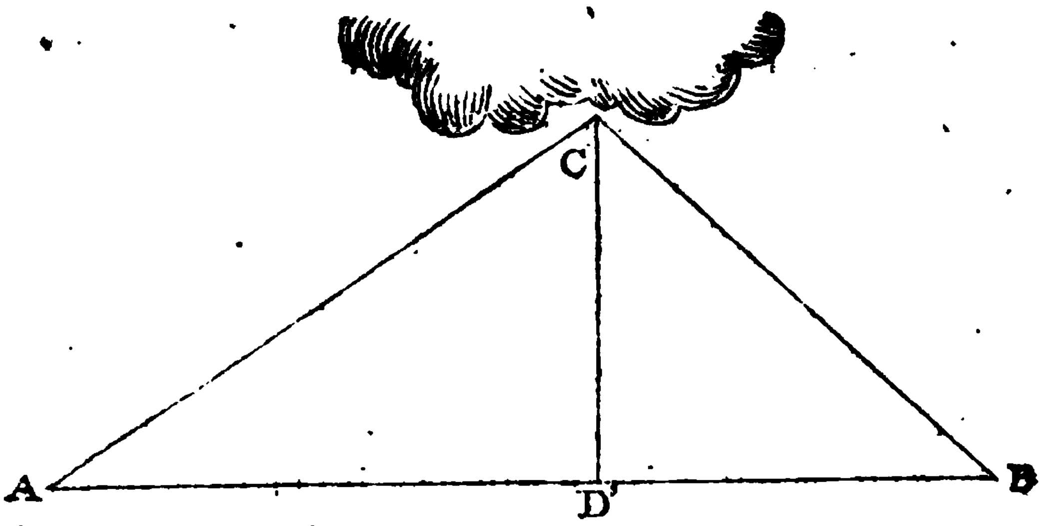
131.5+log. sin 39—log. sin 51°=2.1189258+9.7988718−9.8905026=2.0272950 ∴ DE=106.487 feet=106 feet 5 inches $\frac{844}{1000}$.

Prob. 3. Wanting to calculate the perpendicular height of a cliff, I took its angular altitude 12° 30′, but after measuring 950 yards in a direct line towards its base, I was unexpectedly stopped by a river; here however I again took its altitude 69° 30′; required the height of the cliff, and my distance from the centre of its base?

Let A be the first station, B the second, C the summit of the cliff, and D its base; then AB=950, the angle A=12° 30′, angle ABC=(180°−69° 30′=) 110° 30′ ∴ ang $ACB=(180-\overline{12°\,30'+110°\,30'}=180°-123'=)$ 57°; ∵ in the triangle ABC we have the side AB and the three angles given, to find BC. Now (Art. 67.) AB : BC :: sin ACB : sin A ∴ $BC=\frac{AB.\sin A}{\sin ACB}$, and log BC=log AB+log. sin A—log. sin ACB= (log 950+log. sin 12° 30′—log. sin 57°=) 2.9777236+9.3353368−9.9235914=2.3894690, ∴ BC =245.171; having found BC, there is given in the triangle BCD the right angle BDC, the angle CBD=69° 30′, the angle BCD=(90°−69° 30′=) 20° 30′ and the side BC=245.171, ∵ (Art. 63.) BC : BD :: rad : sin BCD, ∴ $BD=\frac{BC.\sin BCD}{\text{rad}}=$ 85.8608 yards. Also (Art. 63.) BC : CD :: rad : sin CBD; ∴ $CD=\frac{BC.\sin CBD}{\text{rad}}$=229.645 yards.

Prob. 4. Two persons, situated at A and B, distant 2¼ miles, observed a bright spot in a thunder cloud at the same instant; its altitude at A was 46°, and at B 63° 30′; required its perpendicular height from the earth?

First. Angle $ACB=(180°-\overline{46°+63°\,30'}=)$ 70° 30′, then (Art. 67.) AB : BC :: sin ACB : sin BAC, ∴ $BC=\frac{AB.\sin BAC}{\sin ACB}$



Ac

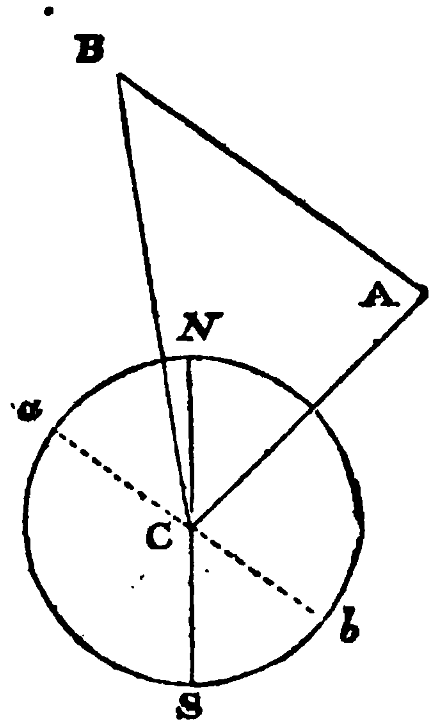
$=2.13612$ miles. Wherefore in the right angled triangle BCD, $BC : CD :: \text{rad} : \sin CBD$ (Art. 63.), $\because CD = \frac{BC.\sin CBD}{\text{rad}} =$

1.5361

1.9117 mile $=$ the height required.

Prob. 5. Two towns, A and B, are invisible and inaccessible to each other, by reason of an impassible mountain, situated between them; but both of them are visible and accessible from the point C, viz. A bears N E from C distance 3 miles, and B bears N b W from C distance $5\frac{1}{4}$ miles; required the bearings and distance of A and B from each other?

First. Since CA lies N E, or 45° on the east of the meridian, and CB lies N b W or $11^{\circ}\ 15'$ on the west, $\because$ angle $C = (45^{\circ} + 11^{\circ}\ 15' =)\ 56^{\circ}\ 15'$; $\because$ (Art. 72.) $CB + CA : CB - CA :: \tan \frac{A+B}{2} : \tan \frac{A-B}{2}$; or $8.25 : 2.25 :: \tan 61^{\circ}\ 52'\frac{1}{2} : \tan 27^{\circ}\ 1'\ 57''$; then (Art. 69.) angle $A = (61^{\circ}\ 52'\ 30'' + 27^{\circ}\ 1'\ 57'' =)\ 88^{\circ}\ 54'\ 27''$, and angle $B = (61^{\circ}\ 52'\ 30'' - 27^{\circ}\ 1'\ 57'' =)\ 34^{\circ}\ 50'\ 33''$; next, (Art. 67.) $CA : AB :: \sin B : \sin C$, $\because AB = \frac{CA.\sin C}{\sin B} = 4.36606$ miles.



Lastly, through the centre C draw ab parallel to AB, and measure the circumference Na, and it will be found to contain $46^{\circ}\ 6'$, which, by referring to the table (Art. 87.), will be found to answer to the N W point nearly; that is, B bears from A N W $1^{\circ}\ 6'$ W distance (4.36606 miles $=$) 4 miles 3 furlongs nearly.

Prob. 6. A general arriving with his army on the bank of a river is desirous of crossing it, but there are two of the enemy's fortresses, *A* and *B*, on the opposite shore, and he wishes to know their bearings and distance from each other; for this purpose two stations *C* and *D* are chosen close to the river side, *C* being directly east, from *D* at $\frac{3}{4}$ mile distance; at *C* the angles are as follow, viz. $ACB=68°$, $BCD=32°$; at *D* the angles are $ADB=62°$, $ADC=64°$. Now suppose he crosses directly from the point *D*, required the bearings and distance of *A* and *B* from each other; the width of the river at the point of crossing.

A

and the distance of the point where he proposes to land from A and B?

First. In the triangle DAC, there are given $DC=\frac{3}{4}$ mile $=$ .75, the angle $ADC=64°$, $DCA=(32°+68°=)$ 100, and DAC $=(180-164=)$ 16°; to find DA. By Art. 67. $DC : DA ::$ $\sin DAC : \sin DCA$, $\because DA=\frac{DC\times\sin DCA}{\sin DAC}=\frac{.75\times\sin 100°}{\sin 16°}=$ 2.67963 miles.

Secondly. In the triangle BDC, there are given $DC=.75$, $BDC=(62°+64°=)$ 126°, $DCB=32°$, and $DBC=(180°-\overline{126°+32°}=)$ 22°, to find BD. By Art. 67. $DC : BD :: \sin DBC : \sin DCB$, $\because BD=\frac{DC\times\sin DCB}{\sin DBC}=\frac{.75\times\sin 32°}{\sin 22°}=$ 1.06095 miles.

Thirdly. In the triangle BDA there are given $DA=$ 2.67963, $BD=1.06095$, and the included angle $ADB=62°$; to find the angles DBA, BAD, and the side BA. Now $\frac{DBA+BAD}{2}$ $=\frac{180°-62°}{2}=59°=$ half the sum of the angles DBA, BAD at the base; also $AD+DB=2.67963+1.06095=3.74058=$ sum of the sides, and $AD-DB=2.67963-1.06095=1.61868=$ diff. of the sides. But (Art. 72.) $AD+DB : AD-BD :: \tan\frac{DBA+BAD}{2} : \tan\frac{DBA-BAD}{2}$; that is, $3.74058 : 1.61868 :: \tan 59° : \frac{1.61868\times\tan 59°}{3.74058}=\tan 35°\ 42'\ 5''=$ half the difference of the angles DBA, BAD at the base.

$\because$ (Art. 69.) $\begin{cases} 59°+35°\ 42'\ 5''=94°\ 42'\ 5'' = \text{the angle } DBA. \\ 59°-35°\ 42'\ 5''=23°\ 17'\ 55''=\text{the angle } BAD. \end{cases}$

Also (Art. 67.) $BD : BA :: \sin BAD : \sin BDA$, $\because BA=\frac{BD\times\sin BDA}{\sin BAD}=\frac{1.06095\times\sin 62°}{\sin 23°\ 17'\ 55''}=2.36842$ miles.

Fourthly. In the triangle DBE there are given the angle E a right angle $DBE=(180°-DBA=180°-94°\ 42'\ 5''=)$ 85° 17′ 55″, the angle $BDE=(90°-DBE=90°-85°\ 17'\ 55''=)$ 4° 42′ 5″, and the side $BD=1.06095$; to find the sides BE and DE.

By Art. 63. $DB : BE ::$ rad : $\sin BDE$, $\because BE=$

$\frac{DB \times \sin BDE}{\text{rad.}} = \frac{1.06095 \times \sin 4° 42' 5''}{\text{rad.}} = .086958$ mile = somewhat more than 150 yards.

Also $DB : DE :: \text{rad} : \sin DBE$, $\therefore DE = \frac{DB \times \sin DBE}{\text{rad.}}$

$= \frac{1.06095 \times \sin 85° 17' 55}{\text{rad.}} = 1.05738$ mile.

Lastly. Since the line CD lies directly east and west, any line CN drawn perpendicular to it will represent the meridian, and the acute angle BNC, which AB makes with CN, will be the bearings of B from A; this angle may be very readily determined in the present instance; for since the two opposite angles DCN and DEN of the quadrilateral $DENC$ are two right angles, the two remaining angles $EDC + ENC = 2$ right angles (cor. 1. 32. 1.); but $EDC = (4° 42' 5'' + 62° + 64° =)$ $130° 42' 5''$, $\because$ $ENC = (180° - BDC = 180° - 130° 42' 5'' =)$ $49° 17' 55''$, which in the table (Art. 87.) answers to S W 4° 17′ 55″ W or S W $\frac{1}{2}$ W nearly; for the bearings of B from A.

Prob. 7. Required the perpendicular height of the spire of a church, the angular altitude of which is 40°; the observer being 137 feet distant, and his eye $5\frac{1}{2}$ feet from the ground? *Answer* 120.457 *feet.*

8. The angular altitude of an observatory is 53°, its perpendicular height 129 feet, and the height of the eye 5 feet; required the distance of the observer? *Ans.* 93.4407 *feet.*

9. A ladder 30 feet long reaches 23 feet up a building; required the angle of inclination at the foot, and its distance from the wall? *Ans. inclination* 50° 3′ 20″; *distance* 19.2613 *feet.*

10. A shore 11 feet long, in order to support a wall, is placed so that the angle at bottom is double the angle at top, how high up the wall does it reach, and how far distant from the wall is its foot? *Ans. height* 9.52628 *feet; distance* $5\frac{1}{2}$ *feet.*

11. Required the altitude of the sun, when the length of a man's shadow is double its height, and likewise when it is only half its height? *Ans.* 26° 34′ 5″ *in the first case, and* 63° 25′ 55″ *in the second.*

12. A maypole being broken by a sudden gust of wind, the upper part (which still adhered by some splinters to the stump) made with the ground at 15 feet distance from the stump, an

angle of $73^\circ\ 30'$; required the height of the maypole and the length of each of the pieces? *Ans. stump* 29.2072 *feet, upper end* 30.4626 *feet, whole length* 59.6698 *feet.*

13. A ship having sailed 234 miles between the south and west, finds herself 96 miles distant from the meridian she sailed from; required her course and difference of latitude [p]? *Ans. course* S S W $2^\circ\ 13'\ 15''$ *west; diff. of latitude* 213.401 *miles south.*

14. There are three towns *A*, *B*, and *C*; from *B* to *C* the distance is 7.625 miles; at *B* the towns *A* and *C* subtend an angle of $51^\circ\ 15'$, and at *C* the towns *A* and *B* make an angle of $37^\circ\ 21'$; required the distance from *A* to each of the other two? *Ans. from A to B* 4.6275 *miles, from A to C* 5.94825 *miles.*

15. Within sight of my house there is a church and a mill, the former is distant 2.875 miles, the latter 4.24625 miles, and they subtend an angle of $47^\circ\ 23'$; required the distance from the mill to the church? *Ans.* 3.125 *miles.*

16. A farmer has a triangular field, the sides of which are as follow, viz. $AB=760$ yards, $AC=690$, and $BC=850$; he is desirous of dividing it into two parts by a hedge from *A*, perpendicular to *BC*; required its length, and likewise whereabouts it will meet the hedge *BC*? *Ans. length* 585.31 *yards; distance from C* 365.2942 *yards.*

17. [q] A man travels from *A* to *B* $5\frac{1}{4}$ miles, then *bending a little to the right hand* of the direct road, he arrives at *C* distant from *B* 3 miles; from *C* both *A* and *B* are visible under an angle of $25^\circ\frac{1}{4}$; what is his distance from home by the shortest cut? *Ans.* 7.796 *miles.*

18. A man having travelled from *A* to *B* $5\frac{1}{4}$ miles, attempts

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[p] The angle which *the direction a ship sails in* makes with the meridian, is called her *course*, whence in the present case, construct a right angled triangle, the hypothenuse of which is $=234$, this will be her *distance*, the base $=96$ will be her *departure*, and the perpendicular will be her *difference of latitude*; and the same in all cases of plain sailing.

[q] Problems similar to this and the following one, are given by Ludlam, to shew how the apparent ambiguity of a problem is sometimes corrected by the wording; particular attention must be paid to '*bending a little to the right*' in prob. 17. and '*attempts to return*' in prob. 18. and the solution will be attended with no difficulty.

to return, but a thick fog coming on, he mistakes his way, and takes a road which *tends a little to the right hand* of his proposed rout; arriving at C, 3 miles from B, he discovers his mistake, and the fog clearing up, he sees both A and B under an angle of $154°\frac{1}{7}$; how far is he distant from home? *Ans.* 2.38 *miles.*

19. In order to measure the breadth of a harbour's mouth, a station was taken at its inner extremity, where the angle made by the two projecting points which form the harbour was observed, viz. 33° 40′; the line bisecting this angle being produced 1200 yards backward and another observation made, the fore-mentioned points were found to subtend an angle of 17° 30′; required the breadth of the said entrance, and how far the harbour extends inland? *Ans. breadth* 751.904 *yards, perp. extent inland* 1242.6 *yards.*

20. Three trees are planted in such a manner that the angle at A is double the angle at B, and the angle at B double that at C, and a line of 234 yards will just reach round them; required their respective distances? [r] *Ans.* $AB=46.3465$ *yards,* $AC=83.5135$ *yards,* $BC=104.14$ *yards.*

21. In order to determine the distance between two inaccessible batteries A and B, two stations X and Z were chosen, distant from each other 4541.8 yards; at X the following angles were taken, viz. $AXB=14°\ 34'$; $BXZ=46°\ 16'$; at Z the angles were $XZA=96°\ 44'$, $XZB=115°\ 23'$; required the distance of the batteries from each other? *Ans.* 3373.1 *yards.*

22. Two ships leave a port together, A steers S W; B steers S S E, and sails twice as fast as A: at the end of f[illegible]ays they arrive at ports 558 miles apart; now, supposing [illegible]ind to have blown equally from one point during the wh[illegible] time; at what rate per hour did the ships run? [s] *Ans.* A 3.1[illegible]5 *miles per hour,* B 6.243.

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[r] If $x=$ the least angle, viz. C; then $2x=B$, and $4x=A$, whence $7x=180$, and $x=\frac{180}{7}=25°\ 42'\frac{6}{7}$. Assume either of the sides of any convenient length, and find (by Art. 67.) the two remaining sides; then say, as the sum of these three sides : to the given sum 234 : : either of the sides : the corresponding side of the proposed triangle.

[s] From any point draw two indefinite lines in the proposed directions, from the table (Art. 87.) Assume any length in the S W line for A's distance, and take double that length in the other line for B's; join these points by a straight line, and find its length (Art. 72, 69, and 67.); then say, as this line :

23. From one of the angles of a rectangular mea
are two straight foot paths, one leading to the oppc
and the other to a stile 110 yards distant from it; th
with the two paths, forms a triangle, of which the
as the numbers 2, 3, and 10; what sum will pay for th
making, and carting of the said meadow at 27*s.* 6*d.*
Ans. 7*l.* 8*s.* 2¼*d.*

24. There are three seaport towns *A*, *B*, and (
E S E, and *C*, E by N from *A*; a telegraph is erected, for the purpose of speedy communication with the metropolis, at 5 miles distance from each of the towns, and in the line *AC*; required the distance of *B* from *A* and *C*, and its bearings from the telegraph? *Ans. from B to A* 8.3147 *miles, from B to C* 5.5557 *miles; and B bears* S E b S *from the telegraph.*

25. A flag-staff is placed on a castle wall 163 feet long, in such a situation that a line of 100 feet in length will reach from its top to one end of the wall, and a line of 89 feet from its top to the other; required the height of the flag-staff, and its distance from the extremities of the wall? *Ans. height* 47.7244 *feet; distance from one extremity* 87.8773 *feet, from the other* 75.1227 *feet.*

26. In the hedge of a circular inclosure 500 yards in diameter three trees *A*, *B*, and *C* were planted in such a manner, that if straight lines be drawn from each to the other two, the angle at *A* will be double the angle at *B*, and the angle at *C* double of *A* and *B* together; required the distance between every two of the trees[1]? *Ans. from A to B* 433.013 *yards, from B to C* 321.394 *yards, and from A to C* 171.01 *yards.*

A's assumed distance : : 558 : *A's real* distance; whence also *B's* distance will be found; and the distance divided by the number of hours, will give the rate of sailing per hour.

[1] To find the angles, see the note on prob. 20. To find the sides; *First*, with the radius 250 describe a circle, and from it cut off a segment containing an angle equal to the greatest angle of the proposed triangle (34. 3.), draw straight lines from the extremities of this chord to the centre, and an isosceles triangle will be formed by these three lines, of which the vertical angle (at the centre) will be double the supplement of the said greatest angle (20 and 22. 3.), and the three angles of this isosceles triangle will be known (32. 1.). *Secondly*, find the base (Art. 67.) which will be the greatest side of the proposed triangle (19. 1.), whence the two remaining sides will likewise be found by Art. 67.

27. An English sloop of war having orders to survey an enemy's port, placed two boats A and B at 1100 fathoms distance apart, A being directly east from B: at the inner extremity of the harbour there is a spire visible from the boats, likewise a castle on one point of the entrance, and a light-house on the other; at A the castle bore S S W, the spire S W by S, and the light-house W S W. At B the castle bore S E, the spire south, and the light-house S by W; required the length and breadth of the harbour? *Ans. length from middle of entrance* 1056 *fathoms; breadth of entrance* 920.59 *fathoms.*

28. On the opposite sides of an impassible wood, two cities A and B are situated; C is a town visible from A and B, distant from the former 3 miles, and from the latter 2, and they make at C an angle of 28°; now, it is desirable to cut a passage from A to B, and an engineer undertakes to make one, 19 feet wide, at 7*s.* 6*d.* per square yard; the inhabitants of A agree to furnish $\frac{4}{5}$ of the expense, which they can accomplish, by every 7 persons paying 31 shillings; those of B can make up the remainder, by every six persons subscribing 33 shillings; required the number of inhabitants in A and B? *Ans.* A 43626, B 8839, *to the nearest unit.*

29. An isosceles triangle has each of the angles at the base double that at the vertex; now, if the vertical angle be bisected, and either of the angles at the base trisected, the segment of the trisecting line, intercepted between the opposite side and the bisecting line, will be three inches; required the sides of the triangle? *Ans. each of the equal sides* 13.8314 *inches; the base* 8.35371 *inches.*

30. In a circle, whose radius is 5, a triangle is inscribed, and the perpendiculars from the centre of the circle to the sides of the triangle are as 1, 3, and 4; required the sides and angles of the triangle?

31. The altitude of a balloon as seen from A was 47°, and its bearings S E; from B, which is $2\frac{1}{4}$ miles south of A, it bore N E b N; required the perpendicular height of the balloon, and its distance from B?

PART X.

THE CONIC SECTIONS.

HISTORICAL INTRODUCTION.

If a solid be cut into two parts by a plane passing through it, the surface made in the solid by the cutting plane, is called A SECTION.

If a fixed point be taken above a plane, and one of the extremities of a straight line passing through it be made to describe a circle on the plane, then will the segments of this line by their revolution, describe two solids (one on each side of the fixed point) which are called OPPOSITE CONES [a].

A plane may be made to cut a cone five ways; *first*, by passing through the vertex and the base; *secondly*, by passing through the cone parallel to the base; *thirdly*, by passing through it parallel to its sides; *fourthly*, by passing through the side of the cone and the base, so as likewise to cut the opposite cone; and *fifthly*, so as to cut its opposite sides in unequal angles [b], or in a position not parallel to the base.

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[a] If the segment of the generating line between the fixed point and the base be of a given length, the cone described by its motion will be A RIGHT CONE, having its axis perpendicular to the base; but if the length of the segment be variable in any given ratio, so as to become in one revolution *a maximum* and *a minimum*, the cone produced will be AN OBLIQUE CONE, and its axis will make an oblique angle with the base.

[b] Of course a right cone is here understood; for if the cone be oblique, the base, which is *a circle*, will cut the opposite sides in *unequal angles*, and the segment made by cutting them in *equal angles* will evidently be an ellipse.

If the plane pass through the vertex and the base, the section is a triangle; if it be parallel to the base, the section is a circle; if parallel to the side of the cone, the section is called A PARABOLA; if the plane pass through the side and cut the opposite cone, the section is called AN HYPERBOLA; and if it cut the opposite sides of the cone at unequal angles, the section is called AN ELLIPSE.

The triangle and circle pertain to common elementary Geometry, and are treated of in the Elements of Euclid; the parabola, the ellipse, and the hyperbola, are the three figures which are denominated THE CONIC SECTIONS.

There are three ways in which these curves may be conceived to arise, from each of which their properties may be satisfactorily determined; *first*, by the section of a cone by a plane, as above described, which is the genuine method of the ancients; *secondly*, by algebraic equations, wherein their chief properties are exhibited, and from whence their other properties are easily deduced, according to the methods of Fermat, Des Cartes, Roberval, Schooten, Sir Isaac Newton, and others of the moderns; *thirdly*, these curves may be described on a plane by local motion, and their properties determined as in other plane figures from their definition, and the principles of their construction. This method is employed in the following pages.

WHEN, or from whom the ancient Greek geometricians first acquired a knowledge of the nature and properties of the cone and its sections, we are not fully informed, although there is every reason to suppose that the discovery owes its origin to that inventive genius, and indefatigable application to science, which distinguished that learned people above all the other nations of antiquity. Some

of the most remarkable properties of these curves were in all probability known to the Greeks as early as the fifth century before Christ, as the study of them appears to have been cultivated (perhaps not as a new subject) in the time of Plato, A. C. 390. We are indeed told, that until his time the conic sections were not introduced into Geometry, and to him the honour of incorporating them with that science is usually ascribed. We have nothing remaining of his expressly on the subject, the early history of which, in common with that of almost every other branch of science, is involved in impenetrable obscurity.

The first writer on this branch of Geometry, of whom we have any certain account, was Aristæus, the disciple and friend of Plato, A. C. 380. He wrote, a treatise consisting of five books, on the Conic Sections; but unfortunately this work, which is said to have been much valued by the ancients, has not descended to us. Menechmus, by means of the intersections of these curves (which appears to have been the earliest instance of the kind) shewed the method of finding two mean proportionals, and thence the duplication of the cube; others applied the same theory, with equal success, to the trisection of an angle; these curious and difficult problems were attempted by almost every geometrician of this period, but the solution (as we have remarked in another place) has never yet been effected by pure elementary Geometry. Archytas, Eudoxus, Philolaus, Denostratus, and many others, chiefly of the Platonic school, penetrated deeply into this branch, and carried it to an amazing extent; succeeding geometers enriched it by the addition of several other curves as the cycloid, cissoid, conchoid, quadratrix, spiral, &c. the whole forming a branch of science justly considered by the ancients

as possessing a more elevated nature than common Geometry, and on this account they distinguished it by the name of THE HIGHER OR SUBLIME GEOMETRY.

Euclid of Alexandria, the celebrated author of the Elements, A. C. 280, wrote four books on the Conic Sections, as we learn from Pappus and Proclus; but the work has not descended to modern times. Archimedes was profoundly skilled in every part of science, especially Geometry, which he valued above every other pursuit; it appears that he wrote a work which is lost, expressly on the subject we are considering, and his writings which remain respecting spiral lines, conoids, and spheroids, the quadrature of the parabola, &c. are sufficient proofs that he was deeply skilled in the theory of the Conic Sections. In his tract on the parabola he has proved by two ingenious methods, that the area of the parabola is two-thirds that of its circumscribing rectangle; which is said to be the earliest instance on record of the absolute and rigorous quadrature of a space included between right lines and a curve. But the most perfect work of the kind among the ancients is a treatise originally consisting of eight books by Apollonius Pergæus of the Alexandrian School, A. C. 230. The first four only of these, have descended to us in their original Greek, the fifth, sixth, and seventh, in an Arabic version; the eighth has not been found, but Dr. Halley has supplied an eighth book in his edition, printed at Oxford, in 1710.

This excellent treatise is the most ancient work in our possession, on the subject; it supplied a model for the earliest writers among the moderns, and still maintains its classical authority: the improvements on the system of Apollonius by modern geometricians are comparatively few, except such as depend on the application of Algebra

and the Newtonian Analysis. Hitherto the ancients had admitted the right cone only (of which the axis is perpendicular to the base) into their Geometry; they supposed all the three sections to be made by a plane cutting the cone at right angles to its side. According to this method, if the cone be right angled (def. 18. 11.), the section will be a parabola; if acute angled, the section will be an ellipse; and if obtuse angled, an hyperbola; hence they named the parabola, *The section of a right angled cone*; the ellipse, *The section of an acute angled cone*; and the hyperbola, *The section of an obtuse angled cone*. But Apollonius first shewed that the three sections depend only on the different inclinations of the cutting plane, and may all be obtained from the same cone, whether it be right or oblique, and whether the angle of its vertex be right, acute, or obtuse. Pappus of Alexandria, who flourished in the fourth century after Christ, wrote valuable lemmata and observations on the writings of Apollonius, particularly on the conics, which are to be found in the seventh book of his *Mathematical Collections*: and Eutocius, who lived about a century later, composed an elaborate commentary on several of the propositions.

In 1522 John Werner published, at Nuremberg, some tracts on the subject; and about the same time Franciscus Maurolycus, Abbot of St. Maria del Porta, in Sicily, published a treatise on the Conic Sections, which has been highly spoken of by some of our best geometers for its perspicuity and elegance. The application of Algebra to Geometry, first generally introduced by Vieta, and afterwards improved and extended by Des Cartes, Fermat, Torricellius, and others, furnished means for the further developement of the nature and properties of curves. The indivisibles of Roberval and Cavalerius;

the *Arithmetic of Infinites*, by Dr. Wallis; the *Theory of Evolutes*, by Huygens; the *Method of Tangents*, by Dr. Barrow, &c. were discoveries which supplied additional means for extending the theory or facilitating the several applications of the doctrine; but that which rendered the most complete and essential service to this department of science, was the discovery of the method of Fluxions by Sir Isaac Newton, which took place about the year 1668.

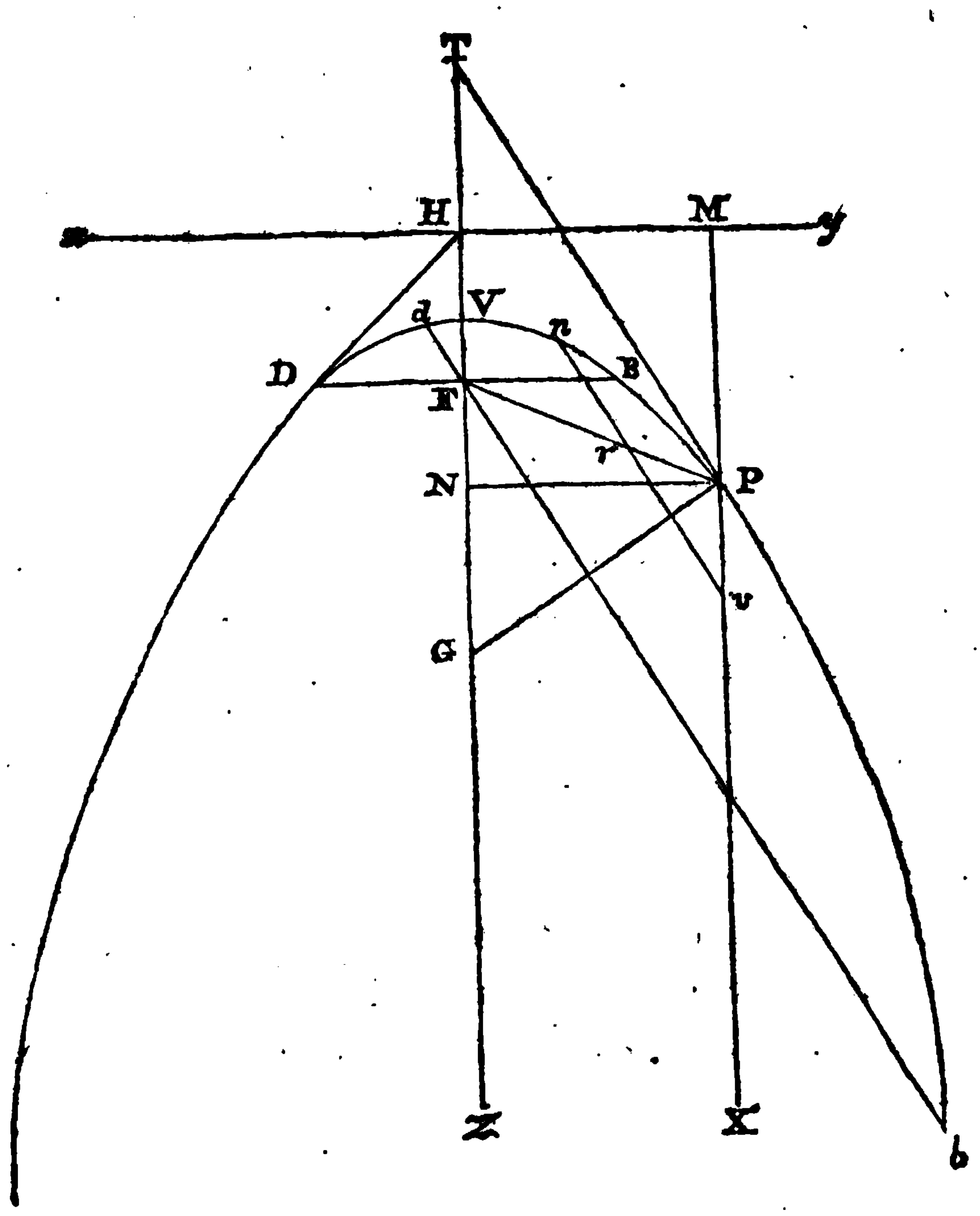
The principal modern writers on the Conic Sections are, Mydorgius, Trevigar, Gregory St. Vincent, De Witte, De la Hire, De l' Hôpital, Dr. Wallis, Milne, Dr. Simson, Emerson, Muller, Steel, Jack, Dr. Robertson, &c. *The Properties of the Conic Sections*, by William Jones, Esq. F.R S. published by Mr. John Robertson, in 1774, is a tract in which is comprised a very great number of properties deduced in a most compendious and general manner, within the narrow compass of 24 pages. Dr. Hamilton's Conic Sections is a very elegant and ample work; Dr. Hutton's treatise on the subject will be found easy and useful. The introductory tracts of the Rev. Messrs. Vince and Peacock are the shortest and plainest elementary pieces which have been put into the hands of students; on the plan of these (especially the latter) the following compendium was drawn up, in which it is hoped there will be found some improvements. A course of Lectures on the Conic Sections has lately been published by the Rev. Mr. Bridge, of the East India College. I have not seen the work, and therefore cannot speak of it, but the talents of the author are well known.

THE PARABOLA.

DEFINITIONS.

1. LET xy be any straight line, and MP a straight line moving parallel to itself at right angles to xy; and if another straight line FP revolve about F, so that FP be always equal to MP, the point P will trace out the curve $DVPb$, which is called A PARABOLA.

2. The straight line xy is called THE DIRECTRIX, and the point F THE FOCUS.



3. If through the focus F, a straight line HZ be drawn perpendicular to the directrix xy, cutting the parabola in V, VZ is called THE AXIS of the parabola, and V, THE VERTEX.

Cor. Hence, because FP is always $= PM$ (Art. 1.), when P by its motion arrives at V, FP becomes FV, and PM becomes VH, $\because FV = VH$.

4. A straight line drawn through the focus F, perpendicular to the axis VZ, and meeting the curve both ways, is called THE LATUS RECTUM, OR PRINCIPAL PARAMETER. *Thus DB is the latus rectum.* In some of the following articles, the latus rectum is denoted by the letter L.

5. Any straight line perpendicular to the axis VZ, meeting the curve, is called AN ORDINATE TO THE AXIS; and the part of the axis intercepted between the vertex V and any ordinate, is called THE ABSCISSA. *Thus NP is an ordinate to the axis, and NV its corresponding abscissa.*

6. A straight line meeting the curve in any point, and which being produced does not cut it, is called A TANGENT to the parabola at that point. *Thus PT is a tangent at the point P.*

7. A tangent drawn from the extremity of the latus rectum, is called THE FOCAL TANGENT. *Thus DH is the focal tangent.*

8. If an ordinate and a tangent be drawn from the same point in the curve, that part of the axis produced, which is intercepted between their extremities, is called THE SUB-TANGENT. *Thus P being any point from whence the tangent PT and the ordinate PN are drawn, NT is the sub-tangent to the point P.*

9. A straight line drawn perpendicular to the tangent from the point of contact, and meeting the axis, is called THE NORMAL. *Thus PG is the normal to the point P.*

10. If a normal and an ordinate be drawn to the same point in the curve, that part of the axis intercepted between them, is called THE SUB-NORMAL. *Thus NG is the sub-normal to the point P.*

11. A straight line drawn from any point in the curve, parallel to the axis, is called A DIAMETER to that point; and the point in which it meets the curve, is called THE VERTEX TO THAT DIAMETER. *Thus PX is a diameter to the point P, and P is its vertex.*

12. A straight line drawn through the focus F, parallel to the tangent at any point, and terminated both ways by the curve, is called THE PARAMETER TO THE DIAMETER of which that point is the vertex. *Thus db is the parameter to the diameter PX.*

13. A straight line drawn from any diameter, parallel to a tangent at its vertex, and meeting the curve, is called AN ORDINATE to that diameter. *Thus vn is an ordinate to the diameter PX.*

PROPERTIES OF THE PARABOLA [b].

14. The straight line FP, drawn from the focus F, to any point P in the curve, is equal to the sum of the segments VF and VN of the axis intercepted between the vertex and the focus, and between the vertex and the ordinate; that is, $FP=VN+VF$.

For $FP=PM$ (Art. 1.) $=HN$ (34. 1.) $=VN+VH=$ (cor. Art. 3.) $VN+VF$. Q. E. D.

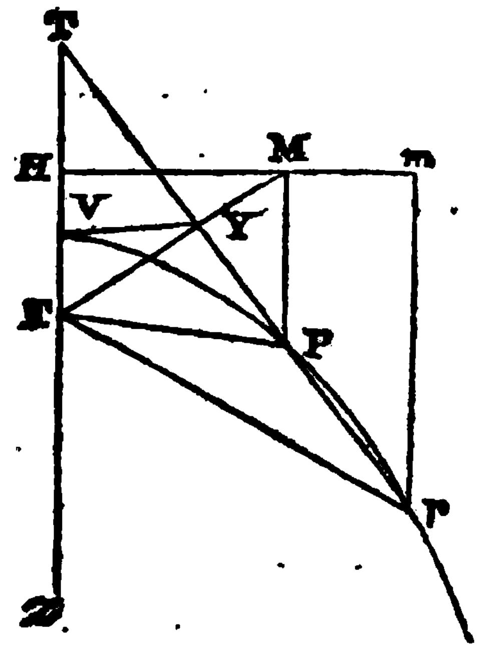
Cor. 1. Hence, when P coincides with B, N will coincide with F, VN will become VF, and FP will become FB; $\because$ $FB=2VF$, and $DB=4VF$, or the latus rectum is equal to four times the distance of the focus from the vertex.

Cor. 2. Hence $FP-FN=FB=$half the latus rectum, for $FP=(VF+VN=)\ 2VF+FN$; $\because$ $FP-FN=2VF=FB$.

15. The straight line PT, which bisects the angle FPM, is a tangent to the parabola at P. *See the following figure.*

For if not, let it cut the curve in P and p, join Fp, FM, pM; draw pm perpendicular to HM, and join FM cutting PT in Y. Then in the triangles FPY, MPY, $FP=MP$ (Art. 1.), PY is common, and the angle $FPY=MPY$ (by hypothesis), $\because$

[b] It will be proper to inform the student before he begins to study the Conic Sections, that he ought to be *thoroughly master* of the first six books of Euclid, and to know something of the eleventh and twelfth; the doctrine of proportion, as delivered in part 4. page 49 to 82 of this volume must likewise be well understood, as its application continually occurs in the following pages.



$FY = MY$, and the angle $FYP = MYP$ (4. 1.) $\therefore$ in the triangles FYp, MYp, the two sides FY, $Yp = MY$, Yp, and the included angles $FYp = MYp$, $\therefore$ $Fp = pM$ (4. 1.); but $Fp = pm$ (Art. 1.), $\therefore$ $pM = pm$; $\therefore$ the angle $pmM = pMm$ (5. 1.), but pmM is a right angle (Art. 1.), $\therefore$ pMm is also a right angle, which is impossible (17. 1.); $\therefore$ PT does not cut the parabola, consequently it is a tangent (Art. 6.) Q. E. D.

16. The tangent VY at the vertex V, is perpendicular to the axis VZ.

For since the tangent PT cuts FM at right angles in whatever point of the curve P be taken (Art. 15.), $\therefore$ when P coincides with V, FP will coincide with FV, PM with VH, and FM with FH; $\therefore$ the tangent VY is perpendicular to (FM, that is, to) FH. Q. E. D.

Cor. 1. Hence, because TF and MP are parallel $FTP = TPM$ (29. 1.) $= FPT$ (Art. 15.), $\therefore$ $FT = FP$.

Cor. 2. Hence FM, VY, and PT intersect each other in the point Y. For $FY = YM$, and (cor. Art. 3.) $FV = VH$, $\therefore$ (2. 6.) VY is parallel to HM, and consequently perpendicular to TZ; $\therefore$ VY is a tangent at V.

17. The focal tangent DH, the directrix xy, and the axis TZ, intersect each other in the point H. (See the figure to Art. 3.)

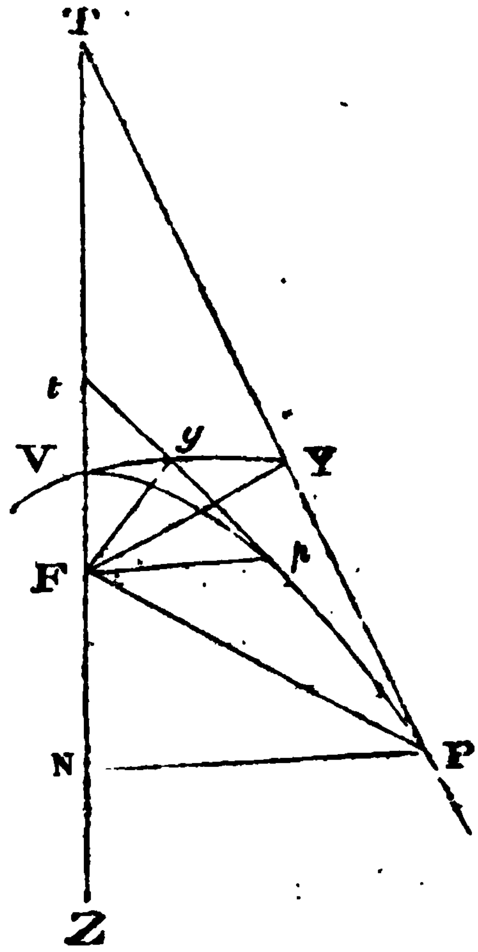
For $FC = FH$ (Art. 14.), $\therefore$ by the preceding corollary, the tangent meets the axis at the point H, where the axis and directrix intersect. Q. E. D.

18. If xv be an ordinate to the diameter PX cutting FP in r, (see the figure to Art. 3.) $Pr = Pv$: for $Prv = rPT$ (29. 1.) $= TPM$ (Art. 15.) $= Pvr$ (29. 1.) $\therefore$ $Pr = Pv$ (6. 1.). Q. E. D.

19. The straight line FY is a mean proportional between FP and FV. *See the figure to Art.* 15.

For since FYT is a right angle (Art. 15.) and YV perpendicular to FT, $TF : FY :: FY : FV$ (cor. 8. 6.), but $TF = FP$ (cor. Art. 16.), $\therefore$ $FP : FY :: FY : FV$. Q. E. D.

Cor. Hence $FP : FV :: FP^2 : FY^2$ (cor. 1, 20. 6.), consequently $FY^2 = FP.FV$ (16. 6.), and $4FY^2 = 4FV.FP$; but $4FV =$ the latus rectum (Art. 14. cor.) which being denoted by L, we have $4FY^2 = L.FP$.

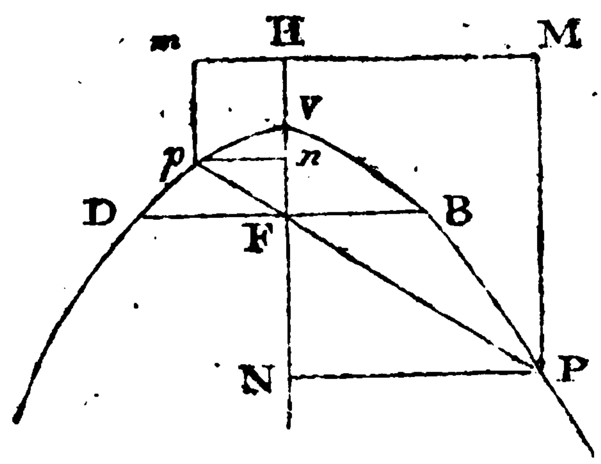


20. The line FP varies as FY^2.

For, let P and p be two points in the curve, from whence the tangents PT, pt are drawn, and let FY and Fy be perpendicular to the tangents respectively. Then, because $FY^2 = FP.FV$, and $Fy^2 = Fp.FV$ (by the preceding cor.) $\because FY^2 : Fy^2 :: (FP.FV : Fp.FV ::$ by 1. 6.) $FP : Fp$, $\therefore FP \propto FY^2$. Q. E. D.

Note. The figure to this Art. is inaccurately cut; VY must be understood as a *straight* line at right angles to TZ.

21. If PF be produced through F and meet the curve again in p, then will $4FP.Fp = L.\overline{FP+Fp}$.



For $FP - FB = PM - FH = NH - FH = FN$. And $FB - Fp = FH - pm = FH - Hn = Fn$, $\because FP - FB : FB - Fp :: FN : Fn ::$ (4. 6.) $FP : Fp$, $\because$ (16. 6.) $FP.Fp - FB.Fp = FP.FB - FP.Fp$; or $2FP.Fp = FP.FB + FB.Fp = FB.\overline{FP+Fp}$ $\because$ (since $2FB = L$ by Art. 4.) $4FP.Fp = L.\overline{FP+Fp}$.

Cor. Hence, if $4a = L$, $X = FP$, and $x = Fp$, the last expression will become $4Xx = 4a.\overline{X+x}$, or $Xx = aX + ax$, $\because \dfrac{1}{a} = \dfrac{1}{x} + \dfrac{1}{X}$.

22. If c be the co-sine of the angle VFP to radius 1, then will $FP = \dfrac{2VF}{1-c}$.

For $FP = VN + VF$ (Art. 14.) $= VF \pm FN + VF = 2VF \pm FN$

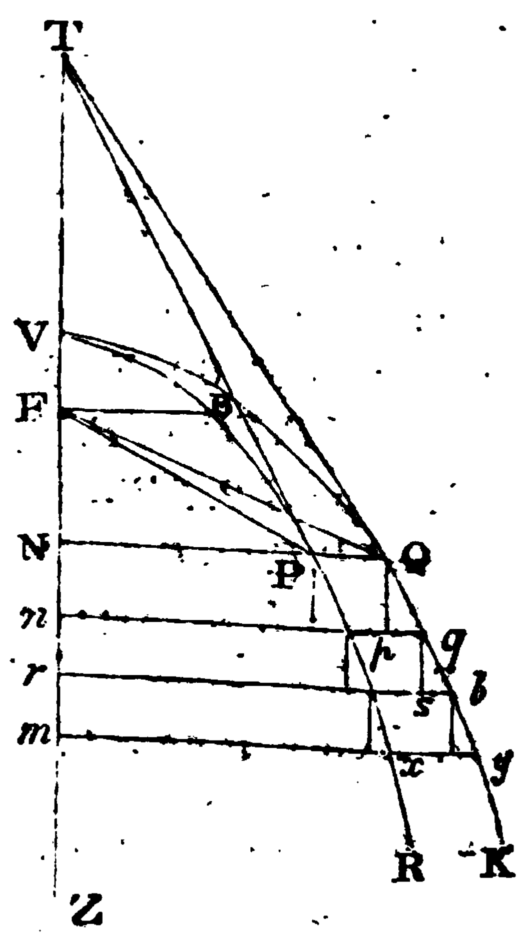
But $\pm FN : FP ::$ (sin FPN : radius :: cos PFN : radius :: cos VFP the supp. of PFN : radius ::) $+c : 1$ by Art. 63. part 9. ∴ (16. 6.) $\pm FN = c.FP$; ∴ $FP = (VN + \overline{VF = 2VF + FN}$ (Art. 14.) $=)$ $2VF + c.FP$; ∴ ($FP - c.FP$, or) $\overline{1-c}.FP = 2VF$, or $FP = \dfrac{2VF}{1-c}$. Q. E. D.

23. The sub-tangent $NT = 2VN$. *See the figure to Art.* 20.

Let VY be a tangent at V meeting PT in Y, then FY being perpendicular to PT (Art. 15.), and $FP = FT$ (cor. Art. 16.); also FY common, to the two triangles FPY, FTY, these triangles are similar and equal (47 and 4. 1.), ∴ $PY = YT$. But VY is perpendicular to the axis VZ (Art. 16.), ∴ it is parallel to the ordinate NP; ∴ $PY : YT :: VN : VT$ (2. 6.); but $PY = YT$ ∴ $VN = VT$ (prop. *A*.5.); ∴ the sub-tangent $NT = 2VN$. Q. E. D.

24. If Rnv be parallel to the tangent PT, and vM perpendicular to the axis VZ, (see the figure to Art. 30.), then $RM = 2VN$; for the triangles TNP, RMv being equiangular (29. 1.) $TN : NP :: RM : Mv$ (4. 6.). But $NP = Mv$ (34. 1.) ∴ $RM = TN$ (14. 5.) $= 2VN$ (Art. 23.) Q. E. D.

25. If two parabolas VR and VK be described on the same axis VZ, and the ordinate NQ meet VR, VK in P and Q, then will the tangents at P and Q intersect the axis VZ produced in the same point T; for VN is the common abscissa to the ordinates NP, NQ of both parabolas, and $NT = 2VN$ in both (Art. 23) Q. E. D.



26. The square of the ordinate is equal to the rectangle contained by the latus rectum and abscissa, or $PN^2 = L.VN$.

For $FP = VN + VF$ (Art. 14.) ∴ $FP^2 = VN^2 + VF^2 + 2VF.VN$ (4. 2.). But $VN^2 + VF^2 = 2VF.VN + FN^2$ (7. 2.), ∴ $FP^2 = 2VF.VN + FN^2 + 2VF.VN = 4VF.VN + FN^2$. But $FP^2 = PN^2 + FN^2$ (47. 1.), ∴ $PN^2 + FN^2 = 4VF.VN + FN^2$; ∴ $PN^2 = 4VF.VN =$ (cor. 1. Art. 14.) $L.VN$. Q. E. D.

Cor. Hence, if any ordinate $PN=y$, its abscissa $VN=x$, and the latus rectum$=4a$, the expression $PN^2=L.VN$ will become $y^2=4ax$; which is the equation of the parabola, considered as a geometrical curve.

27. The abscissa varies as the square of the ordinate.

Let PN and pn be any two ordinates to the axis VZ; then because $PN^2=L.VN$, and $pn^2=L.Vn$ (Art. 26.), $PN^2 : pn^2 :: L.VN : L.Vn ::$ (15.5.) $VN : Vn$, $\because$ (Art. 97. part 4.) $VN \propto PN^2$. Q. E. D.

28. If two parabolas VR and VK be described on the same axis VZ, and the ordinate NQ meets VP in P; then will PN and QN have to one another a given ratio.

Produce np to q, then (Art. 27.) $PN^2 : pn^2 :: VN : Vn :: QN^2 : qn^2$; $\because$ (22. 6.) $PN : pn :: QN : qn$, and (16. 5.) $PN : QN :: pn : qn$. Q. E. D.

29. The area VNP : the area $VNQ :: PN : QN$.

For, let the abscissa VZ be divided into the equal parts Nn nr, rm, &c. and complete the parallelograms Pn, Qn, pr, qr, sm, tm, &c. these having equal altitudes (Nn, nr, rm, &c.) are to one another as their bases (1. 6.).

$\because Pn : Qn :: NP : NQ$

$pr : qr :: np : nq ::$ (Art. 28.) $NP : NQ$

$sm : bm :: rs : rb ::$ (Art. 28.) $NP : NQ$

$\because$ (12. 5.) $Pn+pr+sm+$ &c. : $Qn+qr+bm+$ &c. $:: NP : NQ$ (15. 5.). Wherefore, if the magnitude of the parts Nn, nr, rm, &c. be diminished, and their number increased indefinitely, the sum of all the parallelograms between V and mx will approximate indefinitely near to the area of the curvilineal space Vxm; as the sum of the parallelograms between V and ym will, to the curvilineal space Vym; $\because$ the area $VPxm$: the area $VQym$ :: $NP : NQ$. Q. E. D.

Cor. Hence, if from any point F in the axis, straight lines FP, FQ be drawn, the curvilineal area VFP : the curvilineal area $VFQ :: NP : NQ$.

For the triangle $FPN : FQN :: NP : NQ$ (1. 6.)

And $VPN : VQN :: NP : NQ$ (as shewn above.)

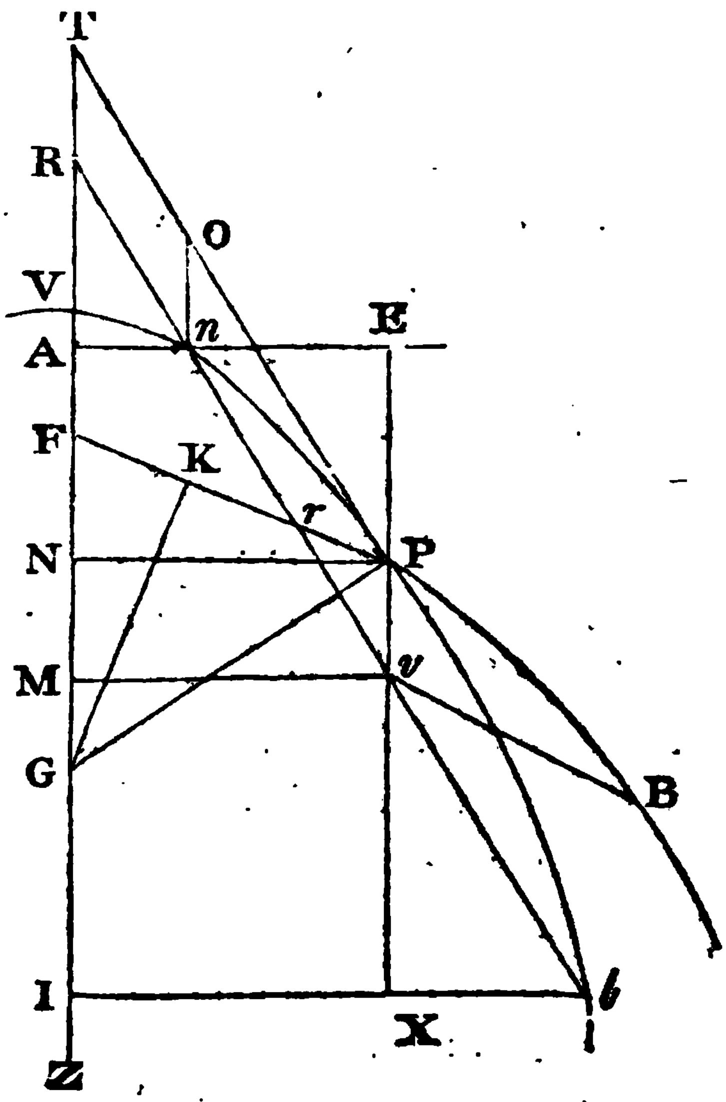
Also $VPN : FPN :: VQN : FQN$ (11. 5.)

$\because VPN-FPN : VQN-FQN : NP : NQ$ (19. 5.).

That is, the area VFP : the area $VFQ :: NP : NQ$.

30. The sub-normal is equal to half the latus rectum, that is, $NG = \frac{1}{2}L$.

For TPG is a right angle (Art. 9. 10.), from which NP is drawn perpendicular to the base TG (Art. 5.), $\because TN : NP :: NP : NG$ (cor. 8. 6.); $\because TN.NG = NP^2$ (17. 6.) $= L.VN$ (Art. 26.); $\because TN : VN :: L : NG$ (16. 6.). But $TN = 2VN$ (Art. 23.), $\because L = 2NG$ (prop. D.5.), and their halves are equal, or $NG = \frac{1}{2}L$. Q. E. D.



Cor. 1. If from F as a centre with the distance $FT = FP$ a circle be described, it will pass through G; for T and P being in the circumference, and TPG a right angle, the point G will likewise be in the circumference (31. 3.); $\because FP = FG$, and the angle $FPG = FGP$ (5. 1.).

Cor. 2. Hence also the angle $TFP = FGP + FPG$ (32. 1.) $= 2FGP$.

31. If GK be drawn perpendicular to FP, then will $PK = \frac{1}{2}L$.

For the triangles PGK, PGN having PKG, PNG right angles, $GPK = PGN$ (cor. 1. Art. 30.), and PG common, are similar and equal (26. 1.); $\because PK = NG = \frac{1}{2}L$ (Art. 30.) Q. E. D.

32. If nv be an ordinate to the diameter PX, then will $nv^2 = 4FP.Pv$.

Because the triangles RAn, RMv are similar, $RM^2 : RA^2 :: (Mv^2 =) NP^2 : An^2$ (4. 6. and 22. 6.) $:: VN : VA$ (Art. 27.); but $RM^2 = \overline{RA + AM}|^2 = RA^2 + AM^2 + 2RA.AM$ (4. 2.), $\because$ (prop. E. 5.) $RM^2 : AM^2 + 2RA.AM :: VN : AN :: (2VN =) RM : 2AN :: RM^2 : 2RM.AN$ (15. 5.); $\because AM^2 + 2RA.AM = 2RM.AN$ (9. 5.), or $AM^2 = 2RM.AN - 2RA.AM$. But $RA = RM - AM$, and $AN = AM - MN$ let these values of RA and AN be substituted in the foregoing expression, and it will be-

come $AM^2 = 2RM.\overline{AM-MN} - \overline{2RM-AM}.AM$ ($=2RM.AM - 2RM.MN - 2RM.AM + 2AM^2$) $= -2RM.MN + 2AM^2$, or $AM^2 = 2RM.MN$. But since $TR = Pv = MN$ (34. 1.), ∴ $RM = TN = 2VN$ (Art. 23.); ∴ the above expression $AM^2 = 2RM.MN = 4VN.MN = 4VN.Pv$.

Now $nv^2 : (vE^2 =) AM^2 :: Rv^2 : RM^2$ (4. 6. and 22. 6.) :: $RM^2 + Mv^2$ (47. 1) : RM^2 :: (since $RM = 2VN$, and $Mv^2 = NP^2 = 4VN.VF$ by Art. 26.) $4VN^2 + 4VN.VF : 4VN^2 :: \overline{4VN + 4VF}.VN : 4VN^2$ :: (Art. 14.) $4FP : 4VN :: 4FP.Pv : 4VN.Pv$; that is, $nv^2 : AM^2 :: 4FP.Pv : 4VN.Pv$; but it has been proved above that $AM^2 = 4VN.Pv$, ∴ (14. 5.) $nv^2 = 4FP.Pv$. Q. E. D.

And in like manner, if $RI - IM$ be substituted for RM, and *composition* be used instead of *conversion*, it may be shewn that $bv^2 = 4FP.Pv$; consequently $nv = bv$; that is, any diameter PX bisects its ordinates.

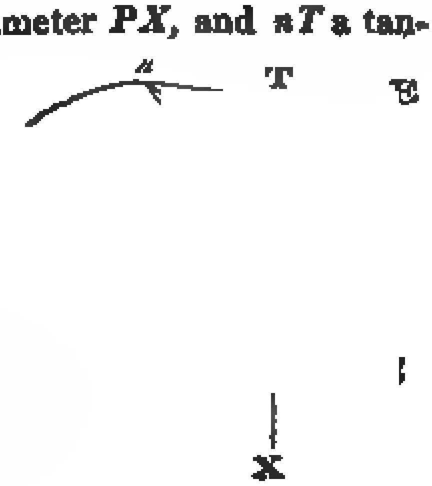
Cor. 1. Because $4FP.Pv = nv^2$, and FP is constant, ∴ $Pv \propto nv^2$, ∴ also $On \propto OP^2$.

Cor. 2. If from any point v in the diameter PX, ordinates vB be drawn cutting PX in a given angle, and having a given ratio to vb; the curve passing through all the points B will be a parabola. For $vb : vB$ being by hypothesis a given ratio, $vb^2 : vB^2$ is likewise given; but (cor. 1.) vb^2 ($= nv^2$) $\propto Pv$, ∴ $vB^2 \propto Pv$.

Cor. 3. Since $AM^2 = 4VN.Pv$, as shewn above, and $Pv = MN$ (34. 1.), ∴ $AM^2 = 4VN.NM$.

Cor. 4. Let P be the parameter to the diameter PX, then when nb passes through the focus F, it becomes the parameter (Art. 12.), and the point v coincides with F; ∴ $Pv = Pv = FP$ (Art. 17.), and because $nv^2 = 4FP.Pv$, ∴ $nb^2 = 4nv^2$ (4. 2.) $= 4 \times 4FP.Pv = 16FP^2$ (since $Pv = FP$), that is $P^2 = 16FP^2$, ∴ $P = 4FP$.

33. If nv be an ordinate to the diameter PX, and nT a tangent at n, the sub-tangent vT will be bisected by the vertex P.

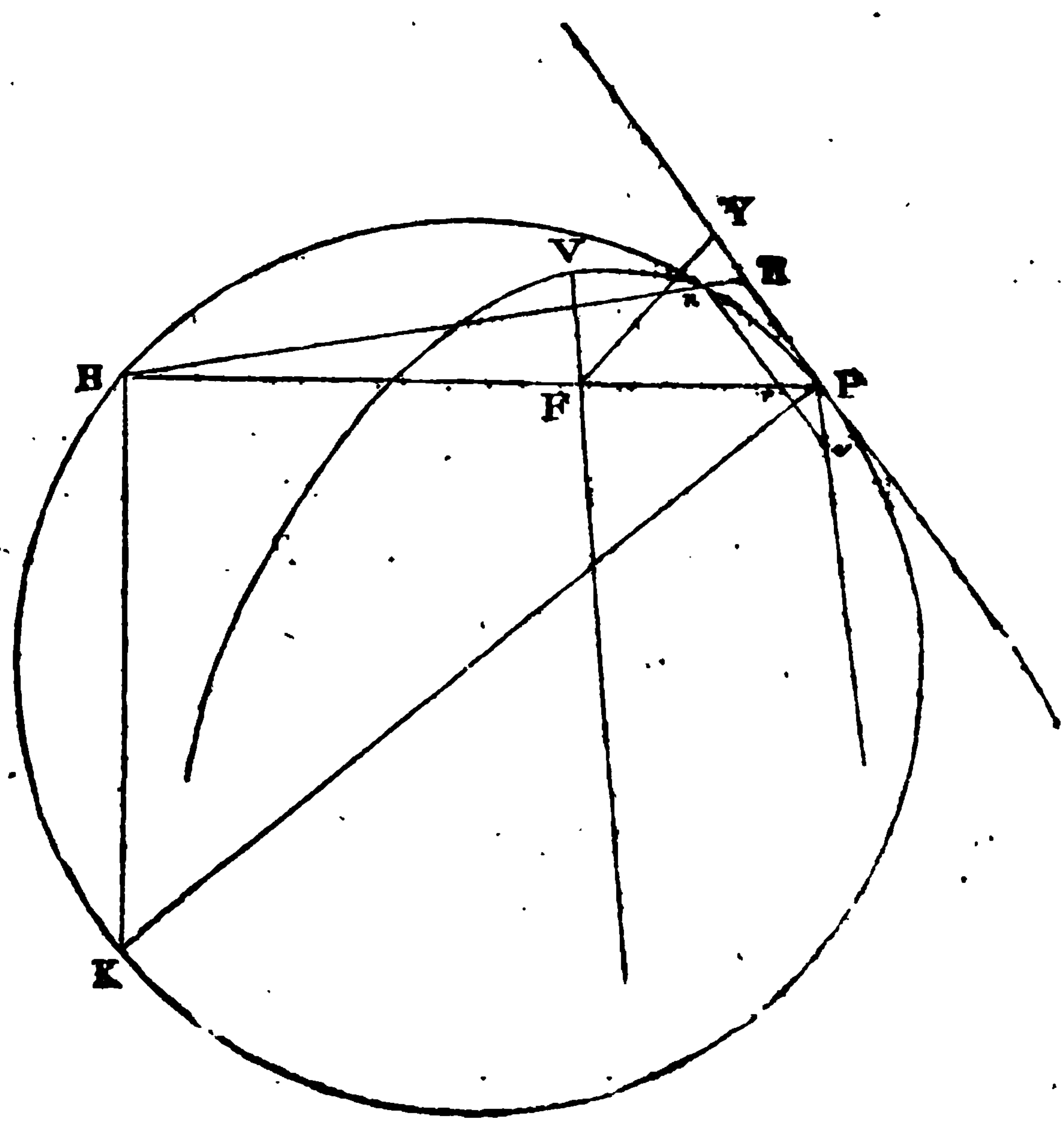


Produce nv to meet the curve in b, produce nT to E, and draw Eb parallel to TX. Then (cor. 1. Art. 32.) $PT : bE :: nT^2 : nE^2 ::$ (4. 6. and 22. 6.) $vT^2 : bE^2$; ∴ (16. 6.) $PT.bE^2 = bE.vT^2$, or $PT.bE = vT^2$, ∴ (17. 6.) $PT : vT :: vT :$

bE : : (4. 6. and 16. 5.) $nv : nb$: : (Art. 32.) $1 : 2$; that is, the sub-tangent vT is bisected in the point P. Q. E. D.

Cor. Hence, if bT be a tangent at b, the two tangents nT, bT and the diameter TX will intersect each other in the same point T; and in like manner, if other parabolas be described upon the diameter PX, by either increasing or decreasing the ordinate nv, or its inclination to the diameter, the tangents will all pass through the point T, as appears from the preceding demonstration.

34. If several circles be described upon as many diameters of different lengths, these circles will have different degrees of curvature, as is plain; and if the diameter be increased and decreased indefinitely, and circles be described from the same centre through every point of the increased or diminished diameter, these circles will possess all possible degrees of curvature. Hence it follows, that if a point be assumed in any curve, a circle may be found which will coincide with an indefinitely small portion of that curve at the assumed point, so that the curve and the circle will have the same tangent, and the same deflection from the tangent at that point; this circle is called THE CIRCLE OF CURVATURE to the proposed point.



35. If F be the focus of a parabola, and P any point in the curve, the chord of curvature to the point P which passes through F is equal to $4FP$.

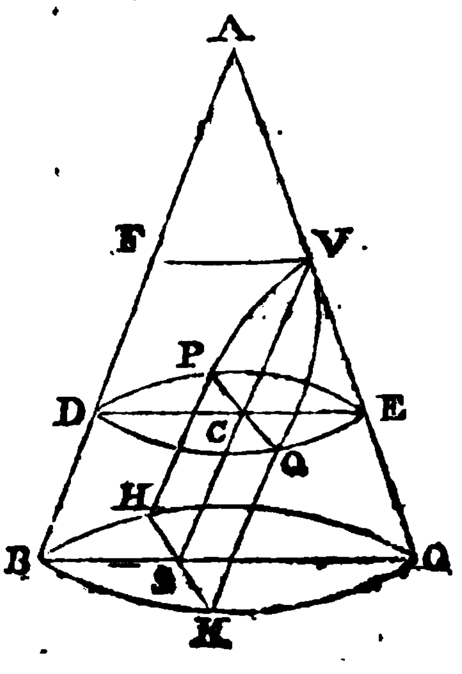
Let Pn be an indefinitely small *arc* of the parabola, coinciding with the circle of curvature PHK (Art. 34.); then the line nR may be considered as common to both; join nP, nH, produce the latter to R, and draw nv parallel to the tangent PY. Then since the angle $RPn = RHP$ (32. 3.), and nP is indefinitely near a coincidence with RP, the triangles PHn, PnR may be considered as equiangular, $\because PH : Pn :: Pn : nR$ (4. 6.) and (27. 6.) $Pn^2 = PH.nR$; but since the *arc* is in its nascent state (or indefinitely small) $Pn = nv$, $\because$ (nv^2 = by cor. Art. 19.) $4FP.Pv = Pn^2 = PH.nR$; but $nR = Pr =$ (Art. 18.) Pv, $\because 4FP.Pv = PH.Pv$, or $PH = 4FP$. Q. E. D.

Cor. 1. Hence, because $4FP$ = the parameter (cor. 4. Art. 32.), $\because$ the chord of curvature passing through the focus is equal to the parameter.

Cor. 2. If the diameter PK be drawn, HK joined, and FY drawn perpendicular to PY, the triangles PHK, PYF will be equiangular, since $YFP = HPK$ (29. 1.) and the angles at H and Y right angles (31. 3. and by construction) $\because FY : FP :: PH : PK ::$ (because $4FP = PH$) $4FP : PK$. Hence, if a tangent be drawn to any point in the parabola, and a perpendicular to the tangent, be drawn from the focus, the diameter of the circle of curvature to that point, will be readily determined.

36. If a cone be cut by a plane parallel to its side, the section will be a parabola.

Let ABO be a cone, and let the plane VHK pass through it; parallel to the side AB, the section $HPVQK$ will be a parabola.



Let the plane HVK be perpendicular to the plane BAO, the common section being VS; $PDQE$ a section of the cone parallel to the base, consequently a circle, PQ and DE its common sections with the fore-mentioned planes, and draw FV parallel to DE. $\because$ since the planes HVK, $PDQE$ are perpendicular to BAC, their common section PQ will be perpendicular to

BAO (19. 11.) and consequently to the lines DE, VS (def. 3. 11.); and because DE the diameter of the circle $PDQE$ cuts PQ at right angles, $PC=CQ$ (3. 3.), $\therefore DC.CE=PC^2$ (14. 2.) Now the triangles VCE, AFV being similar $VC : CE :: AF : FV$ (4. 6.) Let $AF : FV :: FV : L$ (11. 6.) $\therefore VC : CE :: FV : L$ (11. 5.); $\therefore VC.L=CE.FV$ (16. 6.) $=DC.CE$ (34. 1.) $=PC^2$, $\therefore$ (Art. 26.) HVK is a parabola of which PC is an ordinate to the axis, VC the correspondent abscissa, and L the latus rectum. Q. E. D.

THE ELLIPSE.

DEFINITIONS.

37. If two straight lines FP, SP intersecting each other in P, revolve about the fixed points F and S, so that $FP+SP$ be always the same, the point P will trace out the curve $PVKU$, which is called AN ELLIPSE.

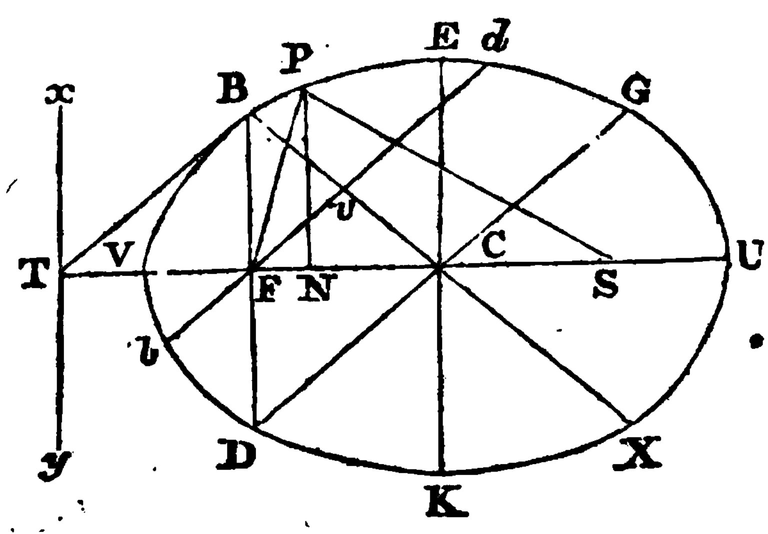
38 The points F and S about which FP and SP revolve, are called THE FOCI.

39. The straight line which joins the foci being produced both ways to the curve, is called THE MAJOR AXIS [c]. *Thus VU is the major axis.*

40. If the major axis VU be bisected in C, C is called THE CENTRE of the ellipse.

41. The straight line drawn through the centre perpendicular to the major axis, and terminated both ways by the curve, is called THE MINOR AXIS [d]. *Thus EK is the minor axis.*

42. Any straight line passing through the centre, and terminated both ways by the curve, is called A DIAMETER. *Thus BX is a diameter of the ellipse.*



[c] It is also named *The transverse axis.*

[d] It is likewise frequently named *The conjugate axis.*

43. The extremity of any diameter is called ITS VERTEX. *Thus V and U are the vertices of the major axis, E and K of the minor axis, and B and X of the diameter BX.*

44. A straight line drawn through the focus, perpendicular to the major axis, and terminated both ways by the curve, is called THE LATUS RECTUM OF PRINCIPAL PARAMETER. *Thus BD is the latus rectum.*

45. A straight line meeting the ellipse in any point, and which being produced does not cut it, is called A TANGENT to that point. *Thus BT is a tangent at the point B.*

46. The tangent to the point *B* or *D*, the extremity of the latus rectum, is called THE FOCAL TANGENT. *Thus BT is the focal tangent.*

47. The straight line drawn perpendicular to the major axis produced, through the point in which the focal tangent meets it, is called THE DIRECTRIX. *Thus xy is the directrix.*

48. Any strai ht line drawn from the curve, perpendicular to the major axis, is called AN ORDINATE to the axis. *Thus PN is an ordinate to the axis.*

49. The parts of the axis intercepted between its vertices and the ordinate, are called ABSCISSAS. *Thus VN and NU are abscissas to the ordinate PN.*

50. If from the vertex of any diameter a tangent be drawn, any straight line parallel to the tangent terminated by the diameter and the curve, is called AN ORDINATE to that diameter; and the intercepted parts of the diameter are called ABSCISSAS. *Thus dv is an ordinate to the diameter BX, and Bv, vX abscissas.*

51. If the ordinate pass through the centre, and meet the curve both ways, it is called THE CONJUGATE DIAMETER *; and if it pass through the focus, it is called THE PARAMETER to that diameter. *Thus DG is the conjugate diameter, and db the parameter, both to the diameter BX.*

PROPERTIES OF THE ELLIPSE.

52. The sum of the two straight lines drawn from the foci of an ellipse to any point in the curve, is equal to the major axis.

* And in general, if each of two diameters be parallel to the tangent at the vertex of the other, these diameters are called *conjugates* to each other.

The sub-tangent, normal, and sub-normal, are the same as in the parabola.

Thus, if P be any point in the curve, then $FP+PS=VU=2VC$. *See the preceding figure.*

For (Art. 37.) $FV+VS=FU+US$; that is, $2FV+FS=2US+FS$ $\because$ $2FV=2US$, and $FV=US$; and (Art. 37.) $FV+VS=FP+SP=VS+US=VU=2VC$. Q. E. D.

Cor. 1. Hence, because $FV+VS=2VC$; by adding VT to both, $ST+TF=2CT$; and by taking $2TF$ from this, $ST-TF=2CT-2TF=2CF$.

Cor. 2. Hence, because (Art. 40.) $CV=CU$, and $FV=US$ (as proved above) $\because$ $CV-FV=CU-US$, or $CF=CS$.

Cor. 3. Hence, $SP=VU-FP=2VC-FP$, and in like manner it appears that $FP=2VC-SP$.

Cor. 4. Hence, because $FP+SP=2VC$, by taking $2SP$ from both $FP-SP=2\,VC-2SP$, or (since $SP=2\,VC-FP$, by cor. 3.) $=2FP-2VC$.

53. The latus rectum is less than $4VF$; for $BF+BS=VU$ (Art. 37.) $=2VF+FS$ (Art. 52.); and since BS is greater than FS, BF must be less than $2VF$, and ($2BF=$) BD less than $4VF$. Q. E. D.

54. A straight line drawn from the focus to the vertex of the minor axis is equal to half the major axis, or $FE=VC$. *See the following figure.*

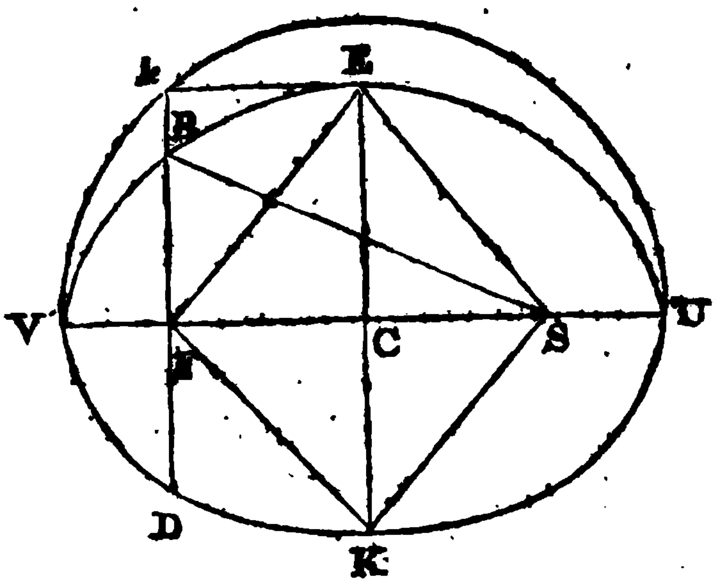
For since $FC=CS$ (cor. 1. Art. 52.) and CE common to the two triangles FCE, SCE (and the angles at C right angles (Art. 41.) $\because$ $FE=ES$ (4. 1.); but (Art. 37.) $FE+ES$, that is $2FE=VU=2VC$, $\because$ $FE=VC$. Q. E. D.

Cor. 1. And in like manner it may be shewn that $FK=KS=ES=EF=VC$, $\because$ in the triangles FEC, FKC, $FK=FE$, the angles at C are right angles, and the side FC is common, whence (26. 1.) $EC=CK$.

Cor. 2. Hence $EC^2=FE^2-FC^2$ (47. 1.) $=VC^2-FC^2=$ (cor. 5. 2.) $\overline{VC-FC}.\overline{VC+FC}=VF.FU$.

55. If on the major axis as a diameter a circle be described, and the latus rectum be produced to meet the circumference in k, then will $Fk=EC$. For (14. 2.) $Fk^2=VF.FU=$ (cor. 2. Art. 54.) EC^2, $\because$ $Fk=EC$.

56. The latus rectum is a third proportional to the major and minor axes, or $VU:EK::EK:BD$.

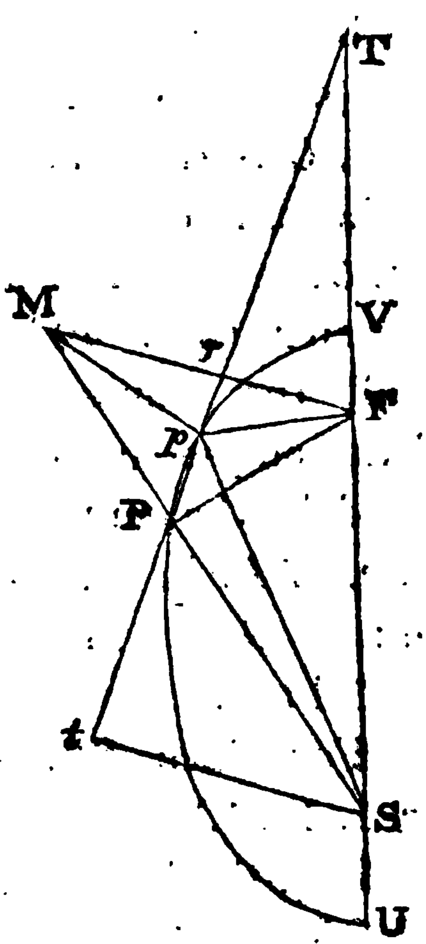


Because $BS = 2VC - BF$ (cor. 3. Art. 52.) ∵ $BS^2 = 4VC^2 + BF^2 - 4VC.BF$. But $BS^2 = BF^2 + FS^2$ (47. 1.) ∵ $4VC^2 + BF^2 - 4VC.BF = BF^2 + FS^2$ ∵ $4VC^2 - 4VC.BF = FS^2 =$ (4. 2.) $4FC^2$ ∵ ($VC^2 - FC^2$ by cor. 2. Art. 54. $=$) $EC^2 = VC.BF$ (Art. 56.); ∵ (17. 6.) $VC : EC :: EC : BF$; whence (15. 5.) $VU : EK :: EK : BD$. Q. E. D.

Cor. 1. If L $(=BD)$ be the latus rectum, then (since $VU = 2VC$) $L.2VC = EK^2$ (17. 6.)

Cor. 2. Hence, of the major and minor axes and latus rectum, any two being given, the third may be found.

57. If FP and SP be drawn from the foci, to any point P in the curve, and FP be produced to M, the straight line PT which bisects the exterior angle FPM is a tangent to the ellipse.



Make $PM = PF$, join MF, let PT if possible, intersect the curve in p, and join Mp, Fp. Then because $MP = FP$, the angle $PMF = PFM$ (5. 1.) $MPr = FPr$ by hypothesis, and Pr common, ∵ (4. 1.) $Mr = Fr$, and the angle $MrP = FrP$; then in the triangles Mpr, Fpr, $Mr = Fr$, pr common, and the angles at r are equal, ∵ (4. 1.) $Mp = Fp$. But (20. 1.) $Sp + pM > SM$, that is $> SP + PM$, that is $> SP + PF$ (because $PF = PM$) that is $> Sp + pF$ (because $Sp + pF = SP + PF$ by Art. 37.); ∵ since $Sp + pM > Sp + pF$, if Sp be taken from both $pM > pF$; but it has been shewn that $pM = pF$, ∵ Mp and pF are both equal and unequal to each other, which is absurd; ∵ PT does not intersect the curve in any other point p; PT is therefore a tangent at P. Q. E. D.

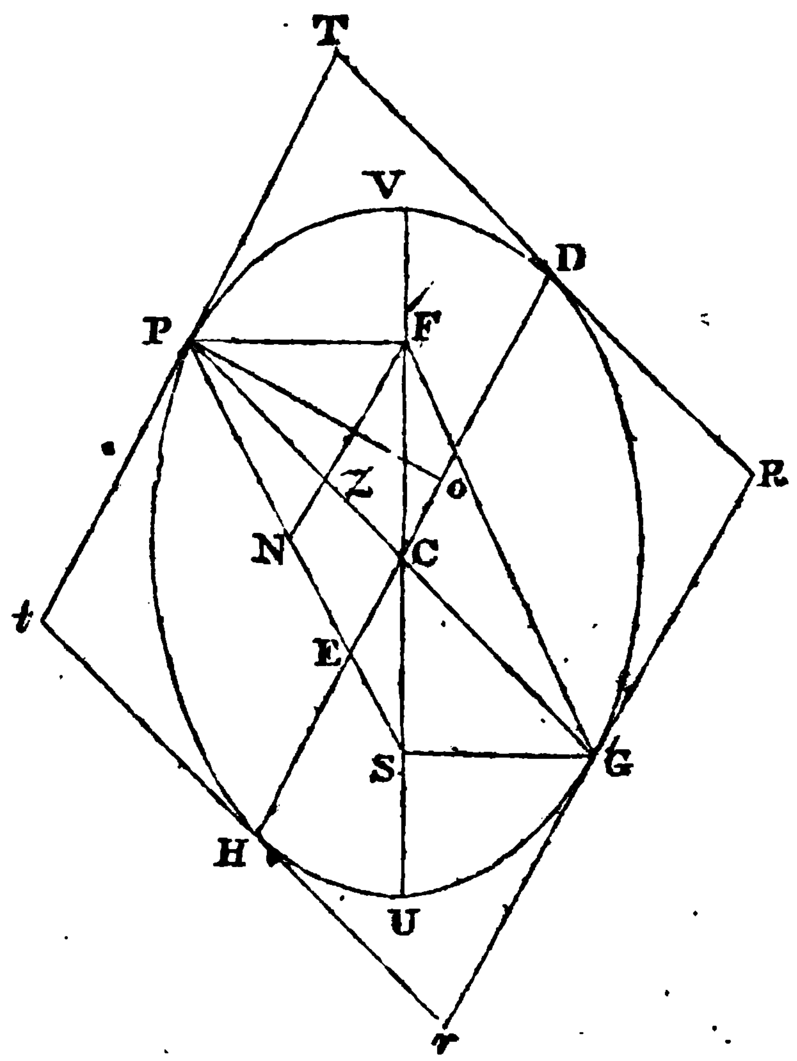
Cor. 1. It is plain that the nearer the point p be to V, the greater will be the angle FpM; and therefore when p coincides with V, the lines Fp, pM will coincide with FV, VT, and the angle FpM will become = two right angles; but the tangent at (p which now coincides with) V bisects this angle, ∵ the tangent at V is at right angles to the axis VU.

Cor. 2. Hence (prop. A. 6.) $ST : TF :: SP : PF$.

Cor. 3. Hence, straight lines drawn from the foci to any point in the curve, make equal angles with the tangent at that point, for the angle $tPS = MPT$ (15. 1.) $= FPT$.

Cor. 4. Hence the triangles FPY, SPt will be similar, and (4. 6.) $SP : St :: FP : FY$.

58. Let P be any point in the ellipse; join FP, SP, then if SG and FG be drawn parallel to these respectively, the point G where they meet will be in the curve.



For since $FPSG$ is a parallelogram, $FG + GS = SP + FP$ (34. 1.) ∵ G is a point in the ellipse by Art. 37. Q. E. D.

Cor. Since PG and FS bisect each other in C (part 8. Art. 241. cor.), C is the centre of the ellipse (cor. 1. Art. 52.), and PG a diameter (Art. 42.), ∵ all the diameters of the ellipse are bisected by the centre.

59. If Rr be a tangent at G, it will be parallel to Tt.

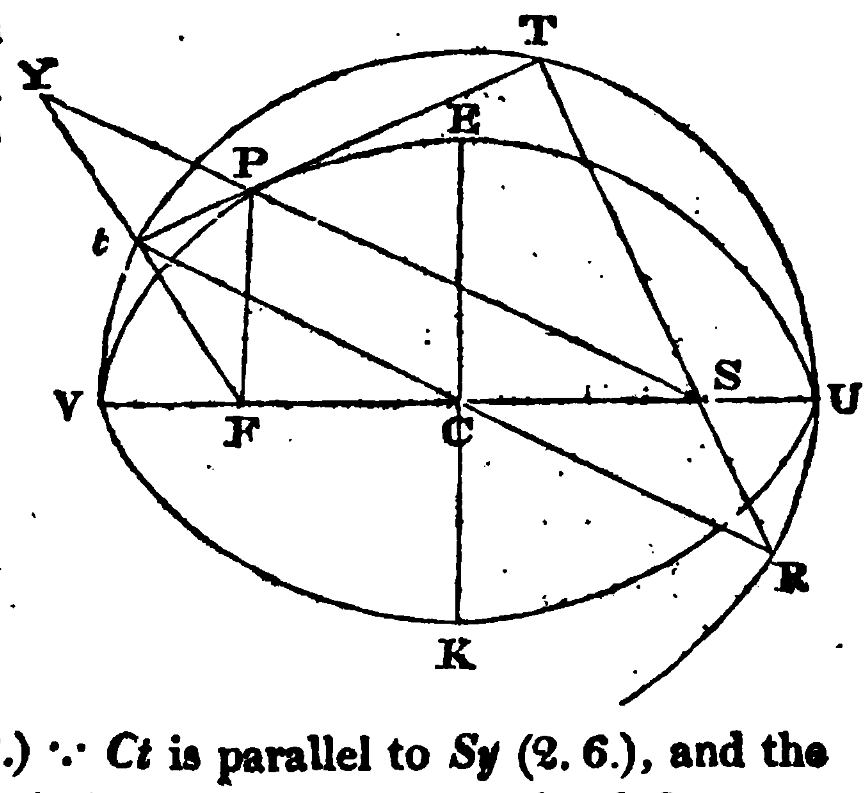
For since $SGr + SGF + FGR = 2$ right angles (13. and cor. 1. 15. 1.), $= SPt + SPF + FPT$, and $SGF = SPF$ (34. 1.), by taking the latter *equals* from the former, the remainders $SGr + FGR = SPt + FPT$, that is, (cor. 3. Art. 57.) $2FGR = 2SPt$, or $FGR = SPt$; but $PGF = GPS$ (29. 1.); add these *equals* to the preceding, and $FGR + PGF = SPt + GPS$; that is, $PGR = GPt$, ∵ (27. 1.) Rr is parallel to Tt. Q. E. D.

Cor. Hence, if HD be a conjugate diameter to PG, tangents at D and H will be parallel, and the four tangents Tt, tr, rR, and RT will form a parallelogram circumscribed about the ellipse.

60. If HD be drawn through the centre, parallel to Tt a tangent at P, cutting SP in the point E, then will $PE=UC$.

Draw FN parallel, and Po perpendicular to HD. Because NF is parallel to tT (30. 1.), and the angles at o right angles, ∵ the angles oPT, oPt are right angles (29. 1.), or $oPT=oPt$, but $FPT=SPt$ (cor. 3. Art. 57.), ∵ by taking the latter from the former $FPo=NPo$, ∵ $PNz=PFz$ (32. 1.), the angles at z (=the angles at o by 29. 1) right angles, and Pz is common to the triangles PzN, PzF, ∵ (26. 1.) $PN=PF$. And since EC is parallel to NF a side of the triangle SNF, and $SC=CF$ (cor. 1. Art. 52.), ∵ $SE=EN$ (2. 6.); ∵ $SP+PF$ (= $SN+NP+PF=2EN+2NP$) $=2PE$. But $SP+PF=2UC$ (Art. 52.), ∵ $2PE=(SP+PF=)\,2UC$, and $PE=UC$. Q. E D.

61. If perpendiculars be drawn from the foci to any tangent, and a circle be described on the major axis as a diameter, the points in which the perpendiculars intersect the tangent shall be in the circumference of the circle.



Let Ft, ST be drawn perpendicular to tT a tangent at P, join SP and produce it to meet Ft produced in Y, and join Ct. Then in the triangles PtF, PtY, the angle $tPF=tPY$ (Art. 57.), the angles at t are right angles, and Pt is common, ∵ (26. 1.) $FP=PY$ and $Ft=tY$; also $FC=CS$ (cor. 1. Art. 52.) ∵ Ct is parallel to Sy (2. 6.), and the triangles FCt, FSY are similar, ∵ $FC : Ct :: FS : SY$ (4. 6.). But $FC=\frac{1}{2}FS$, ∵ $Ct=\frac{1}{2}SY=\frac{1}{2}\overline{SP+PY}=\frac{1}{2}\overline{SP+PF}=$ (Art. 52.) $\frac{1}{2}VU=VC$; ∵ since $Ct=CV$, the points t and V are in the circumference of the circle whose centre is C, and in like manner it may be proved that T is in the circumference. Q. E. D.

61.B. The rectangle $Ft.ST=EC^2$. Produce TS to R and join CR, then because tTR is a right angle, the segment tTR is a semicircle (31.3.), ∴ tC and CR meeting at the centre, will constitute the diameter, and be in the same straight line, ∴ the angle $tCF=SCR$ (15.1.) and tC, $CF=RC$, CS respectively, ∴ (4.1.) $Ft=SR$; ∴ $Ft.ST=SR.ST=$(35.3.) $FS.SU=$ (Art. 54. cor. 2.) EC^2. Q. E. D.

Cor. 1. Hence $Ft : EC :: EC : ST$ (17.6.), ∴ $Ft^2 : EC^2 :: Ft : ST$ (cor. 2. 20.6.) :: $FP : SP$ (4. 6. because the triangles FtP, STP are similar) :: $FP : 2VC-FP$ (because $FP+SP=2VC$, Art. 52) Wherefore putting $VC=a$, $EC=b$, $FP=x$, and $Ft=y$, the analogy $Ft^2 : EC^2 :: FP : 2VC-FP$ becomes $y^2 : b^2 :: x : 2a-x$, ∴ $y^2=\frac{b^2x}{2a-x}$ which equation expresses the nature of the ellipse considered as a spiral, described by the revolution of FP about the centre F.

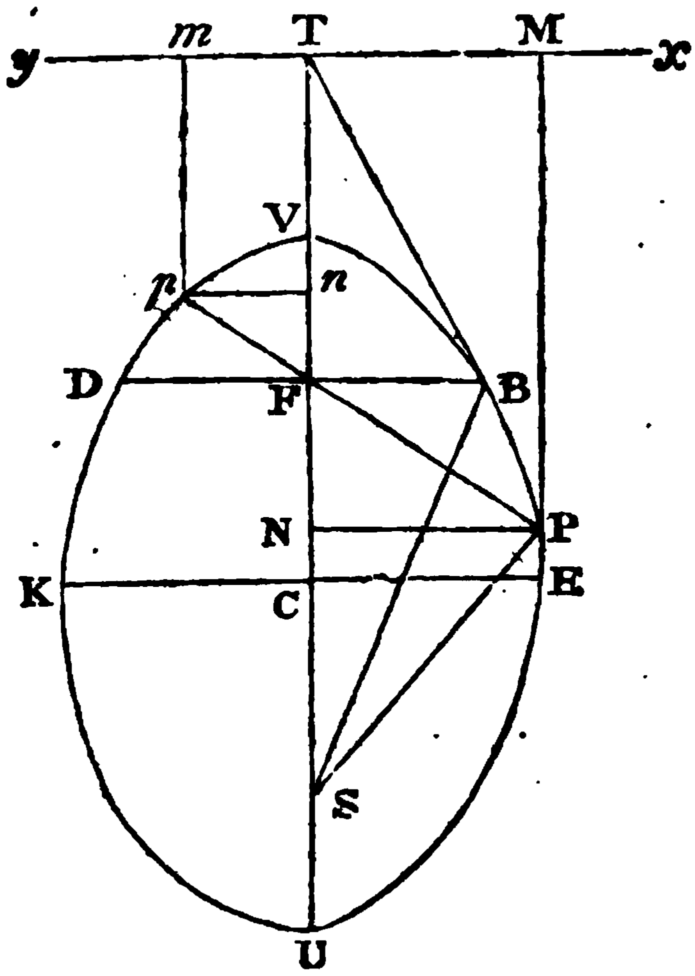
Cor. 2. Because $Ft^2 : EC^2 :: FP : SP$ (cor. 1.) ∴ $4Ft^2 : 4EC^2$ ($=EK^2=L.2VC$, cor. 1. Art. 56.) :: $L.FP : L.SP$, ∴ (16.5.) $4Ft^2 : L.FP :: L.2VC : L.SP :: 2VC : SP :: 2VC : 2VC-FP$.

62. If BT be the focal tangent, then will $CF.CT=VC^2$. *See the following figure.*

Because (cor. 2. Art. 57.) $ST : TF :: SB : BF$, ∴ by composition and division (18. and 17.5.) $ST+TF : ST-TF :: SB+BF : SB-BF$, or (cor. 1. Art. 52.) $2CT : 2CF :: SB+BF : SB-BF$, ∴ (15.5.) $2CT.2CF : 4CF^2$ :: ($\overline{SB+BF}.\overline{SB+BF} : \overline{SB+BF}.\overline{SB-BF}$ ::) $\overline{SB+BF}|^2 : SB^2-BF^2$ (cor. 5. 2.). But BFS is a right angle, ∴ (47. 1.) $SB^2-BF^2=FS^2=$ (4. 2.) $4CF^2$, ∴ (prop. A. 5) $2CT.2CF=\overline{SB+BF}|^2=VU^2$ (Art. 52.)$=4VC^2$ (4.2.), ∴ $CT.CF=VC^2$.

Cor. Hence, because $CT=CF+FT$, ∴ ($CT.CF=$) $CF^2+CF.FT=VC^2$, ∴ $CF.FT=VC^2-CF^2=EC^2$ (cor. 2. Art. 54.)

63. If PM be drawn perpendicular to the directrix yx, then will $FP : PM :: FC : VC$.



Let PN be perpendicular to VU, then $\overline{SN+NF}.\overline{SN-NF} = \overline{SP+PF}.\overline{SP-PF}$ [4], $\therefore$ (16. 6.) $SP-PF : SN-NF :: SN+NF : SP+PF$. But (Art. 52. cor. 4.) $SP-PF = 2VC - 2PF$; and $SN-NF = SC+CN-NF = CF-NF+CN = 2CN$; likewise $SN+NF = 2CF$; and $SP+PF = 2VC$ (Art. 52.); by substituting these values for their equals in the above analogy, it becomes $2VC - 2FP : 2CN :: 2CF : 2VC ::$ (Art. 62.) $2VC : 2CT$; $\therefore$ (15. 5.) $VC-FP : CN :: VC : CT$, subtract the former antecedent from the latter, and the former consequent from the latter, then ($VC-VC+FP : CT-CN :: VC : CT$; that is,) $FP : (NT=) PM :: VC : CT ::$ (Art. 62.) $CF : VC$. Q. E. D.

Cor. Hence, if the centre C be supposed at an infinite distance from V, CF may be considered as equal to VC, $\therefore FP = PM$, and the curve in this case at every finite distance, becomes a parabola. See Art. 1.

64. If PF be produced to meet the curve in p, then will $2FP.Fp = FB.\overline{FP+Fp}$.

Because $FP : NT :: CF : CV$ (Art. 63.), if P be supposed to coincide with B, the point N will coincide with F, and the straight line FP will become FB; $\therefore$ the above proportion will become $FB : FT :: CF : CV$; $\therefore$ since $\left.\begin{matrix} FP : NT \\ FB : FT \end{matrix}\right\} :: CF : CV$,

[4] For (47. 1.) $SP^2 = SN^2 + NP^2$ and $PF^2 = NF^2 + NP^2$, $\therefore SP^2 - PF^2 = SN^2 - NF^2$ or (cor. 5. 2.) $\overline{SP+PF}.\overline{SP-PF} = \overline{SN+NF}.\overline{SN-NF}$ as above.

$FP : NT :: FB : FT$ (11. 5.); but $NT = PM$ (34. 1.), ∴ $FP : PM :: FB : FT$, ∴ $FB : FT :: FP - FB : (PM - FT =) FN$. In like manner it may be shewn that $FB : FT :: Fp : pm$, ∴ $FB : FT :: FB - Fp : (FT - pm =) Fn$; ∴ (11. 5) $FP - FB : FN :: FB - Fp : Fn$. But the triangles FPN, Fpn are similar, ∴ (4. 6.) $FN : FP :: Fn : Fp$, and *ex æquo* (22. 5.) $FP - FB : FP :: FB - Fp : Fp$, ∴ (16. 6.) $FP.Fp - FB.Fp = FB.FP - FP.Fp$, ∴ by transposition $2 FP.Fp = (FB.FP + FB.Fp =) FB.\overline{FP + Fp}$. Q. E. D.

Cor. Hence, if $FB = l$, $FP = X$, and $Fp = x$, the above conclusion expressed algebraically will be $2Xx = l.\overline{X + x}$, or $\frac{2}{l} = \frac{1}{X} + \frac{1}{x}$.

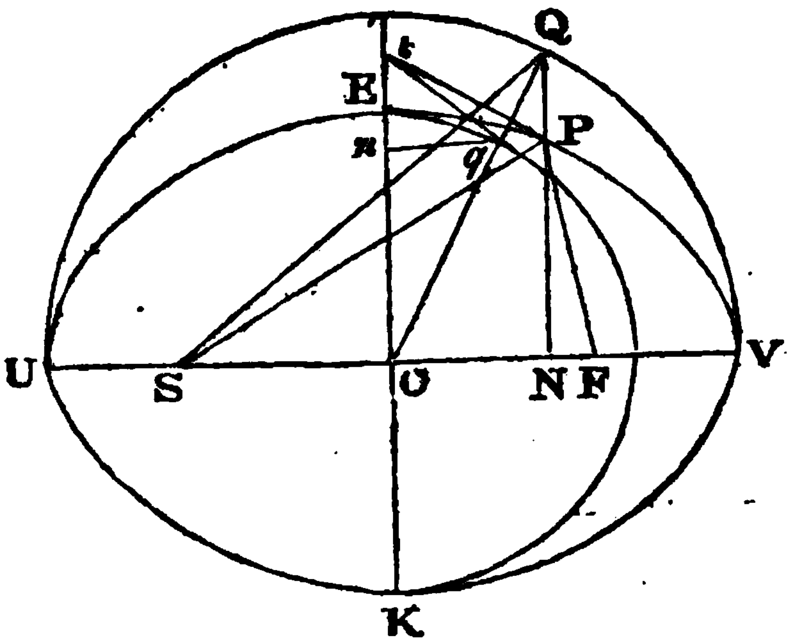
65. If c be the co-sine of the angle UFP to radius 1, then will $FP : EC :: EC : VC - c.CF$.

Because (Art. 63.) $FP : PM :: FC : VC$, ∴ (16. 6.) $FP.VC = FC.PM = FC.\overline{FT \pm FN} = FC.FT \pm FC.FN =$ (because $FC.FT = EC^2$ Art. 62. cor.) $EC^2 \pm FC.FN$. But $FN : FP :: \pm c : 1$; ∴ (16. 6.) $\mp FN = c.FP$, and $\pm FC . FN = c . FC.FP$, ∴ by substituting this latter quantity for its equal in the above equation, it becomes $FP.VC = EC^2 + c.FC.FP$; ∴ $(FP.VC - c.FC.FP =) FP.\overline{VC - c.FC} = EC^2$, ∴ (17. 6.) $FP : EC :: EC : VC - c.FC$. Q. E. D.

Cor. If VC be infinite, FC and VC may be considered as equal, and the above analogy becomes $FP : EC :: EC : \overline{1 - c}.VC$. But (Art. 56.) $EC : \frac{1}{2} L :: VC : EC$, ∴ *ex æquo* (22. 5.) $FP : \frac{1}{2} L :: (VC : \overline{1 - c}.VC ::) 1 : 1 - c$, or (16. 6.) $\overline{1 - c}.FP = \frac{1}{2} L$, and $FP = \frac{\frac{1}{2} L}{1 - c}$ as in the parabola, see Art. 22.

66. If on the major axis as a diameter, a circle be described, and PN an ordinate to the major axis be produced to meet the circumference in Q, and if c be the co-sine of the angle VCQ to radius 1; then will $FP = VC - c.FC$.

Because (Art. 63.) $\overline{SN+NF}.\overline{SN-NF}=\overline{SP+FP}.\overline{SP-FP}$, ∴ $SN+NF : SP-FP :: SP+FP : SN-NF$ (16.6.); ∵ (see Art. 63.) $2CF : 2VC-2FP :: 2VC : 2CN$, or $CF : VC-FP :: VC : CN ::$ (because $QC=VC$) $QC : CN$. But (Art. 63. part 9.) $QC : CN :: 1 : c$, ∴ $CF : VC-FP :: 1 : c$, ∴ (16. 6.) $c.CF=VC-FP$, and $FP=VC-c.CF$. Q. E. D.



67. If PN be an ordinate to the axis, then will $UN.NV : PN^2 :: VC^2 : EC^2$.

For (Art. 63.) $\overline{SN+NF}.\overline{SN-NF}=\overline{SP+FP}.\overline{SP-FP}$, ∴ $SN-NF : SP+PF :: SP-PF : SN+NF$ (16. 6.); but $SN-NF=(SC+CN-NF=CF-NF+CN=)\ 2CN$; likewise $SP+PF=2VC$ (Art. 52.); also $SP-PF=2SP-2VC$ (Art. 52. cor. 4.); and lastly, $SN+NF=2SC$; ∴ substituting these four values for their equals in the above analogy, it becomes $2CN : 2VC :: 2SP-2VC : 2SC$; ∴ (15. 5.) $CN : VC :: SP-VC : SC$.

∴ $\begin{cases} (18.5.)\ UN : VC :: SC+SP-VC : SC. \\ (17.5.)\ VN : VC :: SC-SP+VC : SC. \end{cases}$

From the former of these (12. 5.) $UN : VC :: UN+SC+SP-VC : VC+SC :: SP+SN : UF$;

And from the latter (19. 5.) $VN : VC :: VN-SC+SP-VC : VC-SC :: SP-SN : US$; ∴ (prop. G. 5.) $UN.NV : VC^2 :: \overline{SP+SN}.\overline{SP-SN} : UF.US ::$ (cor. 5. 2. $SP^2-SN^2=$ by 47. 1.) $PN^2 : EC^2$ (because $UF.US=VF.FU$, see cor. 2. Art. 54.) ∴ (16. 5.) $UN.NV : PN^2 :: VC^2 : EC^2$. Q. E. D.

Cor. 1. Hence, because $UN=UC+CN$, and $NV=(VC-CN=)\ UC-CN$, ∴ $UN.NV=\overline{UC+CN}.\overline{UC-CN}=$ (cor. 5. 2.) UC^2-CN^2, ∴ also $VC^2-CN^2 : PN^2 :: VC^2 : EC^2$.

Cor. 2. Hence, if $VC=a$, $EC=b$, $CN=x$, and $PN=y$, the expression $VC^2-CN^2 : PN^2 :: VC^2 : EC^2$ becomes $a^2-x^2 : y^2 :: a^2 : b^2$, ∴ $y^2=\frac{b^2}{a^2}.\overline{a^2-x^2}$ which is the equation of the ellipse.

Cor. 3. Hence $UN.NV \propto PN^2$; that is, the rectangle contained by the abscissæ varies as the square of the ordinate.

68. If Pn be an ordinate to the minor axis EK, then in like manner $En.nK : Pn^2 : EC^2 : VC^2$.

For $Pn=CN$, and $PN=Cn$; $VC-Pn^2 : Cn^2 : VC^2 : EC^2$ (Art. 67. cor. 1.), ∵ (16. 5.) $CV^2-Pn^2 : VC^2 :: Cn^2 : EC^2$, ∵ (17. 5.) $Pn^2 : VC^2 :: EC^2-Cn^2 : EC^2 ::$ (cor. 5. 2.) $\overline{EC+Cn}.\overline{EC-Cn} : EC^2 :: En.nK : EC^2$; ∵ (16. 5.) $Pn^2 : En.nK :: VC^2 : EC^2$, and (prop. B. 5.) $En.nK : Pn^2 :: EC^2 : VC^2$. Q. E. D.

69. If on the major axis UV, as a diameter, a circle UQV be described and NQ an ordinate to the axis be drawn cutting the ellipse in P, and the circle in Q; then will $PN : QN :: EC : VC$.

For $QN^2=UN.NV$ (14. 2.) ∵ (Art. 67.) $QN^2 : PN^2 :: VC^2 : EC^2$, ∵ (22. 6) $QN : PN :: VC : EC$, ∵ (prop. B.5.) $PN : QN :: EC : VC$. Q. E. D.

Cor. 1. In like manner, if on the minor axis EK as a diameter the circle EqK be described, it may be shewn that $Pn : qn :: VC : EC$.

Cor. 2. Hence the area $UPN : UQN :: (UC=) EC : VC$ as in the parabola, (Art. 29.); in like manner $VPN : VQN :: EC : VC$, ∵ $UPV : UQV :: (2EC : 2VC ::) EC : VC$. Also, if any point S be taken in the axis, and SP, SQ be joined, the area UPS : area $UQS :: EC : (UC=) VC$ as in the parabola, cor. Art. 29.

70. If a mean proportional R be found between VC and EC, and with it as radius, a circle be described; the area of this circle will be equal to the area of the ellipse.

For the area UPV : area $UQV :: EC : VC$ (cor. 2. Art. 69.) and since $VC : R :: R : EC$, ∵ (2. 12, and cor. 2, 20. 6.) area of circle UQV whose radius is VC : area of circle whose radius is $R :: VC : EC$; this proportion being compounded with the first, we have $UPV.UQV : UQV \times$ area of circ. whose rad. is $R :: EC.VC : VC.EC$; that is, (15. 5.) elliptical area UPV : circular area whose rad. is $R :: (EC.VC : EC.VC ::) 1 : 1$; or the area of the circle is equal to the area of the ellipse. Q. E. D.

Cor. 1. Since (cor. 2. Art. 69.) $UPV : UQV :: EC : VC ::$ (15. 5.) $EC.VC : VC^2$, ∵ (16. 5) $UPV : EC.VC :: UQV : VC^2$;

∵ (15. 5.) area of ellipse : $EC.VC$: : area of circ. whose diam. is $UV : VC^2$. But the area of the circle varies as VC^2 (2. 12.); ∵ the area of the ellipse varies as $EC.VC$.

Cor. 2. Because $VC : EC :: EC : \frac{1}{2}L$ (Art. 56.), ∵ $VC : \frac{1}{2}L :: VC^2 : EC^2$ (cor. 2, 20. 6.); but $UN.NV : PN^2 :: VC^2 : EC^2$ (Art. 67.), ∵ $UN.NV$ (or $VC^2 - CN^2$, Art. 67. cor. 1.) : PN^2 : : $VC : \frac{1}{2}L$; ∵ since VC and $\frac{1}{2}L$ are constant quantities $UN.NV \propto PN^2$.

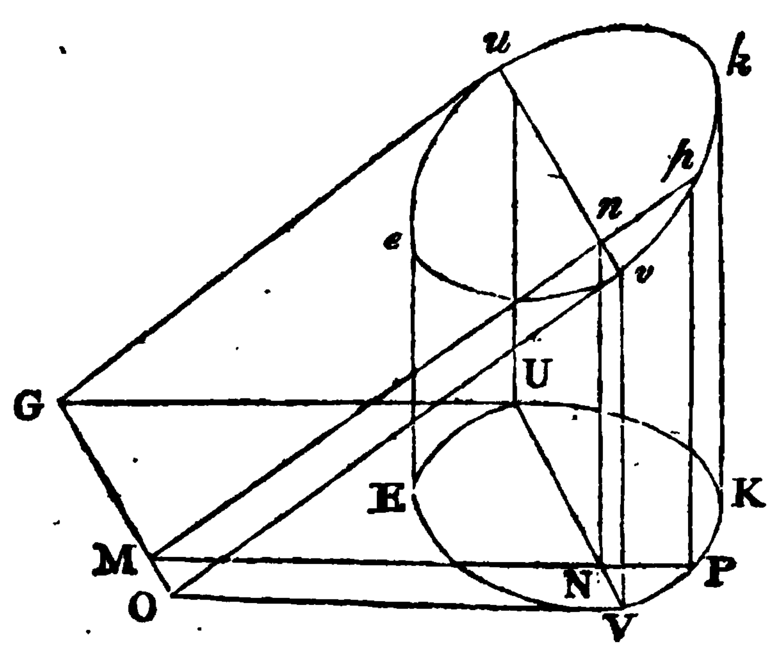
Cor. 3. Hence, if the major axis UV become infinite, the curve at all finite distances from the vertex U will be a parabola; for NV being infinite will be constant, and ∵ $UN \propto PN^2$ which (Art. 27.) is the distinguishing property of the parabola.

Cor. 4. The curve UPV which arises by diminishing the ordinates NQ of the circle in a given ratio, is an ellipse.

For, let $EC : UC :: PN : QN$, then if an ellipse be described on UV as the major axis, having EK for its minor axis, we shall have (Art. 69.) $UC : EC :: QN$: ordinate of the ellipse; and from the preceding analogy (prop. B. 5.) $UC : EC :: QN : PN$ ∵ PN= an ordinate of the ellipse (9. 5.), or the curve passing through P is an ellipse. In like manner it may be shewn, that if the ordinates QN of the circle be increased in any given ratio, the curve described upon UV as a minor axis, and passing through the extremities of the increased ordinates, will be an ellipse.

71. If a plane be inclined in any angle to the plane of a circle, and if straight lines be drawn from every point in the circumference, perpendicular to the inclined plane, the curve which passes through the extremities of all the perpendiculars will be an ellipse.

Let $UEVK$ be a circle, and the perpendiculars Uu, Ee, Vv, Kk, &c. meeting the inclined plane $GuvO$ in the points u, e, v, k; the figure $uevk$ will be an ellipse.



Let UV be a diameter of the circle parallel to GO the common section of the planes, and

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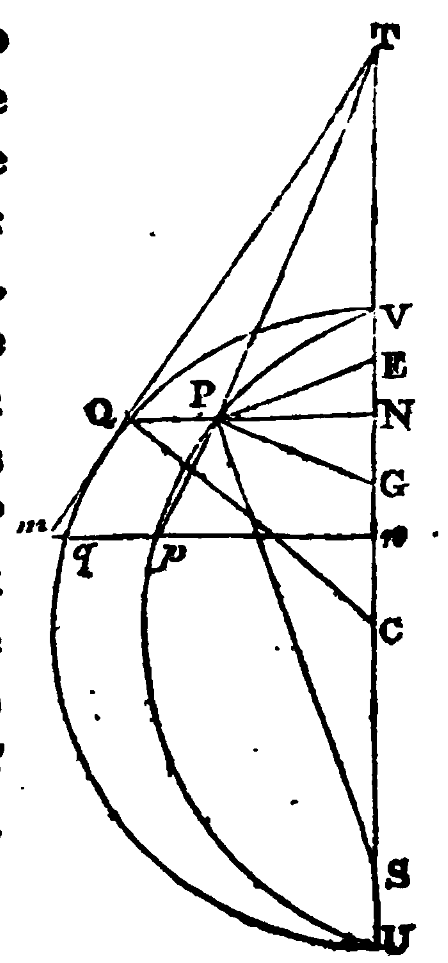
MP at right angles to *UV*; draw *GU*, *OV* each parallel to *MP*, join *Ov* and draw *Mp*, *Gu* parallel to it, join *Nn*, *Pp*. Because *MV* is a parallelogram *MO* is parallel to *NV* (34. 1.), but *NV* is perpendicular to the plane *MNn* by construction (4. 11.) ∵ *MO* is also perpendicular to the plane *MNn* (8. 11.) ∵ *uv* is perpendicular to *MNn* (19. 11.) ∵ *uv* is parallel to *UV* (6. 11.); and since the planes *MpP*, *UuVv* are both at right angles to the plane *GV*, their common section *Nn* is at right angles to it (19. 11.), ∵ *Nn* is parallel to *Pp* (6. 11.); ∵ $pn : nM :: PN : NM$ (2. 6.) and $pn : PN :: nM : NM$ (16. 5.) :: radius : co-sine *PMp* (part 9. Art. 63.) the angle of inclination of the planes, or $pn : PN$ in a given ratio, ∵ (cor. 4. Art. 70.) *uevk* is an ellipse. Q. E. D.

Cor. Hence the oblique section of a cylinder is an ellipse, of which the minor axis is the diameter of the cylinder.

72. If a circle be described on the major axis as a diameter, and any ordinate *NP* be drawn meeting the circle in *Q*, tangents at *P* and *Q* will meet the axis produced in the same point T.

For if possible, let *QT* be a tangent to the circle in *Q*, and *PT* not a tangent to the ellipse, but cut it in *P* and *p*; draw *np* and produce it to meet *TQ* produced in *m*; then since the triangles *TNP*, *Tnp*, as also *TNQ*, *Tnm* are similar (32. 1.) $PN : pn :: NT : nT :: QN : mn$ (4. 6.). But $PN : QN :: pn : qn$ (Art. 69.), ∵ $PN : pn :: QN : qn$ (16. 5.). But by the first analogy $PN : pn :: QN : mn$, ∵ $QN : qn :: QN : mn$, ∵ (9. 5.) $qn = mn$, the less to the greater, which is impossible; ∵ *TP* which meets the ellipse in *P* does not cut it, it must therefore be a tangent to the ellipse. In like manner (see the figure to Art. 66.); since $Pn : qn\ (=nC) :: VC : EC$ (cor. 1. Art. 69.), it may be shewn that tangents at *P* and *q* cut the minor axis in the same point *t*. Q. E. D.



Cor. 1. Because *CQT* is a right angle (18. 3. see the figure to Art. 72.), $CN : CQ :: CQ : CT$ (cor. 8. 6.); but $CV = CQ$, ∵ $CN : CV :: CV : CT$. In like manner it is shewn that (see the figure to Art. 66.) $Cn : CE :: CE : Ct$.

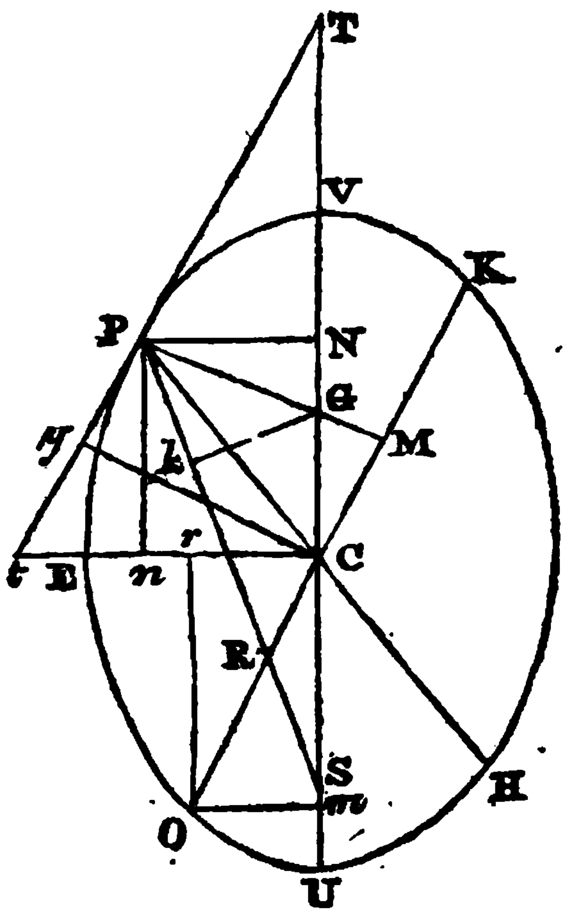
Cor. 2. $TN.NC=QN^2$ (cor. 8. 6. and 17. 6.) $=CQ^2-CN^2$ (47. 1.) $=VC^2-CN^2=$ (cor. 1. Art. 67.) $VN.NU$.

Cor. 3. The sub-tangent NT is greater than $2VN$; for since (by the preceding) $TN.NC=VN.NU$, $\because$ (16. 6.) $NT:VN::NU:NC$; but $CU>NC$, $\because$ $(NC+CU=)\ NU>2NC$, $\because$ $NT>2VN$.

Cor. 4. If PG be the normal, then (cor. 8. 6. and 17. 6.) $TN.NG=PN^2$, and $TN.NC:TN.NG::VC:\frac{1}{2}L$ (cor. 1. Art. 67. and cor. 1. Art. 72.) $\because$ $NC:NG::VU:L$ (15. 5.).

73. If HP be a diameter and KO its conjugate, then PM being drawn perpendicular to KO cutting the axis VU in G, the rectangle $PM.PG=EC^2$.

For if Cy be drawn parallel to PM, the angle $PGN=yCG$ (29. 1.), but $yCG+yCt=(GCt=)$ a right angle, and $ytC+yCt=$ a right angle (32. 1.), $\because$ $yCG+yCt=ytC+yCt$; take away the common angle yCt, and the remainder $yCG=ytC$, $\because$ $PGN=(yCG=)\ ytC$, and $PNG=Cyt$ being right angles; $\because$ the triangles PGN, Cyt are equiangular (32. 1.); and $PG:(PN=$ by 34. 1.) $Cn::Ct:(Cy=)\ PM$ (4. 6.); $\because$ $PM.PG=Cn.Ct$ (16. 6.) $=EC^2$ by cor. 1. Art. 72. Q. E. D.



74. Join PS, then if PG be drawn perpendicular to Tt, and Gk perpendicular to PS, $Pk=\frac{1}{2}L$.

For the angles at k and M being right angles, and the angle kPM common, the triangles PMR, PkG are equiangular (32. 1.) $\because$ $PR:PM::PG:Pk$ (4. 6.), and $PR.Pk=PM.PG$ (16. 6.) $=EC^2$ (Art. 73.), $\because$ ($PR=$ by Art. 60.) $VC:EC::EC:Pk$ (16. 6.). But $VC:EC::EC:\frac{1}{2}L$ (Art. 56.), $\because$ $Pk=\frac{1}{2}L$ (9. 5.). Q. E. D.

75. If PC, CO be semi-conjugate diameters, and PN, Om be perpendicular to the axis, then will $CN^2+Cm^2=VC^2$.

For $VC^2-Cm^2:Om^2::VC^2:EC^2$ (cor. 1. Art. 67.) $::VC^2-CN^2:PN^2$ (Art. 67.) But OC being parallel to tT, and the angles at m and N right angles, $\because$ (29. 1.) the triangles COm,

PNT are similar, and (4. 6.) $Om : Cm :: PN : NT$; ∵ (22. 6.) $Om^2 : Cm^2 :: PN^2 : NT^2$, ∵ from this and the first analogy (22. 5.) $VC^2 - Cm^2 : Cm^2 :: VC^2 - CN^2 : NT^2$. But $CN.NT : NT^2 :: CN : NT$ (1. 6.) ∵ by inversion $Cm^2 : VC^2 - Cm^2 :: NT : CN$, and by composition $VC^2 : VC^2 - Cm^2 :: CT : CN ::$ (1. 6.) $CN.CT : CN^2$. But $VC^2 = CN.CT$ (cor. 1. Art. 72.), ∵ $VC^2 - Cm^2 = CN^2$ (14. 5.), ∵ $VC^2 = CN^2 + Cm^2$. Q. E. D.

Cor. 1. Hence $VC^2 - CN^2 = Cm^2$, ∵ $Cm^2 : PN^2 :: VC^2 : EC^2$ by the first analogy in the proposition, and $Cm : PN :: VC : EC$ (22. 6.). In like manner, because $VC^2 - Cm^2 = CN^2$, ∵ $CN^2 : Om^2 :: VC^2 : EC^2$, and $CN : Om :: VC : EC$.

Cor. 2. Hence also $Cm : PN :: CN : Om$, ∵ (16. 6.) $Cm.Om = PN.CN$.

76. If PN, Om be perpendicular to the axis VU, and PC, CO semi-conjugate diameters, then will $PN^2 + Om^2 = EC^2$.

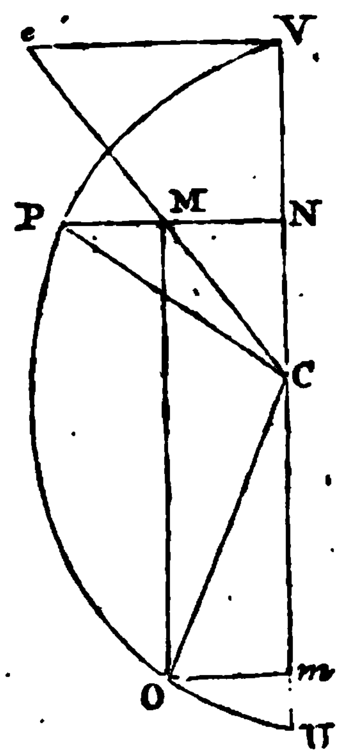
For $CN^2 : Om^2 :: VC^2 : EC^2$ (cor. 1. Art. 75.), $:: VC^2 - CN^2 : PN^2$ (cor. 1. Art. 67.) ∵ summing the antecedents and consequents (12. 5.) $VC^2 : Om^2 + PN^2 :: VC^2 - CN^2 : PN^2 ::$ (Art 67.) $VC^2 : EC^2$, ∵ $Om^2 + PN^2 = EC^2$ by 14. 5. Q. E. D.

Cor. 1. Because CP and CO are semi-conjugate diameters to each other, ∵ CP will be parallel to a tangent at O; and $Cn^2 + Cr^2 = (Om^2 + PN^2$ 34. 1.$=)$ EC^2; and hence the same relation subsists between the ordinates and abscissas to the minor axis, that does between those to the major axis.

77. $CP^2 + CO^2 = VC^2 + EC^2$.

For $VC^2 = CN^2 + Cm^2$ (Art. 75.), and $EC^2 = PN^2 + Om^2$ (Art. 76.); ∵ $VC^2 + EC^2 = (CN^2 + PN^2 + Cm^2 + Om^2 =)$ $CP^2 + CO^2$ (47. 1.). Q. E. D.

78. If Ve a tangent to the major axis, be made equal to the semi-minor axis, and eC be joined cutting PN, any ordinate to the major axis in M; then will $MN^2 + PN^2 = Ve^2$.



For the triangles eVC and MNC being similar (2. 6.) $Ve : MN :: CV : CN$, and $Ve^2 : MN^2 :: CV^2 : CN^2$ (22. 6.), $\because Ve^2 : Ve^2 - MN^2 :: CV^2 : CV^2 - CN^2$ (prop. E. 5.) $:: Ve^2 : PN^2$ (cor. 1. Art. 67.); $\because Ve^2 - MN^2 = PN^2$ (14. 5.), and consequently $MN^2 + PN^2 = Ve^2$. Q. E D.

Cor. Because $MN^2 + PN^2 = (Ve^2 =) EC^2$, *see also the figure to Art.* 73. and $Om^2 + PN^2 = EC^2$ (Art. 76.); $\because MN = Om$, and MO being joined, it will be parallel to the axis VU (33. 1.). Hence, if a straight line OC be drawn from the extremity O of the parallel MO, through the centre C, it will be the conjugate diameter to PC; and hence by this proposition, having any diameter of an ellipse given, the position of its conjugate may be readily determined.

78. If PC, CO be semi-conjugate diameters, and PM be drawn perpendicular to CO (see the figure to Art. 73.) then will $CO.PM = VC.EC$.

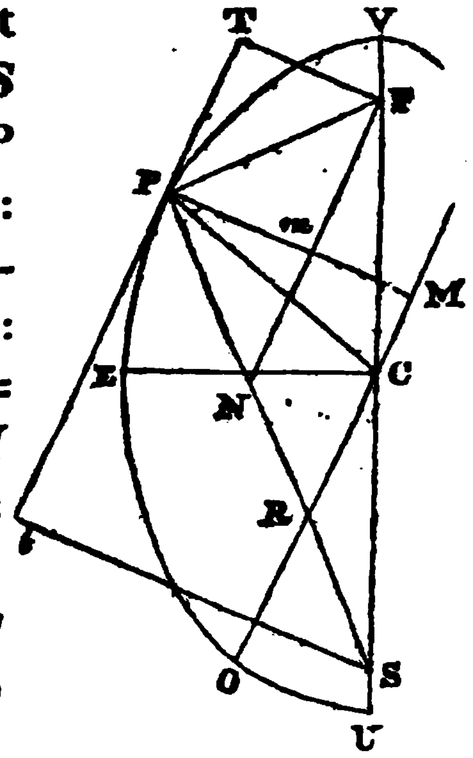
Because PN, Om are perpendicular to the axis, and Cy perpendicular to the tangent, $\because$ (cor. 1. Art. 75.) $CN : Om :: VC : EC$, and (16. 5.) $CN : VC :: Om : EC$; and the triangles TCy, OCm being similar $CT : Cy :: CO : Om$ (4. 6.), the two latter analogies being compounded (prop. F. 5.) $CN.CT : VC.Cy :: CO : EC$; but (because $CN.CT = VC^2$, cor. 1. Art. 72.) $VC^2 : VC.Cy :: VC : Cy :: CO : EC$; $\because$ (16. 6.) $VC.EC = OC.Cy = OC.PM$ (34. 1.) Q. E. D.

Cor. 1. Let $VC = a$, $EC = b$, $PC = x$, $Cy = y$, then (Art. 77.) $CO^2 = (VC^2 + EC^2 - PC^2 =)\ a^2 + b^2 - x^2$, $\because y^2 = \left(Cy^2 = \frac{VC^2 . EC^2}{CO^2} =\right) \frac{a^2 b^2}{a^2 + b^2 - x^2}$.

Cor. 2. Hence, if at the vertices of two diameters which are conjugates to each other, tangents be drawn, a parallelogram will be circumscribed about the ellipse, the area of which is $4CO.PM$ a constant quantity. See the figure to Art. 58.

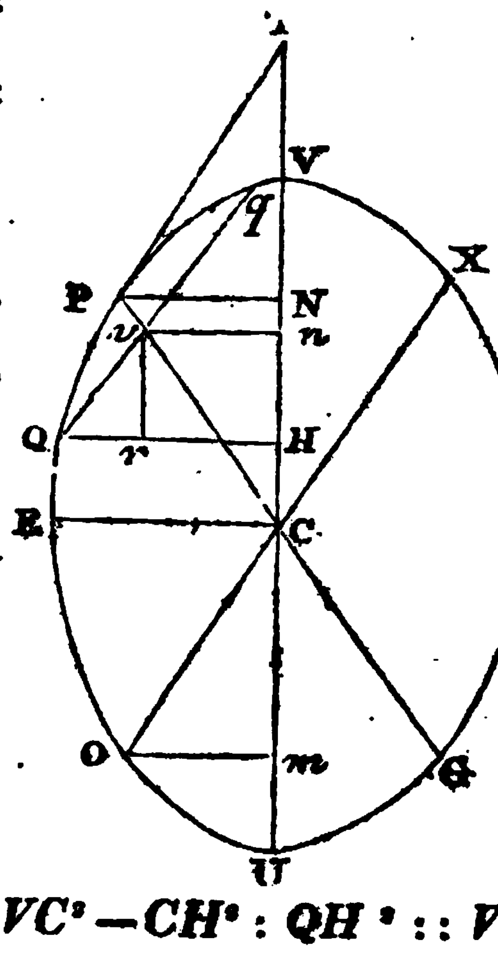
79. If CP, CO be semi-conjugate diameters, then will $FP.SP = CO^2$.

For the triangles SPt, PRM, FPT are similar, because TF, PM, and tS are parallel, the angles at T, M, and t right angles, and $TPF = tPS$ (cor. 3. Art. 57.) $= PRM$ (29. 1.); $\because$ $SP : St :: PR : PM$, and $FP : FT :: PR : PM$ (4. 6.), these analogies being compounded (prop. F. 5.) $SP.FP : St.FT :: PR^2 : PM^2$. But (Art. 78.) $VC.EC = OC.PM$, $\because$ (VC = by Art. 60.) $PR : PM :: OC : EC$ (16. 6.); and $PR^2 : PM^2 :: OC^2 : EC^2$ (22. 6.); $\because$ from above $SP.FP : St.FT :: OC^2 : EC^2$; but $St.FT = EC^2$ (Art. 61. B.) $\because$ $SP.FP = OC^2$ (14. 5.) Q. E. D.



80. Let OX be the conjugate and Qv an ordinate to the diameter PG, then will $Pv.vG : Qv^2 :: PC^2 : CO^2$.

Draw PN, vn, QH, and Om perpendicular to the axis VU, and vr parallel to it. Then because PN is parallel to Qr, vr to TN, and Qv to PT, the triangles PTN, Qvr are equiangular, and (4. 6.) Qr : (rv = by 34. 1.) Hn :: $PN : NT$, $\because$ $Qr : \frac{CN}{NT}.Hn :: PN : (\frac{CN}{NT}.NT =) CN$ (part 4. Art. 75.); but $vn : Cn :: PN : CN$ (2. 6.); $\because$ by adding the antecedents together, and the consequents together (12. 5.) in the two last analogies, $Qr + vn : \frac{CN}{NT}.Hn + Cn :: 2PN : 2CN$, or $QH : \frac{CN}{NT}.Hn + Cn :: PN : CN$ (15. 5.), and $QH^2 : \overline{\frac{CN}{NT}.Hn + Cn}|^2 :: PN^2 : CN^2$ (22. 6.).

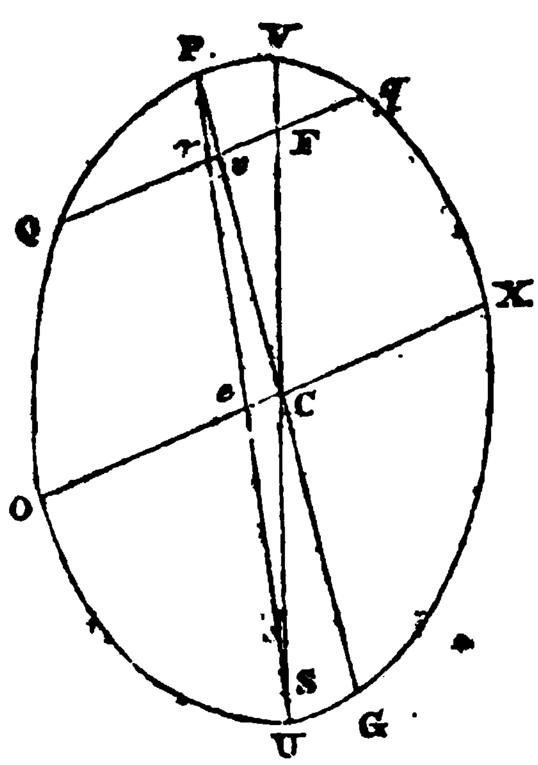


But (cor. 1. Art. 67.) $VC^2 - CH^2 : QH^2 :: VC^2 - CN^2 : PN^2$ (being each as $VC^2 : EC^2$) $\because$ *ex æquo* (22. 5.) $VC^2 - CH^2 : \overline{\frac{CN}{NT}.Hn + Cn}|^2 :: VC^2 - CN^2 : CN^2$ :: (cor. 2. Art. 72.) $CN.NT : CN^2$ :: (15. 5.) $NT : CN$; $\because$ (since CN :

VC :: $VC : CT$ by cor. 1. Art. 72.; whence, by cor. 2, 20. 6, $CN : CT :: CN^2 : VC^2 = \frac{CT}{CN}.CN^2)$ $VC^2 - CH^2$ or its equal $\frac{CT}{CN}.CN^2 - \overline{Cn-Hn}|^2 = \overline{\frac{NT}{CN}.Cn + \frac{CN}{NT}.Hn}|^2$ (16. 6, and part 4. Art. 69.); ∴ $\frac{CT}{CN}.CN^2 - Cn^2 - Hn^2 = \frac{NT}{CN}.Cn^2 + \frac{CN}{NT}.Hn^2$ (by actually squaring and multiplying;) ∴ $\frac{CT}{CN}.CN^2 - \frac{CT}{CN}.Cn^2 = \frac{CT}{NT}.Hn^2$ (by reduction, and from the figure); ∴ $CN^2 - Cn^2 = \frac{CN}{NT}.Hn^2$ (by dividing by $\frac{CT}{CN}$), or $NT.\overline{CN^2 - Cn^2} = CN.Hn^2$; ∴ (16. 6.) $CN^2 - Cn^2 : Hn^2 :: CN : NT ::$ (by inversion in the 7th analogy, above) $CN^2 : VC^2 - CN^2$; ∴ (16. 5.) $CN^2 - Cn^2 : CN^2 :: Hn^2 : VC^2 - CN^2$; but (2. 6.) $CN : Cn :: CP : Cv$, ∴ $CN^2 - Cn^2 : CN^2 :: CP^2 - Cv^2 : CP^2$ (part 4, Art. 69.). Also, (by similar triang. and 22. 6.) $rv^2 = Hn^2 : (Cm^2 =$ by Art. 75.) $VC^2 - CN^2 :: Qv^2 : CO^2$; ∴ ($CP^2 - Cv^2 =$ cor. 5. 2.) $Pv.vG : CP^2 :: Qv^2 : CO^2$, and (16. 5.) $Pv.vG : Qv^2 :: PC^2 : CO^2$. Q. E. D.

Cor. Hence it may likewise be shewn by similar reasoning, that if Qv be produced to meet the curve again in q, $Pv.vG : qv^2 :: PC^2 : CX^2$, ∴ $Qv : qv :: CO : CX$. But $CO = CX$ (cor. Art. 58.), ∴ $Qv = qv$.

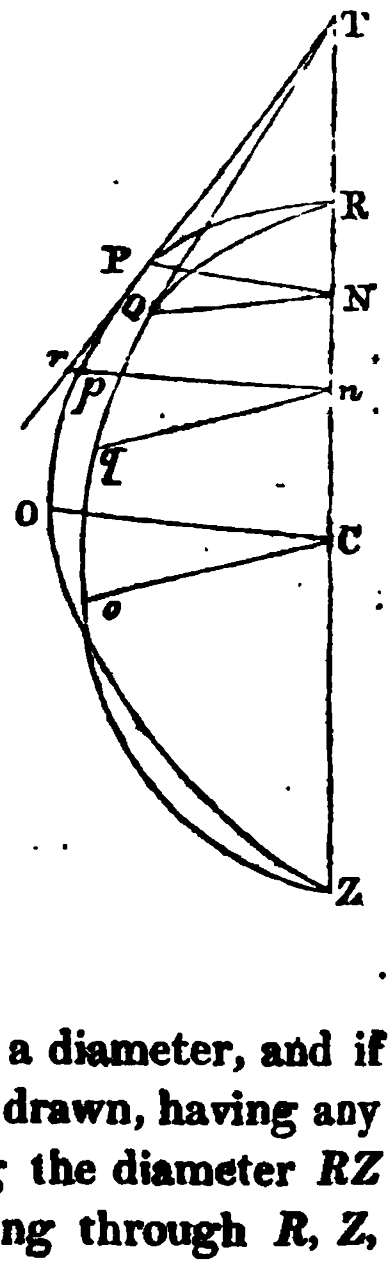
81. The parameter P to any diameter PG is a third proportional to the major axis and conjugate diameter; that is, $VU : OX :: OX : P$.



Let the ordinate Qv passing through the focus F meet the curve again in q; then will Qq be the parameter to the diameter PG, and (cor. Art. 80) $Qv = \frac{1}{2}P$. Because ($Pv.vG =$) $PC^2 - Cv^2 : Qv^2 :: PC^2 : CO^2$ (Art. 80.) ∴ $Qv^2 : PC^2 - Cv^2 :: CO^2 : PC^2$ (prop. B.5.) But because Ce is parallel to vF (Art. 60.) $Pe = VC$, ∴ $PC^2 - Cv^2 : (Pe^2 - Se^2 =) Pe^2 - er^2 :: Pv^2 : Pr^2 :: PC^2 : Pe^2$ ∴ *ex æquo* (22. 5.) Qv^2

$: \overline{Pe^2 - er^2} :: CO^2 : (Pe^2 =) VC^2$. But $Pe^2 - (Se^2 =) er^2 = \overline{Pe + er} . \overline{Pe - er}$ (cor. 5. 2.) $=$ (Art. 60.) $CP . SP =$ (Art. 79.) CO^2; $\therefore Qv^2 : CO^2 :: CO^2 : VC^2$ and (22. 6.) $Qv : CO :: CO : VC$, $\therefore$ (15. 5.) $2Qv : 2CO :: 2CO : 2VC$, that is $P : OX :: OX : VU$ or $VU : OX :: OX : P$. Q. E. D.

82. If two ellipses RPZ, RQZ have a common diameter RZ, from any point N in which NP and NQ an ordinate to each of them be drawn, then will the tangents at P and Q meet the diameter RZ produced in the same point T.



Draw TP a tangent to the ellipse RPZ and join TQ; TQ shall be a tangent to the ellipse RQZ. For if not, let TQ meet the curve again in q and draw the ordinates nq, np and produce np, TP to meet in r. Then $PN^2 : pn^2 :: RN.NZ : Rn.nZ :: QN^2 : qn^2$ (cor. 3. Art. 70.), $\therefore PN : pn :: QN : qn$ (22. 6.). But the triangles PNT, rnT are similar, as are also QNT, qnT; $\therefore PN : rn :: NT : nT$ (4. 6.) $:: QN : qn$, $\therefore PN : pn :: PN : rn$ (11. 5.), $\therefore pn = rn$ (14. 5.), the less equal to the greater, which is absurd; $\therefore TQ$ meets the curve no where but in Q, consequently touches it in Q. Q. E. D.

Cor. Hence, if RZ be bisected in C, the point C will be the centre of both ellipses, and (cor. 1. Art. 72.) $CN : CR :: CR : CT$.

83. If RPZ be an ellipse, of which RZ is a diameter, and if from every point in RZ, straight lines QN be drawn, having any given ratio to the ordinates PN, and cutting the diameter RZ in any given angle, then shall the curve passing through R, Z, and all the points Q be an ellipse.

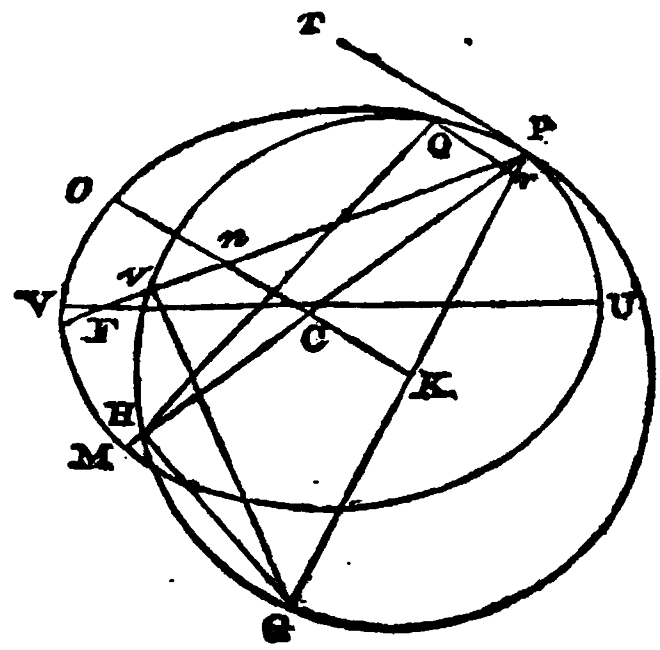
For since by hypothesis $PN : QN :: OC : oC$, (22. 6.) $PN^2 : QN^2 :: OC^2 : oC^2$. But (Art. 80.) $RN.NZ : PN^2 :: CR^2 : OC^2$, $\therefore$ *ex æquo* (22. 5.) $RN.NZ : QN^2 :: CR^2 : Co^2$ which (by Art. 80.) is the property of the ellipse; $\therefore$ the curve $RQoZ$ is an ellipse. Q. E. D.

84. If $PQMG$ be the circle of curvature at the point P in the ellipse PVU, PG the diameter of curvature, and PH, Pv

the chords of curvature passing through the centre C, and focus F respectively; then will

$$CP : CO :: CO : \tfrac{1}{2}PH.$$
$$PK : CO :: CO : \tfrac{1}{2}PG.$$
$$VC : CO :: CO : \tfrac{1}{2}Pv.$$

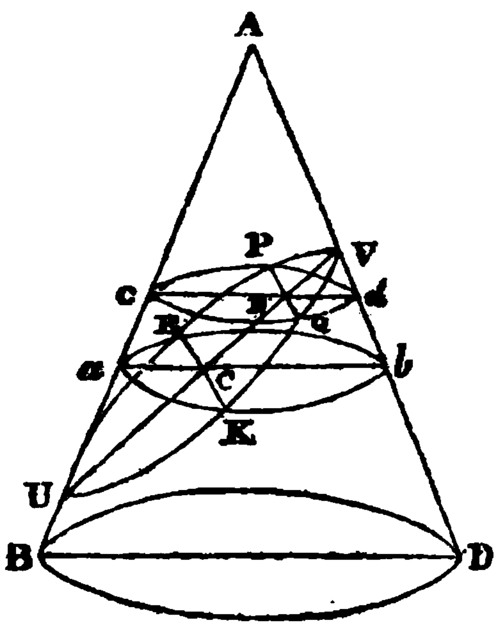
Join PC and produce it to M, and join Gv, HQ and QP; draw the tangent TP, and through Q and C draw Qr, OCK each parallel to TP, then will OC be the semi-conjugate diameter and Qr an ordinate to PH, and let QP be the *arc* in its nascent state, which may therefore be considered as common to the circle and ellipse. Then because the angle $TPQ=PHQ$ (32. 3.) $=PQr$ (29. 1.) and QPr is common to the two triangles QPr, QPH, these triangles are equiangular (32. 1.), ∴ $Pr : PQ :: PQ : PH$ (4. 6.); ∴ $Pr.PH = PQ^2 =$ (since the *arc* QP is indefinitely small, see Art. 35.) Qr^2; ∴ $Pr.rM : Pr.rH :: PC^2 : CO^2$ (Art. 80.), ∴ ($rM : rH$, that is since r and P are indefinitely near coinciding) $2PC : PH :: PC^2 : CO^2$; ∴ (15. 5.) $PC : \tfrac{1}{2}PH :: PC^2 : CO^2$, ∴ (cor. 2, 20. 6.) $PC : CO :: CO : \tfrac{1}{2}PH$. Since CK is parallel to TP, and TP perpendicular to PG (cor. 16. 3.), CKP is a right angle (29. 1.), also PHG is a right angle (31. 3.), and the angle HPG common to the triangles PKC, PHG ∴ these triangles are equiangular, and (4. 6.) $PK : PC :: PH : PG :: \tfrac{1}{2}PH : \tfrac{1}{2}PG$. But $PC : CO :: CO : \tfrac{1}{2}PH$, ∴ *ex æquo* (22. 5.) $PK : PC :: CO : \tfrac{1}{2}PH$, and $PC : CO :: \tfrac{1}{2}PH : \tfrac{1}{2}PG$, ∴ $PK : CO :: CO : \tfrac{1}{2}PG$. Again, the triangles PnK, PvG having the angles at K and v right angles, and the angle at P common, are similar (32. 1.); ∴ ($Pn=$by Art. 60.) $VC : PK :: PG : Pv :: \tfrac{1}{2}PG : \tfrac{1}{2}Pv$, and $PK : CO :: CO : \tfrac{1}{2}PG$ ∴ *ex æquo* $VC : CO :: CO : \tfrac{1}{2}Pv$. Q. E. D.



Cor. Hence $VU : 2CO :: 2CO : Pv$, that is, the chord of curvature Pv which passes through the focus F, is a third proportional to the major axis, and the conjugate diameter, and is consequently equal to the parameter of the diameter PM. (Art. 81.)

85. If a plane cut a cone so as neither to meet the base nor be parallel to it, the section will be an ellipse.

Let ABD be a cone, and let the section $VEUK$ be perpendicular to ABC the plane of the generating triangle, VU being their common section, and the section $PcQd$ be parallel to the base and therefore a circle, and let its common sections with ABD and $VEUK$ be cd and PQ; let $aEKb$ be a section likewise parallel to the base, bisecting VU in C, having EK and ab for its common sections with the planes $VEUK$ and ABD. Because ABD and $PcQd$ are both perpendicular to $VEUK$, their common section PQ is perpendicular to ABD (19. 11.) and therefore perpendicular to VU and cd (conv. 4. 11.), in like manner



it may be shewn that EK is perpendicular to VU and ab, ∵ EK and PQ are bisected in C and N (3. 3.); and since cd and ab are parallel (16. 11.), ∵ the triangles UNc, UCa are equiangular, and $UN : Nc :: UC : Ca$, also $NV : Nd :: (CV=) \; UC : Cb$, ∵ by compounding the terms of these analogies $UN.NV : Nc.Nd :: UC^2 : Ca.Cb$. But $Nc.Nd = PN^2$ and $Ca.Cb = EC^2$ (14. 2.), ∵ $UN.NV : PN^2 :: UC^2 : EC^2$ which (by Art. 67.) is the property of the ellipse; therefore $VEUK$ is an ellipse, consequently if a cone be cut by a plane which neither meets the base nor is parallel to it, the section will be an ellipse. Q. E. D.

THE HYPERBOLA.

DEFINITIONS.

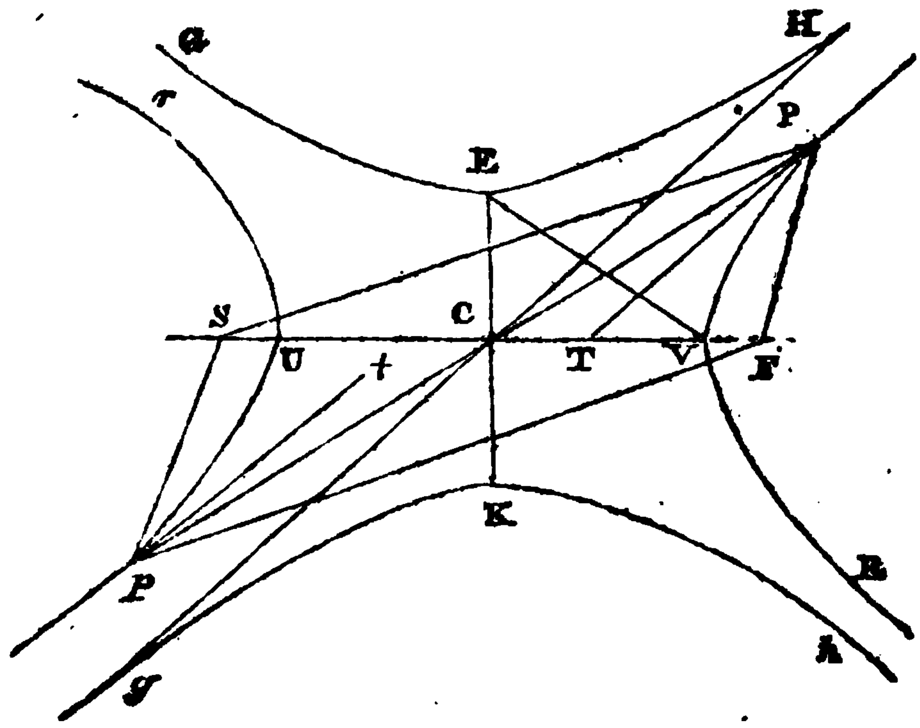
86. If two straight lines FP, SP revolve about the fixed points F and S, and intersect each other in P, so that $SP - FP$ may always equal any given straight line Z, the point P will describe the figure PVR which is called AN HYPERBOLA.

87. If two straight lines Fp, Sp revolve in like manner about F and S, so that $Fp - Sp$ may always equal the given straight line Z, the point p will likewise describe an hyperbola pUr; this figure and the former, with respect to each other, are called OPPOSITE HYPERBOLAS.

88. The fixed points F and S about which the straight lines FP and SP, Fp and Sp revolve, are called THE FOCI.

89. If F, S be joined, the straight line UV intercepted between the opposite hyperbolas is called THE MAJOR AXIS, and the points U, V are called THE PRINCIPAL VERTICES.

90. If UV be bisected in C, the point C is called THE CENTRE.



91. If through the centre C the straight line EK be drawn perpendicular to the major axis UV, and if from V as a centre, with the distance CF a circle be described, cutting EK in the points E and K, the straight line EK is called THE MINOR AXIS.

Cor. Hence $EC=CK$ (3. 3.).

92. If $EC=CV$, that is, $EK=UV$ the hyperbola is called EQUILATERAL.

93. If with EK as a major axis, and UV as a minor axis two opposite hyperbolas GEH, gKh be described, these are called CONJUGATE HYPERBOLAS.

94. Any straight line passing through the centre C, and terminated by the two opposite hyperbolas, is called A DIAMETER.

Thus Pp is a diameter to the point P, or p.

95. A straight line meeting the curve at any point, and which being produced does not cut it, is called A TANGENT to that point.

Thus PT is a tangent at the point P.

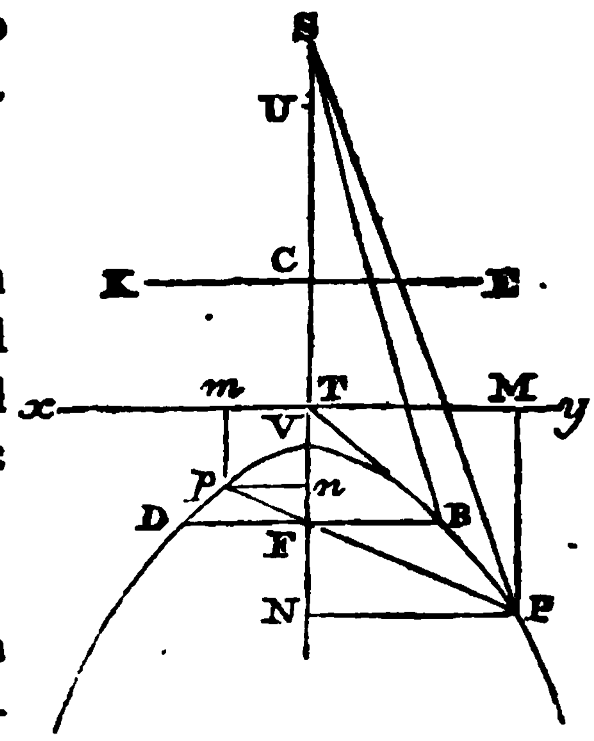
96. If Pp be a diameter, and PT a tangent at the point P, and through the centre C a straight line Hg be drawn parallel

to the tangent PT, the line Hg is called THE CONJUGATE DIAMETER to Pp.

97. If through the focus F a straight line DB be drawn, perpendicular to the axis FS, meeting the curve in B and D, DB is called THE LATUS RECTUM or principal parameter.

98. A tangent at the extremity of the latus rectum produced to meet the axis, is called THE FOCAL TANGENT.

Thus BT is the focal tangent.



99. A straight line drawn through the point where the focal tangent meets the axis, and parallel to the latus rectum, is called THE DIRECTRIX.

Thus xy is the directrix.

100. A straight line drawn from any point in the curve, perpendicular to the axis, is called AN ORDINATE TO THE AXIS at that point.

Thus PN is an ordinate to the axis at the point P.

100 B. The segments of the axis, intercepted between the ordinate and the vertices of the opposite hyperbolas, are called ABSCISSAS.

Thus U and V being the vertices, and PN the ordinate, VN and NU are the abscissas.

101. If PG be a diameter and PT the tangent at the point P, a straight line drawn from any point Q in the curve, parallel to PT, and meeting PG produced in v, is called AN ORDINATE to the diameter PG; see the figure to Art. 141.

102. If the ordinate to any diameter pass through the focus, and meet the curve on the opposite side, the ordinate thus produced is called THE PARAMETER to that diameter.

Thus bd is the parameter to the diameter PG. See the figure to Art. 141.

103. An asymplote is a straight line passing through the centre, which continually approaches the curve, but does not meet it, except at an infinite distance from the vertex; or, it is a tangent to the curve at an infinite distance.

Thus (see the figure to Art. 134.) *CX, Cx are the asymptotes.*

PROPERTIES OF THE HYPERBOLA.

104. The difference of the two straight lines drawn from the foci to any point in the curve, is equal to the major axis; that is, $SP-FP=UV=2VC$. (See the figure to Art. 89.)

For since $SP-FP$ is a constant quantity in whatever point of the curve P be taken (Art. 86.), let the points P, p be supposed to arrive at V and U respectively, then SP will become SV, and FP will become FV, ∴ $SP-FP$ will become $SV-FV$; in like manner $Fp-Sp$ will (by the arrival of the point p at U) become $FU-SU$, ∴ $SV-FV=FU-SU$ (Art. 87.); but $SV=VU+SU$ and $FU=VU+FV$, ∴ $VU+SU-FV=VU+FV-SU$ ∴ $2SU=2FV$ and $SU=FV$; ∴ $SP-FP=SV-FV=SV-SU=UV=$(Art. 90.) $2VC$. Q. E. D.

Cor. 1. Hence the foci are equally distant from the centre and likewise from the vertices, that is, $SC=FC$, $SU=FV$, and $SV=FU$.

Cor. 2. Hence $SC=UV+FP=2VC+FP$; and $SP+FP=2VC+2FP$.

Cor. 3. Because $BS-BV=UV$ (see the figure to Art. 97.) $=FS-2VF$, and $BS \succ FS$ ∴ $BF \succ 2VF$ and $(2BF=)\ BD \succ 4VF$ ∴ the latus rectum is greater than four times the distance of the focus F from the vertex V.

105. The rectangle $VF.FU=EC^2$ (see the figure to Art. 89.)

For $EC^2=VE^2-VC^2$ (47. 1.) $=FC^2-VC^2$ (Art. 91.) $=\overline{FC+VC}.\overline{FC-VC}$ (cor. 5. 2.). But $FC+VC=FU$ (cor. 1. Art. 104.) and $FC-VC=VF$, ∴ $VF.FU=EC^2$. Q. E. D.

For the same reason $US.SV=EC^2$.

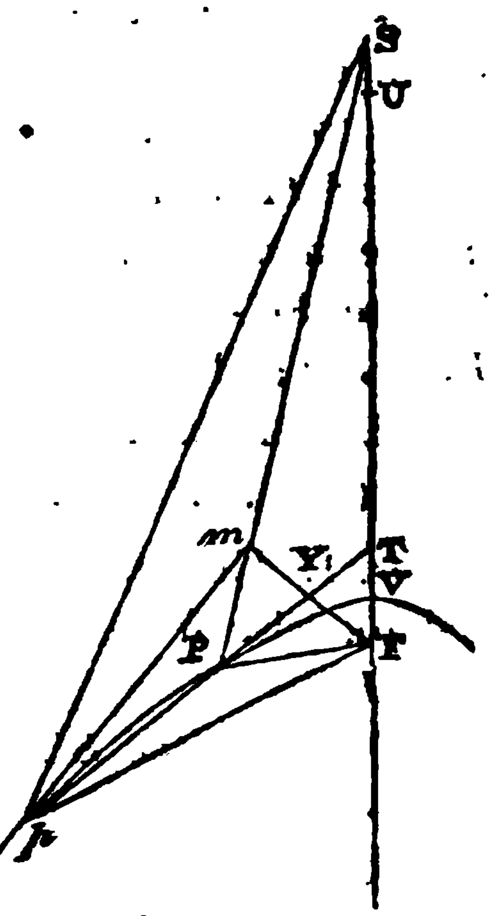
106. The latus rectum is a third proportional to the major and minor axis; or $VU : EK :: EK : BD$ (see the figure to Art. 97.).

Because $BS^2=\overline{2VC+FB}^2$ (cor. 2. Art. 104.) $=4VC^2+FB^2+4VC.FB$ (4. 2.). And $BS^2=FS^2+FB^2$ (47. 1.) $=4FC^2+FB^2$ (4. 2.), ∴ $4VC^2+4VC.FB=4FC^2$; and $VC^2+VC.FB=FC^2$; ∴ $VC.FB=FC^2-VC^2=$(Art. 105.) EC^2, ∴ $VC : EC :: EC : FB$ (17. 6.), ∴ $VU : EK :: EK : BD$ (15. 5.) Q. E. D.

Cor 1. Hence $EC^2 = \frac{1}{4}L.VC$, and $EK^2 = L.VU$.

Cor. 2. Hence, in the equilateral hyperbola, because $VU = EK$ (Art. 92.) ∵ $BD = EK$ (prop. A. 5.); that is, the major axis, minor axis, and latus rectum, are equal to each other.

107. If FP, SP be drawn from the foci to any point P in the curve, the straight line PT which bisects the angle FPS will be a tangent at P.



For if not, let PT meet the hyperbola again in p, draw FY perpendicular to PT meeting it in Y, produce FY to m, and join pS, pm, and pF.

In the triangles FPY, mPY, the angle $mPY = FPY$ by hypothesis, the angles at Y right angles by construction, and PY common, ∵ (26. 1.) $FY = mY$; ∵ in the triangles FpY, mpY, the sides FY, $Yp = mY$, Yp each to each, and the included angles at Y right angles, ∵ (4. 1.) $Fp = mp$; ∵ $Sp - pF = Sp - pm$. But $Sp - pF = SP - PF$ (Art. 86.) $= SP - Pm = Sm$, ∵ $Sp - pm = Sm$, and $Sp = Sm + pm$ which (20. 1.) is absurd, ∵ TP cannot possibly meet the hyperbola again in any point p, ∵ TP touches the curve. Q. E. D.

Cor. 1. Hence the tangent at the vertex V is perpendicular to the axis SF. *See cor.* 1. *Art.* 57.

Cor. 2. Hence (3. 6.) $ST : TF :: SP : PF$.

108. All the diameters of the hyperbola are bisected by the centre C. (See the figure to Art. 89.)

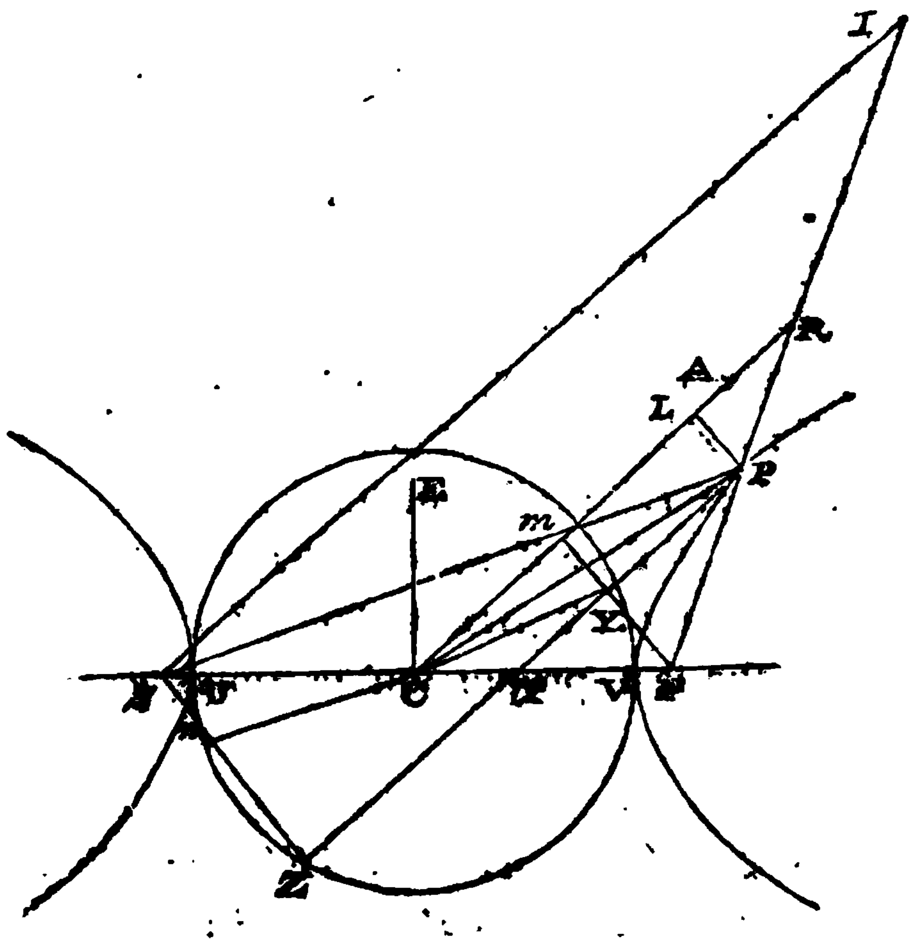
Complete the parallelogram $PSpF$, then (34. 1.) $Sp = FP$ and $SP = pF$, ∵ $Fp - pS = SP - PF$, ∵ (Art. 87.) the point p is in the opposite hyperbola; join Pp, ∵ (part 8. Art. 241. cor.) $SC = CF$ and $pC = CP$, and the like may be shewn of any other diameter. Q. E. D.

Cor. 1. Hence the tangents PT, pt at the points P and p are parallel, for since (34. 1.) $SPF = SpF$ and these angles are bisected by PT and pt (Art. 107.) their halves will be equal; that is, $TPp = tpP$, ∵ (27. 1.) PT is parallel to pt.

Cor. 2. Hence, if tangents be drawn at the extremities of two conjugate diameters Pp, Hg, the four tangents will form a parallelogram.

109. If CR be parallel to a tangent PT, cutting FP produced in R, then will $PR=VC$.

Draw SI parallel to CR, and join PS. Then because the angle $PSI=SPY$ (29. 1.) $=YPF$ (Art. 107.) $=FIS$ (29. 1.), $\because$ $PS=PI$ (6. 1.) $=PR+RI$; then (2. 6.) $FC:CS::FR:RI$. But (cor. 1. Art. 104.) $FC=CS$, $\because$ (prop. A. 5.) $FR=RI$; $\because$ $PS=PR+FR=PR+FP+PR=2PR+FP$. But (cor. 2. Art. 104.) $PS=2VC+FP$, $\because$ $2PR+FP=2VC+FP$, $\because$ $PR=VC$. Q. E. D.



110. If the tangent PT be produced, and straight lines SZ, FY be drawn from the foci perpendicular to it, the points Y and Z will be in the circumference of the circle described on the major axis UV as a diameter.

Join CY and produce FY to meet SP in m, then since the triangles mPY, FPY are equal and similar (Art. 107.), $FY=mY$ and $FP=MP$; $\because$ $Sm=(SP-Pm=)$ $SP-FP=2VC$ (Art. 109.). Next, because $FC=CS$ and $FY=Ym$, $\because$ (2. 6.) CY is parallel to Sm, and the triangles FCY, FSm are equiangular (29. 1.), $\because$ $FC:CY::FS:Sm$ (4. 6); but $FC=\frac{1}{2}FS$, $\because$ $CY=\frac{1}{2}Sm$ (16. and prop. D. 5.) $=VC$ by what has been shewn above, $\because$ V and Y are in the circumference of the circle of which C is the centre. Produce YC to n, then since $CY=\frac{1}{2}Sm$, $\because$ $nY=Sm$, and they are parallel, $\because$ (33. 1.) Sn, mY are equal and parallel, and if Sn be produced to meet YZ in Z, then $SZY+mYZ=2$ right angles (29. 1.); but mYZ is a right angle

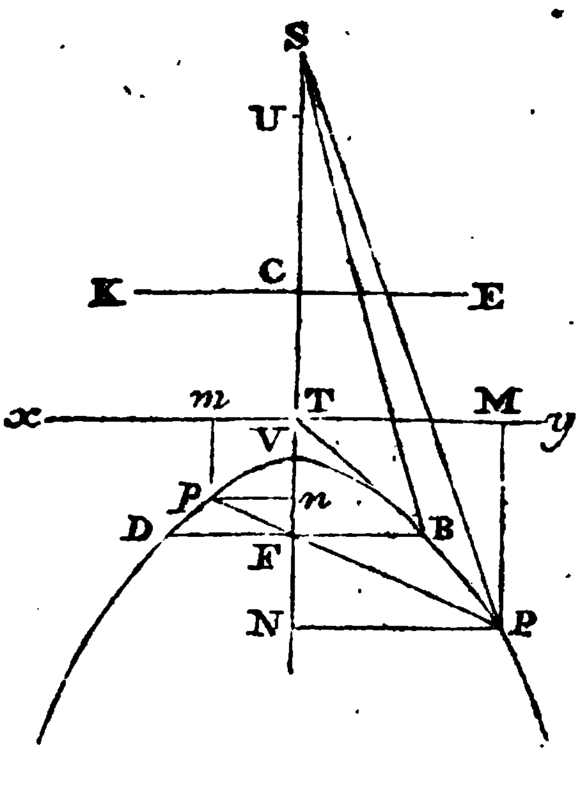
$\because$ SZY is a right angle; that is, the straight line passing through S and n is perpendicular to YZ; and since nY is a diameter of the circle, and nZY a right angle, Z is in the circumference (31. 3.) Q. E. D.

111. The rectangle $FY.SZ=EC^2$.

For since Z is a right angle (Art. 110.), and nC, CY meet at the centre C, they are both in the same straight line (31. 3.) $\because$ $FCY=SCn$ (15. 1.), also $SC=CF$ (cor. 1. Art. 104.), and $nC=CY$, $\because$ (4. 1.) $FY=Sn$. But (cor. 36. 3.) $SZ.Sn=VS.SU$; that is, $FY.SZ=VS.SU=$(Art. 105.) EC^2. Q. E. D.

Cor. 1. Because the triangles FPY, SPZ have the angles at P equal (Art. 107.) and the angles at Y and Z right angles, they are equiangular (32. 1.), and $FY : FP :: SZ : SP$ (4. 6.), $\because$ $FY : SZ :: FP : SP$ (16. 5.). But $FY.SZ=EC^2$; $\because$ (17. 6.) $FY : EC :: EC : SZ$, and (cor. 2, 20. 6.) $FY^2 : EC^2 :: FY : SZ$. But since $FY : SZ :: FP : SP$ $\because$ $FY^2 : EC^2 :: FP : (SP=)$ $2VC+FP$ (cor. 2. Art. 104.) If $VC=a$, $EC=b$, $FP=x$, and $FY=y$ the last proportion becomes $y^2 : b^2 :: x : 2a+x$, $\because$ $y^2=\frac{b^2x}{2a+x}$.

Cor. 2. Hence $4FY^2 : 4EC^2 :: FP : SP :: L.FP : L.SP$ (15. 5.), $\because$ $4FY^2 : L.FP :: 4EC^2 : L.SP$ (16. 5.) $:: L\times 2VC$* $: L.SP :: 2VC : (SP=)\ 2VC+FP$ (cor. Art. 104.)



112. If BD be the focal tangent, then will the rectangle $CF.CT=VC^2$.

For since (cor. 2. Art. 107.) $ST : FT :: SB : BF$, $\because$ (18. and 17. 5.) $ST+FT$ or $2CF : ST-FT$ or $2CT :: SB+BF : SB-BF$, $\because$ (15. 5.) $\overline{4CF^2 : 4CF.CT} :: \overline{SB+BF}.\overline{SB-BF} : (\overline{SB-BF}.\overline{SB-BF}=)\ \overline{SB-BF}|^2 :: SB^2-BF^2 : (UV^2=)\ 4VC^2$, see Art. 86. But since SFB is a right angle, $4CF^2\ (=SF^2)=SB^2-BF^2$ (47. 1.), $\because$ $4CF.CT=4VC^2$ (14. 5.) and $CF.CT=VC^2$. Q. E. D.

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* For $2VC : 2EC :: 2EC : L$ (Art. 106.) $\because$ $L.2VC=4EC^2$.

Cor. 1. Hence (17.6.) $CF : VC :: VC : CT$.

Cor. 2. Because $CT=CF-FT$, $\therefore CF.CF-CF.FT=CF.CT=VC^2$; $\therefore CF.FT=CF^2-VC^2=$(Art. 105.) $EC^2 \therefore CF : EC :: EC : FT$.

113. If from any point P in the curve, PM be drawn perpendicular to the directrix xy, then will $FP : PM :: CF : CV$.

Join SP and draw PN perpendicular to the axis UV produced, then because (47. 1.) $SP^2=SN^2+NP^2$ and $FP^2=FN^2+NP^2$, by taking the latter from the former $SP^2-FP^2=SN^2-FN^2$, that is (cor. 5. 2.) $\overline{SP+FP}.\overline{SP-FP}=\overline{SN+FN}.\overline{SN-FN}$; $\therefore$ (16. 6.) $SP+FP : SN+FN :: SN-FN : SP-FP$. But (cor. 2. Art. 104.) $SP+FP=2VC+2FP$; also $SN+FN=SC+CN+FN=CF+CN+FN=2CN$, and $SN-FN=SF=2CF$, likewise (Art. 104.) $SP-FP=2VC$; $\therefore$ if instead of the terms of the above analogy, their equals be substituted, we shall have $2VC+2FP : 2CN :: 2CF : 2VC$, or $VC+FP : CN :: CF : VC ::$ (cor. 1. Art. 112.) $VC : CT$, $\therefore$ (cor. 19. 5.) $FP : (NT=) PM :: VC : CT :: CF : VC$. Q. E. D.

Cor. Hence, if P be supposed to coincide with B, FP will become FB and PM will$=FT$; $\therefore$ the above analogy becomes $(FP : PM ::) FB : FT :: FC : VC$.

114. If PF be produced to meet the curve again in p, then will $2FP.Fp=FB.\overline{FP+Fp}$.

Because (cor. Art. 113.) $FP : PM :: FB : FT$, $\therefore$ (16 and cor. 19. 5.) $FP-FB : (PM-FT=) FN :: FB : FT$. But (cor. Art. 113.) $FB : FT :: Fp : pm$, $\therefore FB-Fp : (FT-pm=) Fn :: FB : FT$; $\therefore FP-FB : FN :: FB-Fp : Fn$. But the triangles $FPN.Fpn$ are similar, $\therefore FN : FP :: Fn : Fp$, $\therefore$ (22. 5.) $FP-FB : FP :: FB-Fp : Fp$, $\therefore$ (16. 5.) $FP.Fp-FB.Fp=FB.FP-FP.Fp$; or $2FP.Fp=FB.\overline{FP+Fp}$. Q. E. D.

Cor. Hence, if $FB=l$, $FP=X$, and $Fp=x$, we shall have $2Xx=l.\overline{X+x}$, and $\dfrac{2}{l}=\dfrac{X+x}{Xx}=\dfrac{1}{x}+\dfrac{1}{X}$.

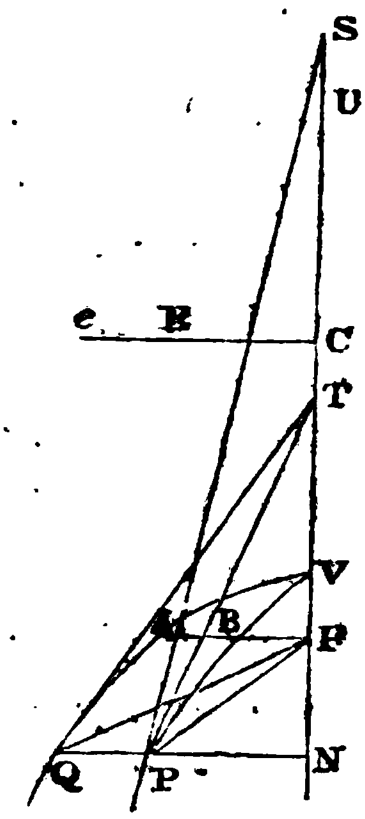
115. If c be the co-sine of the angle PFU to the radius 1, then will $FP : EC :: EC : VC+c.CF$.

For (Art. 113.) $FP : PM :: CF : VC$, $\therefore$ (16. 6.) $FP.VC=CF.PM=$ (34. 1.) $CF.TN=CF.\overline{TF\mp FN}=CF.TF\mp CF.FN$. But (cor. 2. Art. 112.) $CF.TF=EC^2$, and (Art. 63. Part 9.) FN;

$FP :: \pm c : 1$; $\because FN = \pm c.FP$ (16. 6.), and $\mp CF.FN = -c.FP.CF$, $\therefore$ from the first equation by substitution $FP.VC = EC^2 - c.FP.CF$, or $FP.VC + c.FP.CF = EC^2$, that is $FP.\overline{VC+c.CF} = EC^2$; $\therefore$ (16.6.) $FP : EC :: EC : VC + c.CF$. Q. E. D.

116. If PN be an ordinate to the major axis VU, then will $VN.NU : PN^2 :: VC^2 : EC^2$.

For (Art. 113) $\overline{SP+PF}.\overline{SP-PF} = \overline{NS+NF}.\overline{NS-NF}$, $\therefore$ (16. 6.) $NS+NF : SP-PF :: SP+PF : NS-NF$ which by substitution (as in the latter part of Art. 113.) becomes $2CN : 2VC :: 2VC+2FP : 2CF$, $\therefore CN : VC :: VC+FP : CF$; whence by composition (18. 5.), and division (17. 5.) we obtain the following analogies, viz.



First $CN-VC : VC :: VC+FP-CF : CF$.

Secondly $CN+VC : VC :: VC+FP+CF : CF$.

By adding the antecedents and consequents in the first, and subtracting in the second (12. and 19. 5.) we have

$CN-VC : VC :: CN+FP-CF : CF+VC$

$:: FP+FN : CF+VC$

$CN+VC : VC :: FP+CF-CN : CF-VC$

$:: FP-FN : CF-VC$

$\therefore$ compounding the ratios (23. 6.)

$\overline{CN-VC}.\overline{CN+VC} : VC^2 :: \overline{FP+FN}.\overline{FP-FN} : \overline{CF+VC}.\overline{CF-VC}$, or $VN.NU : VC^2 :: (FP^2 - FN^2 =) PN^2 : (CF^2 - VC^2 =$ by Art. 105.) EC^2 $\therefore$ alternately $VN.NU : PN^2 :: VC^2 : EC^2$. Q. E. D.

Cor. Hence, because $VN.NU = \overline{CN-VC}.\overline{CN+VC} = CN^2 - VC^2$ (cor. 5. 2.) $\therefore$ by substitution $CN^2 - VC^2 : PN^2 :: VC^2 : EC^2$; wherefore, if $VC = a$, $EC = b$, $CN = x$, and $PN = y$, we shall have $x^2 - a^2 : y^2 :: a^2 : b^2$, whence $y^2 = \frac{b^2}{a^2}.\overline{x^2 - a^2}$.

117. If two hyperbolas VP, VQ be described on the same major axis, having eC and EC respectively for the semi-minor axes; and if NP be produced to Q, then will $QN : PN :: eC : EC$.

For (Art. 116.) $\begin{cases} VN.NU : PN^2 :: VC^2 : EC^2 \\ VN.NU : QN^2 :: VC^2 : eC^2 \end{cases}$

$\therefore$ ex æquo $QN^2 : PN^2 :: eC^2 : EC^2$, and (26. 6.) $QN : PN :: eC : EC$. Q. E. D.

Cor. 1. Hence it may be shewn, as in Art. 72. that tangents at P and Q will meet the axis produced in the same point T; that the area VQN : area VPN :: $eC : EC$, and that if F be any point in the axis, the area VQF : area VPF :: $eC : EC$.

Cor. 2. Hence, if VQ be an equilateral hyperbola, or $VC=eC$ (Art. 92.); then since $VN.NU : QN^2 :: VC^2 : eC^2$ (Art. 116.) $VN.NU=QN^2$ (prop. A.5.)

118. In the equilateral hyperbola, the latus rectum is equal to the minor axis, that is $2Fb=2eC$.

For since (Art. 105.) $VF.FU=eC^2$, if the point N be supposed to coincide with F, the expression (cor. 2. Art. 117.) $VN.NU=QN^2$ will become $VF.FU=Fb^2$, $\therefore Fb^2=eC^2$, $Fb=eC$, and $2Fb=2eC$. Q. E. D.

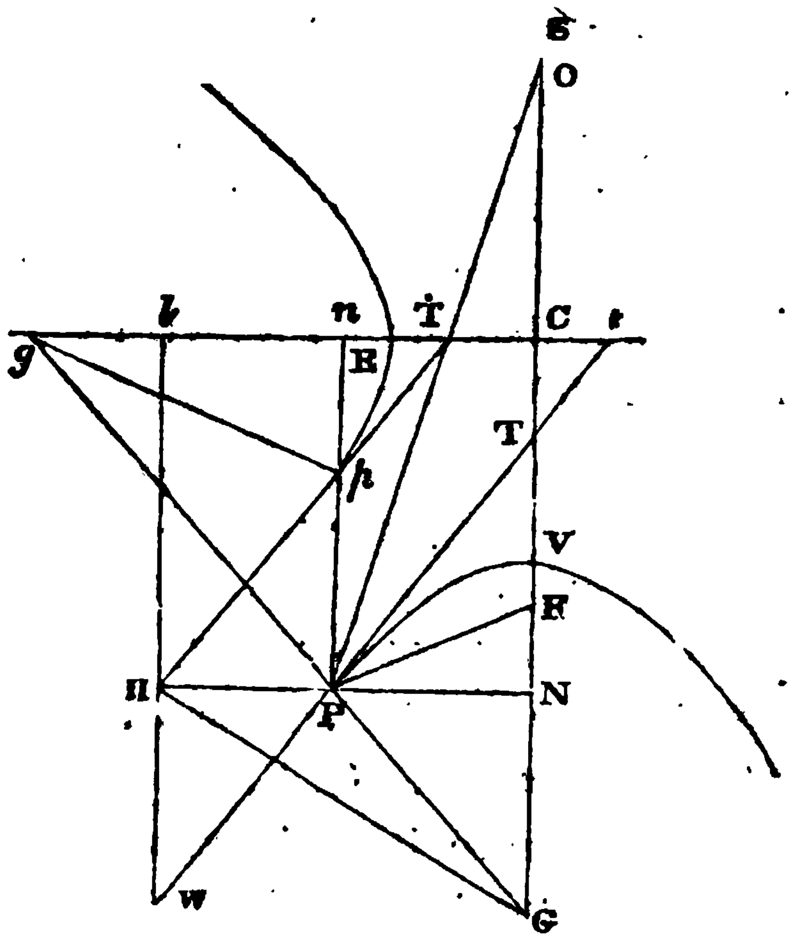
Cor. 1. Hence it again appears that the major axis, minor axis, and latus rectum of an equilateral hyperbola, are equal to each other.

Cor. 2. Hence, because (Art. 106.) $VC : EC :: EC : BF$, $\therefore$ (cor. 2, 20. 6.) $VC : BF :: VC^2 : EC^2$. But (Art. 116.) $VC^2 : EC^2 :: VN.NU$ or $CN^2-CV^2 : PN^2$, $\therefore VN.NU$ or $CN^2-CV^2 : PN^2 :: VC : BF$.

119. If Pn be an ordinate to the minor axis EC, then will $Cn^2+EC^2 : PN^2 :: EC^2 : VC^2$ (see the following figure.)

For (34. 1.) $Pn=NC$ and $Cn=NP$ $\therefore$ (cor. Art. 116.) $Pn^2-VC^2 : Cn^2 :: VC^2 : EC^2$, $\therefore$ by adding antecedents and consequents $Pn^2 : Cn^2+EC^2 :: VC^2 : EC^2$ and by inversion $Cn^2+EC^2 : Pn^2 :: EC^2 : VC^2$. Q. E. D.

120. If PN any ordinate to the major axis be produced to meet the conjugate hyperbola in π, then will $\pi N^2-PN^2=2EC^2$.



Because (cor. Art. 116.) $Cb^2 - EC^2 : \Pi b^2 :: EC^2 : VC^2$ ∵ (16. 5.) $Cb^2 - EC^2 : EC^2 :: (\Pi b^2 =) CN^2 : VC^2$, and (17. 5.) $Cb^2 - 2EC^2 : CE^2 :: CN^2 - CV^2 : CV^2 ::$ (by alternation and inversion in cor. 2. Art. 118.) $PN^2 : EC^2$, ∵ (9. 5.) $Cb^2 - 2EC^2 = PN^2$, ∵ $Cb^2 - PN^2 = 2EC^2$, but (34. 1.) $Cb - PN = \Pi N - PN$, ∵ $\Pi N^2 - PN^2 = 2EC^2$; and in like manner it may be shewn, that if $b\Pi$ be produced to meet the hyperbola VP in the point w, $wb^2 - \Pi b^2 = 2VC^2$. Q. E. D.

121. If PT be a tangent at the point P, then will $CN.CT = VC^2$.

Because (cor. 2. Art. 107.) $ST : TF :: SP : PF$, ∵ dividendo et componendo) $ST - TF : ST + TF :: SP - PF : SP + PF$; that is (see Art. 112.) $2CT : SF :: 2VC : SP + PF$. But (Art. 113.) $\overline{SN - NF} . \overline{SN + NF} = \overline{SP - PF} . \overline{SP + PF}$, ∵ since $SN - NF = SF$, $SP - PF = 2VC$ (Art. 104.), and $SN + NF = 2CN$, by substitution $SF . 2CN = 2VC . \overline{SP + PF}$ ∵ (16. 6.) $SF : 2VC :: SP + PF : 2CN$; but it has been shewn that $2CT : SF :: 2VC : SP + PF$ ∵ *ex æquo* $2CT : 2VC :: 2VC : 2CN$, that is $CT : VC :: VC : CN$, ∵ (17. 6.) $CN.CT = VC^2$. Q. E. D.

Cor. 1. Because $NT = CN - CT$, ∵ $CN.NT = CN.\overline{CN - CT} = CN^2 - CN.CT = CN^2 - VC^2$.

Cor. 2. Because in the equilateral hyperbola $CN^2 - VC^2 = PN^2$ (because $VC = EC$, see the cor. to Art. 116.) ∵ $CN.NT = (CN^2 - VC^2 =) PN^2$.

Cor. 3. Hence also, in the conjugate hyperbola $E\Pi$, if pn be an ordinate to the axis Eg, and $p\dot{T}$ a tangent at p, then will $Cn.C\dot{T} = EC^2$.

122. If Pn be an ordinate to the minor axis EC, and the tangent Pt meet EC in t, then will $Cn.Ct=EC^2$.

Because (Art. 121.) $CN.CT=VC^2$, ∵ (17. 6.) $CN : VC :: VC : CT$, ∵ (cor. 2, 20. 6.) $CN : CT :: CN^2 : VC^2$, ∵ (17. 5.) $NT : CT :: CN^2-VC^2 : VC^2$ :: (because by cor. Art. 116. $CN^2-VC^2 : PN^2 :: VC^2 : EC^2$, by alternation) $PN^2 : EC^2$. But the triangles TPN, TtC are similar, ∵ (4. 6.) $NT : CT :: PN : Ct$; ∵ (from above) $PN : Ct :: PN^2 : EC^2$, ∵ (16. 6.) $PN.EC^2=Ct.PN^2$, or $EC^2=Ct.PN$; But (34. 1.) $PN=Cn$, ∵ $Cn.Ct=EC^2$. Q. E. D.

Cor. Hence, because $Cn.C\dot{T}=EC^2$ (cor. 3. Art. 121.) ∵ $Cn.Ct=Cn.C\dot{T}$ and $Ct=C\dot{T}$; that is, if the perpendicular Pn cut the conjugate hyperbola in p, and tangents be drawn at P and p, the points t and $\dot{T}$ where they meet the minor axis, will be equally distant from the centre C; and conversely, if $Ct=C\dot{T}$, the perpendicular Pn will pass through the point p.

123. The same things remaining $nt : n\dot{T} :: nP^2 : np^2$.

For by the preceding corollary $Cn.C\dot{T}=EC^2$, ∵ (17. 6.) $Cn : EC :: EC : C\dot{T}$, ∵ (cor. 2, 20. 6.) $Cn : C\dot{T} :: Cn^2 : EC^2$, ∵ (componendo et dividendo) $Cn+C\dot{T}$ or $nt : Cn-C\dot{T}$ or $n\dot{T} :: Cn^2+EC^2 : Cn^2-EC^2$. But (Art. 119.) $Cn^2+EC^2 : Pn^2 :: EC^2 : VC^2$ and (cor. Art. 116.) $Cn^2-CE^2 : np^2 :: EC^2 : VC^2$ ∵ (11. 5.) $Cn^2+EC^2 : nP^2 :: Cn^2-EC^2 : np^2$, ∵ (alternando) $Cn^2+EC^2 : Cn^2-EC^2 :: nP^2 : np^2$; that is, $nt : n\dot{T} :: nP^2 : np^2$. Q. E. D.

124. The normals at P and p will meet the minor axis in the same point g.

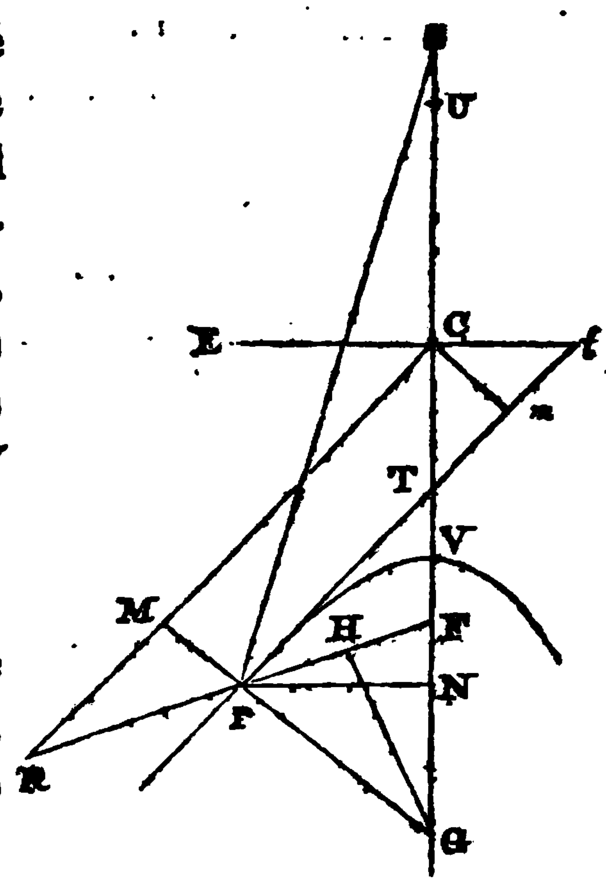
For the angles $gp\dot{T}$, gPt being right angles $nP^2=nt.ng$ and $np^2=n\dot{T}.ng$ (14. 2.) ∵ $nP^2 : np^2 :: nt.ng : n\dot{T}.ng$, ∵ (Art. 123.) $nt : n\dot{T} :: nt.ng : n\dot{T}.ng :: ng : ng$; that is, the normals at P and p cut the minor axis at equal distances from n or in the same point g. Q. E. D.

Cor. In like manner it is shewn, that if NP be produced to meet the conjugate hyperbola in Π, the normals from these points will meet the major axis in the same point G.

125. If CR be parallel to a tangent at P, and MPG perpendicular to it, then will the rectangle $PM.PG=EC^2$.

Let PN be the ordinate, and draw Cm perpendicular to the tangent Pt. Because in the triangles PTG, CTt, the angles at

P and C are right angles, and the vertical angles at T equal, $\therefore Ctm = PGN$, and the angles at m and N being right angles, the remaining angle $tCm = NPG$, $\therefore Cmt$, PNG are equiangular, and (4. 6.) $Cm : Ct :: PN : PG$, $\therefore$ (16. 6.) $Cm.PG = Ct.PN$, but $Cm = PM$ (34. 1.), $\therefore PM.PG = Ct.PN = EC^2$ (Art. 122.) Q. E. D.

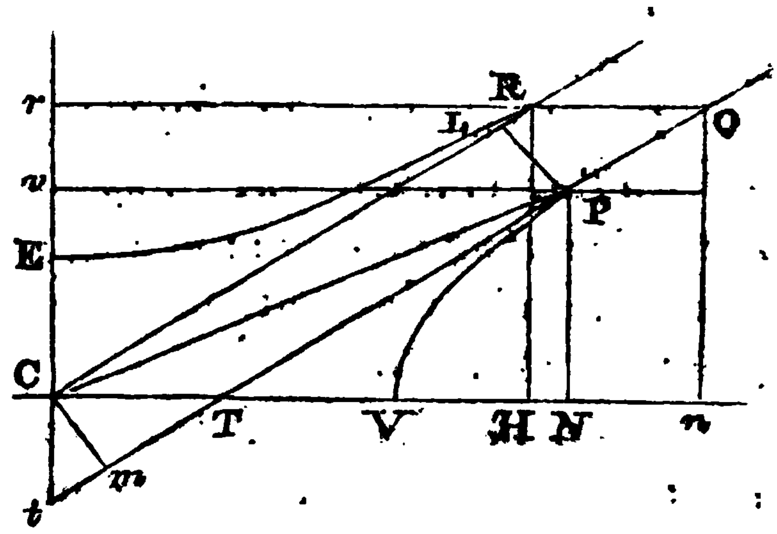


126. If from the point P the normal PG be drawn, PF joined, and GH drawn perpendicular to PF, then will $PH = \frac{1}{2}L$.

Produce GP, FP to M and R, then because the angles at H and M are right angles and those at P vertical, the triangles PHG, PMR are equiangular, and (4. 6.) $PG : PH :: PR : PM$, $\therefore$ (16. 6.) $PH.PR = PG.PM =$ (Art. 125.) $EC^2 =$ (cor. 1: Art. 106.) $\frac{1}{2}L.VC$. But (Art. 109.) $PR = VC$, $\therefore PH.PR = \frac{1}{2}L.PR$, or $PH = \frac{1}{2}L$. Q. E. D.

127. If CR be parallel to the tangent at P, and PN, RH perpendicular to the major axis, then will $CN^2 - CH^2 = VC^2$.

Draw rR an ordinate to the minor axis, and produce it to Q, and draw the ordinate Qn, Then (cor. Art. 116.) $Cn^2 - CV^2 : Qn^2 :: CN^2 - CV^2 : PN$ and $Qr^2 - VC^2 : RH^2 :: CN^2 - CV^2 : PN^2$. But (Art. 120.) $Qr^2 - Rr^2 = 2CV^2$, $\therefore Qr^2 - VC^2 = VC^2 + Rr^2 =$ (34. 1.) $VC^2 + CH^2$, $\therefore$ by substitution $VC^2 + CH^2 : RH^2 :: CN^2 - VC^2 : PN^2$. But the triangles CRH, TPN are similar, $\therefore$ (4. 6.) $RH : CH :: PN : TN$, and (22. 6.) $RH^2 : CH^2 :: PN^2 : TN^2$, $\therefore$ *ex æquo* $VC^2 + CH^2 : CH^2 :: CN^2 - VC^2 : TN^2 ::$ (cor. 1. Art. 121.) $CN.NT : TN^2 :: CN : TN$, $\therefore$ by conversion (prop. E. 5.) $VC^2 + CH^2 : VC^2 :: CN : (CN - TN =) CT ::$ (1. 6.) $CN^2 : CN.CT$. But (Art. 121.) $VC^2 = CN.CT$, $\therefore$ (14. 5.) $VC^2 + CH^2 = CN^2$, $\therefore CN^2 - CH^2 = VC^2$. Q. E. D.



Cor. Hence $CH^2 (=CN^2 - VC^2) : PN^2 :: VC^2 : EC^2$ (cor. Art. 116.) and $CH : PN :: VC : EC$ (22. 6.)

128. The same things remaining $CN : RH :: VC : EC$.

For (Art. 127.) $VC^4 + CH^2 : RH^2 :: CN^2 - VC^2 : PN^2 ::$ (cor. Art. 127.) $VC^2 : EC^2$, and $VC^2 + CH^2 = CN^2$, $\therefore CN^2 : RH^2 :: VC^2 : EC^2$ and (22. 6.) $CN : RH :: VC : EC$. Q. E. D.

129. If CR be parallel to the tangent PT and PN, RH ordinates to the major axis, then will $RH^2 - PN^2 = EC^2$.

Because (Art. 128.) $CN^2 : RH^2 :: VC^2 : EC^2 :: CN^2 - VC^2 : PN^2$ by subtracting antecedents and consequents $VC^2 : RH^2 - PN^2 :: CN^2 - VC^2 : PN^2 :: VC^2 : EC^2$, $\therefore$ (14. 5.) $RH^2 - PN^2 = EC^2$. Q. E. D.

Cor. Because $Cr^2 - Cv^2 = RH^2 - PN^2$ (34. 1.) $= EC^2$, and $CN^2 - CH^2 = VC^2$ (Art. 127.), $\therefore$ if CP be conjugate to CR, CR is also conjugate to CP.

130. If CP and CR be semi-conjugate diameters, then will $CP^2 - CR^2 = VC^2 - EC^2$.

Because (Art. 127.) $CN^2 - CH^2 = VC^2$, and (Art. 129.) $RH^2 - PN^2 = EC^2$, $\therefore$ by subtracting the latter from the former $CN^2 + PN^2 - CH^2 - RH^2 = VC^2 - EC^2$. But (47. 1.) $CP^2 = CN^2 + PN^2$, and $CR^2 = CH^2 + RH^2$, $\therefore$ $(\overline{CN^2 + PN^2} - \overline{CH^2 + RH^2} =)\ CP^2 - CR^2 = VC^2 - EC^2$. Q. E. D.

131. The same things remaining, if PL be drawn perpendicular to CR, then will $CR.PL = VC.EC$.

Draw Cm parallel to PL, then because (Art. 128.) $CN : RH :: VC : EC$, $\therefore$ (16 5.) $CN : VC :: RH : EC$. But the triangles CTm, RCH (having the alternate angles RCH, CTm equal (29. 1.), and the angles at H and m right angles) are similar, and (4. 6.) $CT : Cm :: CR : RH$, $\therefore$ (compounding the two latter proportions,) $CN.CT$ (=by Art. 121.) $VC^2 : VC.Cm :: RH.CR : RH.EC :: CR : EC$, $\therefore$ (15. 5.) $VC : Cm :: CR : EC$, $\therefore$ (16. 6.) $= CR.Cm = VC.EC$; but $Cm = PL$ (34. 1.), $\therefore CR.PL = VC.EC$. Q. E. D.

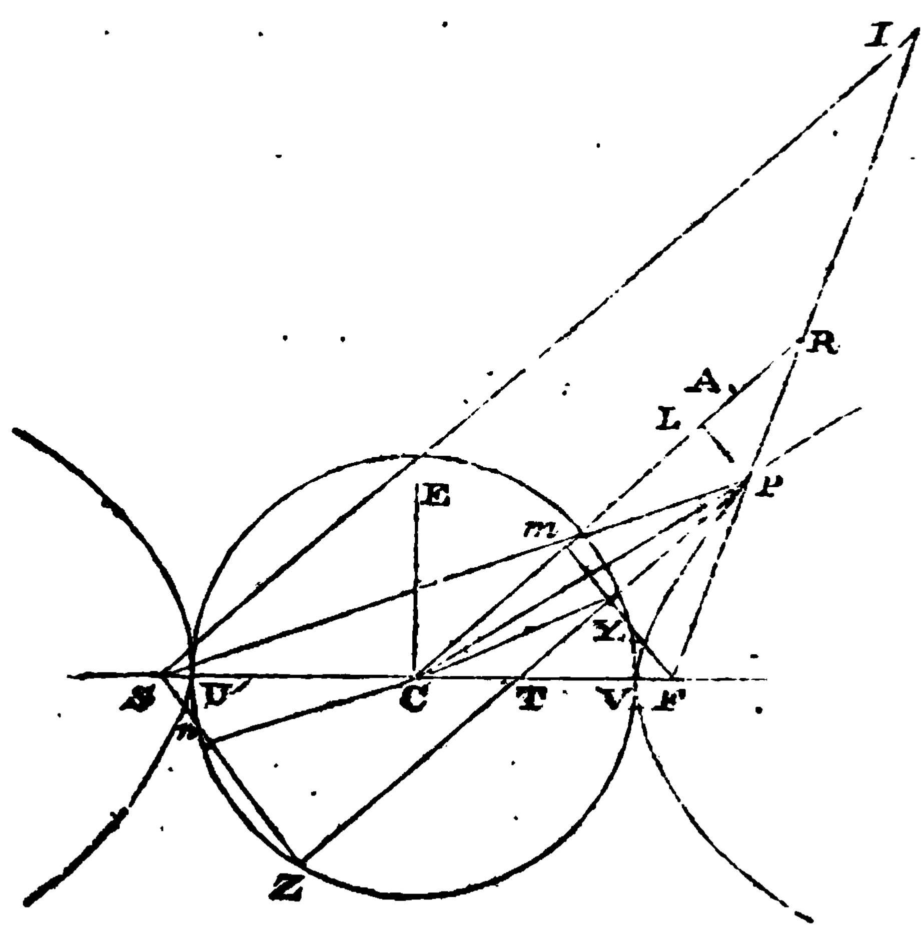
Cor. 1. Hence (16. 6.) $VC : PL :: CR : EC$, and (22. 6.) $VC^2 : PL^2 :: CR^2 : EC^2$.

Cor. 2. Let $VC = a$, $EC = b$, $CP = x$, and $PL = y$; then because $ab = CR.y$, $\therefore y^2 = \frac{a^2 b^2}{CR^2}$. But (Art. 130.) $x^2 - CR^2 = a^2 - b^2$, $\therefore CR^2 = x^2 - a^2 + b^2$ $\therefore y^2 = \frac{a^2 b^2}{x^2 - a^2 + b^2}$.

Cor. 3. Hence, if tangents be drawn at the extremities of any two conjugate diameters (cor. 2. Art. 108.) a parallelogram will be formed, and all the parallelograms that can be formed by the tangents in this manner are equal to each other, as appears from the foregoing demonstration, being each equal to $2VC.2EC = VU.EK$; see the figure to Art. 133.*

132. If CA be a semi-conjugate to CP, then will $FP.PS = CA^2$.

Let FP and CA be produced to meet in R, and draw FY, SZ perpendicular to the tangent at P. Then the triangles FPY, PRL, and SPZ being equiangular, (4. 6.) $FP : FY :: PR : PL$ and $SP : SZ :: PR : PL$, ∴ compounding these proportions $FP.SP : FY.SZ :: PR^2 : PL^2 ::$



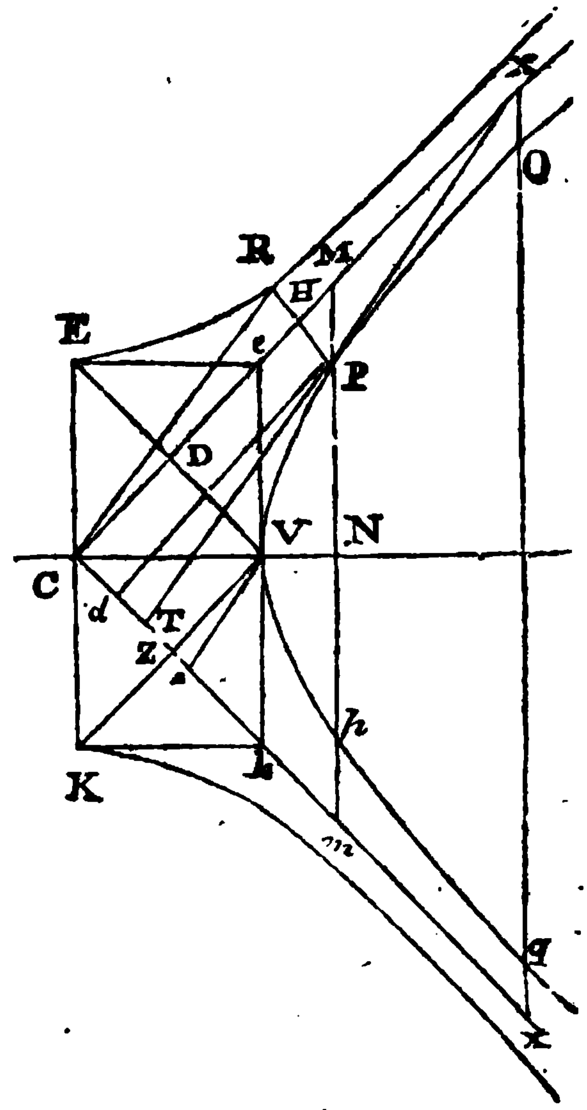
(Art. 109.) $VC^2 : PL^2 ::$ (cor. 1. Art. 131.) $CA^2 : EC^2$. But (Art. 111.) $FY.SZ = EC^2$, ∴ (14. 5.) $FP.SP = CA^2$. Q. E. D.

133. If through the vertex V the straight line ek be drawn equal and parallel to the minor axis EK, and from the centre C straight lines CM, Cm be drawn through e and k meeting any ordinate (PN) to the major axis, produced in M and m; then will $PM.Pm = Ve^2$. *See the following figure.*

Because (cor. Art. 116.) $CN^2 - VC^2 : PN^2 :: VC^2 : EC^2$ and (4. and 22. 6.) $CN^2 : NM^2 :: VC^2 : (Ve^2 =) EC^2$, ∴ (19. 5.) $VC^2 : NM^2 - PN^2 :: VC^2 : EC^2$, ∴ (14. 5.) $NM^2 - PN^2 = EC^2 = Ve^2$. But (cor. 5. 2.) $NM^2 - PN^2 = \overline{NM+PN}.\overline{NM-PN} = PM.Pm$; ∴ $PM.Pm = Ve^2$. Q. E. D.

Cor. 1. Hence, in like manner $pm.pM$ may be shewn to be equal to $Vk^2 = Ve^2$, ∴ $PM.Pm = pm.pM$; and if any other line

Xx be drawn parallel to Mm cutting the curve in Qq, then by similar reasoning it is shewn that $PM.Pm = QX.Qx = qx.qX$.



134. The straight lines CM, Cm continually approach the curve but do not meet it at any finite distance from the centre C, and therefore (Art. 103.) CM and Cm are asymptotes to the hyperbola.

Because $PM.Pm = Ve^2$ (Art. 133.), $PM = \frac{1}{Pm}$ (Art. 111 Part 4.) that is PM and Pm are inversely as each other, or as Pm increases, PM decreases; and when Pm becomes infinitely great, PM becomes infinitely small; that is, at any finite distance it does not entirely vanish. For the same reason as pM increases, pm decreases; and at an infinite distance from C becomes infinitely small, but does not vanish; $\because$ CM and Cm continually approach the curve, but do not meet it at any finite distance, they are therefore asymptotes.

Cor. 1. Hence it appears that CM, Cm are likewise asymptotes to the conjugate hyperbolas; for Ve, Vk being respectively equal and parallel to EC, CK, $\because$ (33. 1.) Ee, Kk will each be equal and parallel to VC; and by the same reasoning it is plain that CM, Cm continually approach the conjugate hyperbolas, but do not meet them at any finite distance from the centre.

Cor. 2. If VE be joined, the right angled triangles VEC, VeC having $CE = Ve$ and VC common, are equal in all respects (4. 1.) $\because$ $VE = eC$, and the angle $CVE = VCe$. In like manner it follows that $VK = Ck$, and since $EC = CK$ (Art. 108.) $\because$ the right angled parallelograms $CEeV$, $CKkV$ are equal (36. 1.) and consequently similar, and the four diameters Ce, EV, Ck, KV are equal, $\because$ (cor. Art. 241. Part 8.) CD, De, ED, DV, CZ, Zk, KZ, ZV are equal to each other; and because $Vk = CK = EC$ $\because$ (33. 1.) EV and Ck are parallel; in like manner it is plain that KV and Ce are parallel.

135. The position of any diameter with respect to the axis being given, that of its conjugate may be determined. For (Art. 133.) $NM^2-PN^2=EC^2$, and (Art. 129.) $RH^2-PN^2=EC^2$ $\therefore NM=RH$, $\therefore$ if CP be a semi-diameter, PN an ordinate at P to the major axis produced to the point M in the asymptote, and MR be drawn parallel to the major axis, then if RC be joined, RC will be conjugate to CP by cor. to Art. 129. And in the same manner the position of the conjugate to any other diameter is known. Q. E. D.

136. If a straight line Xx be drawn in any position cutting the curve in Qq, and the tangent TPt be parallel to it, then will $QX.Qx=PT.Pt$. *See the figure to Art.* 141.

Through Q and P draw Ww, Zz perpendicular to the axis; then the triangles XQW, TPZ, wQx, and zPt being similar $QW : QX :: PZ : PT$ (4. 6.) and $Qw : Qx :: Pz : Pt$; these proportions being compounded $QW.Qw : QX.Qx :: PZ.Pz : PT.Pt$. But (cor. Art. 133.) $QW.Qw=PZ.Pz$ $\therefore$ (14. 5.) $QX.Qx=PT.Pt$. Q. E. D.

Cor. By similar reasoning $qx.qX=PT.Pt$, $\therefore QX.Qx=qx.qX$.

137. The same construction remaining $QX=qx$.

For $QX.Qx=QX.\overline{Qq+qx}=QX.Qq+QX.qx$. And $qx.qX=qx.\overline{qQ+QX}=qx.qQ+qx.QX$; $\therefore$ (since $QX.Qx=qx.qX$ by the preceding corollary) $QX.Qq+QX.qx=qx.qQ+qx.QX$, from these equals take away $QX.qx$, and the remainders are equal, viz. $QX.Qq=qx.Qq$, divide both sides by Qq, and $QX=qx$. Q. E. D.

Cor. Hence, if Xx move parallel to itself so as to coincide with Tt, the points Q and q will each coincide with P, and Qq will vanish; also QX and qx will coincide with, and be equal to TP and tP respectively; $\therefore$ (since $QX=qx$) $TP=tP$, $\therefore QX.Qx=TP^2$.

138. The same construction remaining if through P, the diameter Gv be drawn, $Qv=qv$.

Because the triangles XvC, TPC are similar, and also xvC, tPC; $\therefore$ (4. 6. and 16. 5.) $vX : PT :: vC : PC :: vx : Pt$. But $PT=Pt$ by the preceding cor. $\therefore$ (14. 5.) $vX=vx$. But (Art. 137.) $QX=qx$, $\therefore vX-QX=vx-qx$ or $Qv=qv$. Q. E. D.

Cor. Hence $vX^2-vQ^2=PT^2$. For (cor. 5. 2.) $vX^2-vQ^2=\overline{vX-vQ}.\overline{vX+vQ}=QX.Qx=$(cor. Art. 137.) TP^2.

139. *If PH, VD be parallel to an asymptote Cx, then will $PH.CH=VD.CD$.* *See the figure to Art.* 134.

Through the points V and P draw the straight lines ek, Mm each perpendicular to the axis CN, and Pd, Vv parallel to CX. Because the triangles PMH, VeD, Pdm, and Vvk are similar, ∴ (4.6.) $PH : PM :: VD : Ve$, and $(Pd=)\ CH : Pm :: (Vv=)\ CD : Vk$ and by compounding $PH.CH : PM.Pm :: VD.CD : Ve.Vk$. But (Art. 133.) $PM.Pm=(VE^2=)\ Ve.Vk$, ∴ (14.5.) $PH.CH=VD.CD$. Q. E. D.

Cor. 1. Hence, because (cor. 2. Art. 134.) $CD=VD$, ∴ $PH.CH=CD^2=VD^2$.

Cor. 2. Hence also, if PH be produced to meet the conjugate hyperbola in R, $RH.CH=ED.CD=VD.CD=CD^2$ or VD^2.

Cor. 3. Hence, because $PH.CH=(CD^2=)\ RH.CH$, by dividing these equals by CH, $PH=RH$.

140. If PT be a tangent at P meeting the asymptotes in T and X, and CR be joined, then will CR and TX be parallel and $CR=TP=PX$.

For PH being parallel to CT one side of the triangle CXT, ∴ (2.6) $PX : PT :: XH : HC$. But (cor. Art. 137.) $PX=PT$, ∴ (prop. A.5.) $XH=HC$; ∴ In the triangles PXH, RCH there are the two sides XH, $HP=CH$, HR respectively, and the vertical angles at H equal (15.1.) ∴ $PX=(PT=)\ CR$; also the angle $HRC=HPX$ (4.1.) ∴ CR and TPX are parallel (27.1.) Q. E. D.

141. If PG and DO be conjugate diameters, and Qv an ordinate to PG, then will $Pv.vG : Qv^2 :: CP^2 : CD^2$.

At the point P draw the tangent PT, and produce the ordinate vQ to meet the asymptote in X.

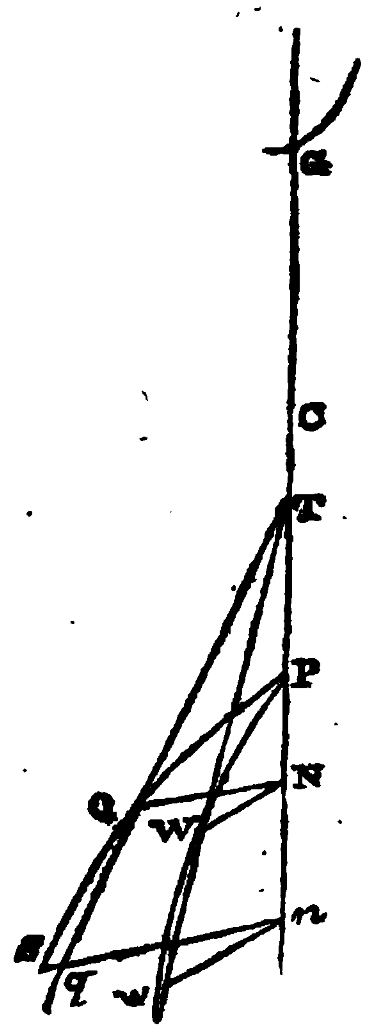
Then since CD, PT, and vX are parallel (Art. 96, 101.), TP is therefore parallel to Xv a side of the triangle XCv, $\therefore$ (2. 6.) $Cv : vX :: CP : PT$, and (22. 6.) $Cv^2 : vX^2 :: CP^2 : PT^2$; $\therefore$ (19. 6.) $Cv^2 - CP^2 : vX^2 - PT^2 :: CP^2 : PT^2$. But 1. $Cv^2 - CP^2 =$ (cor. 5. 2.) $\overline{Cv - CP} . \overline{Cv + CP} = Pv.vG$. 2. (cor. Art. 138.) $vX^2 - Qv^2 = PT^2$ or $vX^2 - PT^2 = Qv^2$. 3. (Art. 140.) $PT = CD$; $\therefore$ substituting these results for their equals in the above analogy, it becomes $Pv.vG : Qv^2 :: CP^2 : CD^2$. Q. E. D.

Cor. Hence $Pv.vG \propto Qv^2$.

142. The parameter P to any diameter PG is a third proportional to the major axis VU, and the conjugate DO to the diameter PG; that is, $P : DO :: DO : VU$.

Let Mm be the ordinate to the diameter PG which passes through the focus F, which is therefore the parameter P (Art. 102.); then will $Mr = \frac{1}{2}P$ (Art. 138.). Then because CD, FM are parallel, $Cr : CP :: Fe : Pe$ (2. 6.), and $Cr^2 : CP^2 :: Fe^2 : Pe^2$ (22. 6.), $\therefore$ dividendo $Cr^2 - CP^2 : CP^2 :: Fe^2 - Pe^2 : Pe^2$. But (Art. 141.) $Pr.rG : Mr^2 :: CP^2 : CD^2$; $\therefore$ alternando ($Pr.rG =$) $Cr^2 - CP^2 : CP^2 :: Mr^2 : CD^2$, $\therefore Mr^2 : CD^2 :: Fe^2 - Pe^2 : Pe^2$. But $Fe^2 - Pe^2 = \overline{Fe - Pe} . \overline{Fe + Pe}$ (cor. 5. 2.) $=$ $FP.PS$ (Art. 109.) $= CD^2$ (Art. 132.); $\therefore Mr^2 : CD^2 :: CD^2 : Pe^2$ and (22. 6.) $Mr : CD :: CD :$ ($Pe =$ by Art. 169.) VC; $\therefore$ (15. 5.) $2Mr$ or $P : DO :: DO : VU$. Q. E. D.

143. If two hyperbolas PQq, PWw be described on the same diameter GP and from any point N in it the ordinates NQ, NW be drawn, NQ shall have a given ratio to NW.



In GP produced take any other point n, and from it draw the ordinates nq, nw; then (cor. Art. 141.) $PN.NG : Pn.nG ::$ $NQ^2 : nq^2 :: NW^2 : nw^2$; $\therefore NQ : nq :: NW : nw$ (22. 6.), and $NQ : NW :: nq : nw$ (16. 5.). Q. E. D.

Cor. 1. Hence, as in the parabola (Art. 29, and cor.) and the ellipse (Art. 69. cor. 2.) the area NQP : area NWP in a given ratio. Also, if any point v be taken in the axis and vQ, vW be joined, the area PQv : the area PWv in a given ratio.

Cor. 2. Hence, if PQq be an hyperbola, and from every point N, n, &c. in the diameter, ordinates NQ, nq, &c. be drawn, and if straight lines NW, nw, &c. be drawn from the points N, n, &c. making a given angle with NQ, nq, &c. and having a given ratio to each other, the curve PWw passing through P, and the extremities of those lines, will be an hyperbola, having PG for its diameter.

For $NQ^2 : NW^2 :: nq^2 : nw^2 :: PN.NG : Pn.nG$, that is, $nq^2 \propto PN.NG$ (cor. Art. 141.) which is the property of the hyperbola.

144. If two hyperbolas PQq, PWw be described on the same diameter PG, and NQ, NW an ordinate to each be drawn from the same point N, tangents at Q and W will intersect the diameter PG in the same point T.

Let QT be a tangent at Q, and join TW; TW is a tangent; for if not, let it meet the hyperbola again in w, draw the ordinates nw, nq, and produce nq to meet the tangent TQ produced in s. Then because the triangles QTN, sTN are similar, as also TNW, Tnw, $\because$ (4. 6.) $NQ : ns$ ($:: TN : Tn$) $:: NW : nw$. But (Art. 143.) $NQ : nq :: NW : nw$, $\because$ $NQ : ns :: NQ : nq$ $\because$ (9. 5.) $ns = nq$, the greater equal to the less, which is absurd; $\because$ TW which meets the hyperbola, cannot cut it; TW is therefore a tangent. Q. E. D.

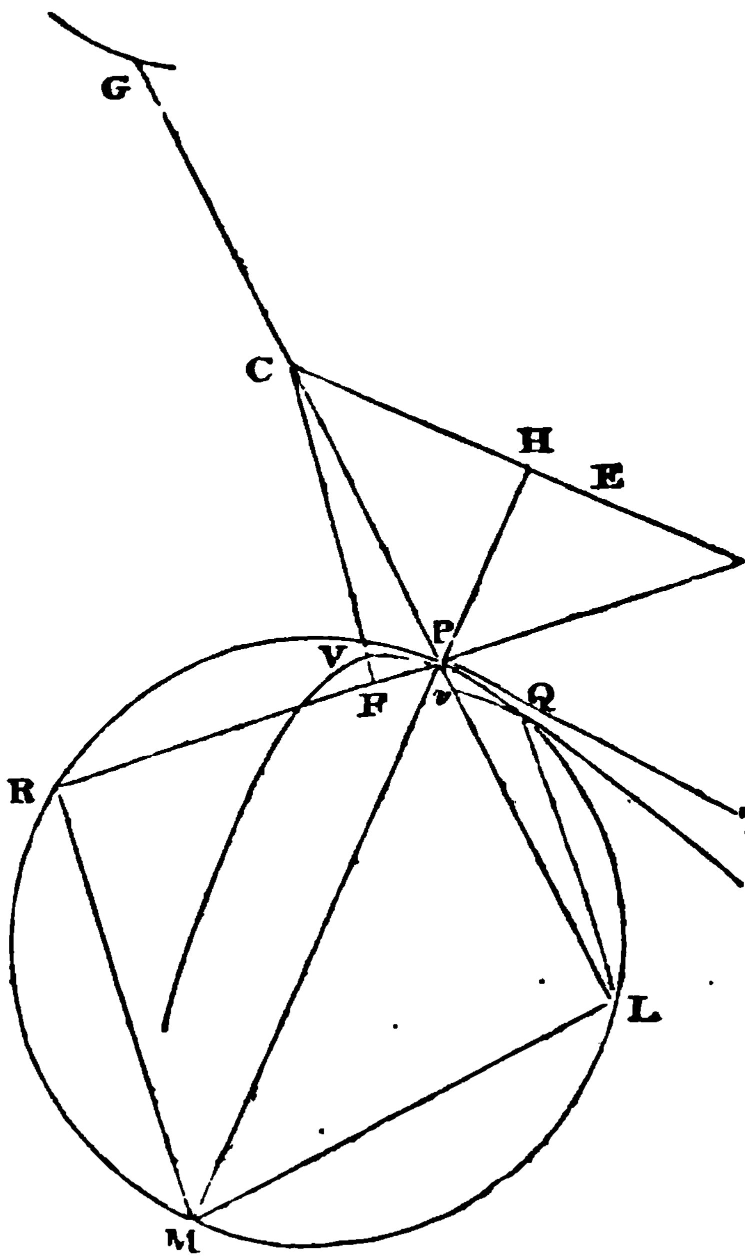
Cor. Hence, if GP be the major axis of the hyperbola PQp, since (cor. 1. Art. 117.) tangents at Q and W will in like manner meet the axis GP in the same point T, $\because$ (Art. 121.) $CN.CT = CP^2$, $\because$ (17. 6.) $CN : CP :: CP : CT$.

145. If PM be the diameter of curvature at the point P, and PL, PR chords of curvature, the former passing through the centre C, and the latter through the focus F, then will MP produced be perpendicular to the semi-conjugate diameter EC, and

$$PC : CE :: CE : \tfrac{1}{2} PL$$
$$PH : CE :: CE : \tfrac{1}{2} PM$$
$$VC : CE :: CE : \tfrac{1}{2} PR$$

First. Let PQ be a nascent *arc* common to the hyperbola and circle of curvature, draw Qv parallel to the tangent PT, join

CF, and draw the chords *PQ*, *QL*, *LM*, *MR*. Then the triangles *QPv*, *QPL* having the angle *QPv* common, and (29. 1.) *PQv*= *TPQ*=(32. 3.) *QLP*, are equiangular, ∵ (4. 6.) *Pv* : *PQ* :: *PQ* : *PL*, ∵ (17. 6.) *Pv.PL* $=PQ^2$; but since the *arc PQ* is indefinitely small, *Qv* and *PQ* will be indefinitely near a coincidence, and therefore may be considered as equal, ∵ *Pv.PL*= $PQ^2=Qv^2$, also for the same reason *vC*= *PC*.



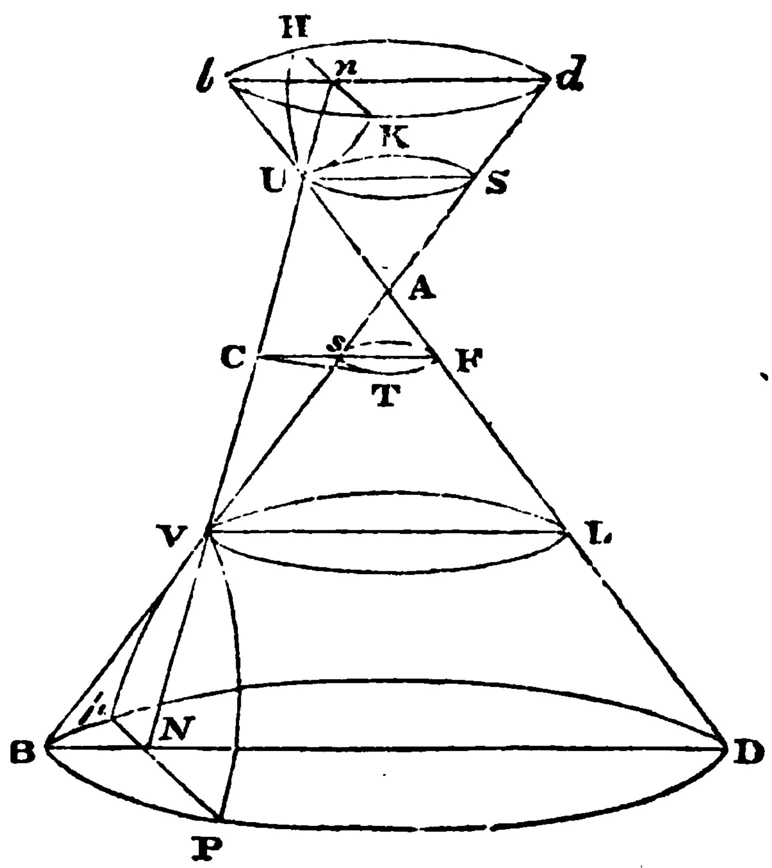
But (Art. 141.) *Pv.vG* : (Qv^2=) *Pv.PL* :: PC^2 : CE^2, ∵ (15. 5.) (*vG*=) 2*PC* : *PL* :: *PC* : ½*PL* :: PC^2 : CE^2, ∵ (cor. 2, 20. 6.) *PC* : *CE* :: *CE* : ½*PL*.

Secondly. The triangles *PCH*, *PML* having the vertical angles at *P* equal (15. 1.) and likewise the angles at *H* and *L* right angles (31. 3. and construction), are equiangular, and *PH* : *PC* :: *PL* : *PM* :: ½*PL* : ½*PM*; but by the former case *PC* : *CE* :: *CE* : ½*PL*, ∵ *ex æquo PH* : *CE* :: *CE* : ½*PM*.

Thirdly. The triangles *PKH*, *PMR* are similar (15. 1, 31. 3. and construction) ∵ *PK* : *PH* :: *PM* : *PR* (4. 6.) :: ½*PM* : ½*PR* (15. 5.). But, as in the preceding case *PH* : *CE* :: *CE* : ½*PM*, ∵ *ex æquo* (*PK*=by Art. 109.) *VC* : *CE* :: *CE* : ½*PR*. Q. E. D.

Cor. Hence, because 2*VC* : 2*CE* :: 2*CE* : *PR* by the above, and 2*VC* : 2*CE* :: 2*CE* : the parameter (Art. 142.) ∵ the chord of curvature *PR*, passing through the focus, is equal to the parameter.

146. If a cone ABD be cut by a plane PVp which meets the opposite cone Abd in any point U except the vertex, the section PVp will be an hyperbola.



Let $dHbKA$ be the opposite cone, let BD be perpendicular to pP; bisect UV in C, draw VL, CF, US, and bd parallel to the diameter BD of the base, then will the section passing through VL, CF, US, and bd parallel to the base be circles (13. 12.) and HK, Pp the intersections of the cutting plane with the planes of the circles $HbKd$, $pBPD$ will be parallel (16. 11.). Draw CT a tangent to the circle TFs, then (36. 3.) $BN.ND = PN^2$ and $bn.nd = Kn^2$, also $sC.CF = CT^2$. Now the triangles VNB, sCV are similar, as are UND, UCF, $\therefore$ (4. 6.) $VN : NB :: VC : Cs$ and $UN : ND :: UC : CF$, $\therefore$ (compounding these analogies) $VN.UN : BN.ND :: VC.UC : Cs.CF$; that is, $VN.NU : PN^2 :: VC^2 : CT^2$ $\therefore$ (Art. 116.) the figure PVp is an hyperbola, C the centre, CV the semi-major axis, and CT the semi-minor axis. Q. E. D.

Cor. Hence the section HUK will be the opposite hyperbola to PVp and similar to it; for $Vn : nd :: VC : Cs$ and $Un : nb :: UC : CF$, $\therefore$ (compounding) $Vn.nU : dn.nb :: UC.VC : Cs.CF$, or (as above) $Vn.nU : nK^2 :: VC^2 : CT^2$.

The foregoing are the principal and most useful properties of the Conic Sections; a branch of knowledge, which is absolutely necessary to prepare the Student for the Physico Mathematical Sciences; many more properties of these celebrated curves might have been added, if our prescribed limits had permitted; but it would require a large volume, to treat the subject in that comprehensive and circumstantial manner, which its importance demands; we must therefore refer the reader, for a

more ample detail, to the writings of Apollonius, De l'Hôpital, Hamilton, Emerson, &c. observing in conclusion, that what is here given will, as far as relates to this subject. be fully sufficient to enable him to understand Sir Isaac Newton's Principia, or any other work usually read by Students, on Mathematical Philosophy and Astronomy.

THE END.

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